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A NOTE ON CONTROLLABILITY OF IMPULSIVE NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY A ROSENBLATT PROCESS IN A HILBERT SPACE

SALAH HAJJI[1] AND EL HASSAN LAKHEL[2]

ABSTRACT. In this paper we study the controllability results of impulsive neutral stochastic functional differential equations with variable delays driven by Rosenblatt process in a Hilbert space. The controllability results are obtained by the Banach fixed point theorem. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained result.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory and plays important role in both deterministic and stochastic control systems. Roughly speaking, controllability generally means that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. If the system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details reader may refer [7, 15, 16] and references therein.

Impulsive systems arise naturally in various fields, such as mechanical systems, economics, engineering, biological systems and population dynamics, undergo abrupt changes in their state at certain moments between intervals of continuous evolution. Since many evolution process, optimal control models in economics, stimulated neural networks, frequency- modulated systems and some motions of missiles or aircrafts are characterized by the impulsive dynamical behavior. Recently, there has been increasing interest in the analysis and synthesis of impulsive systems due to their significance both in theory and applications. Thus the theory of impulsive differential equations has seen considerable development (see [1, 12, 5, 18] and references therein).

In this paper, we study the controllability of neutral functional stochastic differential equations with impulses described in the form

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[1] Corresponding author.

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$$\begin{cases} d[x(t)+g(t,x(t-\theta(t)))] = [Ax(t)+f(t,x(t-\rho(t)))+Bu(t)]dt+\sigma(t)dZ_H(t),\ t\in[0,T],\ t\neq t_k, \\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=I_k x(t_k^-), \quad t=t_k,\ k=1,2,....,m, \\ x(t)=\varphi(t),\ -\tau\le t\le 0 \quad a.s. \quad \tau>0. \end{cases} \tag{1.1}$$

Here A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X. Z_H is a Rosenblatt process on a real and separable Hilbert space Y and the control function $u(.)$ takes values in $L^2([0,T],U)$, the Hilbert space of admissible control functions for a separable Hilbert space U. The symbol B stands for a bounded linear operator from U into X. Moreover, the fixed moments of time t_k satisfy $0<t_1<t_2<...<t_m$. $\Delta x(t_k)$ denotes the jump in the state x at time t_k with $I(.) : X \longrightarrow X$ determining the size of the jump. Suppose that $\theta,\ \rho : [0,+\infty)\to[0,\tau]$ $(\tau>0)$ are continuous and $f,g : [0,+\infty)\times X\to X,\quad \sigma : [0,+\infty)\to\mathcal{L}_2^0(Y,X),$ are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

In recent years the stochastic functional differential equations driven by a fractional Brownian motion (fBm) have attracted the attention of many authors and many valuable results on existence, uniqueness and the stability of the solution have been established, see [2, 3, 4]. The very large utilization of the fBm in practice are due to its self-similarity, stationarity of increments and long-range dependence; one prefers in general fBm before other processes because it is Gaussian and the calculus for it is easier; but in concrete situations when the gaussianity is not plausible for the model, one can use for example the Rosenblatt process. Although defined during the 60s and 70s [17, 21] due to their appearance in the Non-Central Limit Theorem, the systematic analysis of Rosenblatt processes has only been developed during the last ten years, motivated by their nice properties (self-similarity, stationarity of the increments, long-range dependence). Since they are non-Gaussian and self-similar with stationary increments, the Rosenblatt processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. There exists a consistent literature that focuses on different theoretical aspects of the Rosenblatt processes. Let us recall some of these works. For example, the rate of convergence to the Rosenblatt process in the Non Central Limit Theorem has been given by Leonenko and Ahn [9]. Tudor [22] studied the analysis of the Rosenblatt process. The distribution of the Rosenblatt process has been given in [10]. An existence and uniqueness result of mild solutions for a class of neutral stochastic differential equation with finite delay driven by Rosenblatt process in Hilbert space has been recently established in Ren and Shen [19].

On the other hand, to the best of our knowledge, there is no paper which investigates the study of controllability for impulsive neutral stochastic differential equations with delays driven by a Rosenblatt process. Thus, we will make the first attempt to study such problem in this paper. Our results are inspired by the one in [19] where the existence and uniqueness of mild solutions to model

(1.1) is studied with $B = 0$ and without impulses, as well as some results on the asymptotic behavior.

The outline of the paper is as follows: In Section 2 we introduce some notations, concepts, and basic results about Rosenblatt process, Wiener integral over Hilbert spaces with respect to it, and we recall some preliminary results about analytic semi-groups and fractional power associated to its generator. In Section 3, we derive the controllability of neutral impulsive stochastic differential systems driven by a Rosenblatt process. Finally, in Section 4, paper concludes with an example is to illustrate the obtained results.

2. Preliminaries

In this section, we collect some definitions and lemmas on Wiener integrals with respect to an infinite dimensional Rosenblatt process and we recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators, which will be used throughout the whole of this paper. For details of this section, we refer the reader to [22, 13] and references therein.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Self-similar processes are invariant in distribution under suitable scaling. They are of considerable interest in practice since aspects of the self-similarity appear in different phenomena like telecommunications, turbulence, hydrology or economics. An interesting class of processes with long range dependence that have "moderate" tails ("moderate" means that the tails are heaver than those of normal distribution but lighter than those of a stable distribution), known as Rosenblatt processes, appears in the so-called Non-Central Limit Theorem see [21]. We briefly recall the Rosenblatt process as well as the Wiener integral with respect to it.
Let us recall the notion of Hermite rank. Denote by $H_j(x)$ the Hermite polynomial of degree j given by $H_j = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{\frac{-x^2}{2}}$ and let g be a function on $\mathbb{R}$ such that $\mathbb{E}[g(\zeta_0)] = 0$ and $\mathbb{E}[g(\zeta_0)^2] < \infty$. Assume that g has the following expansion in Hermite polynomials

$$g(x) = \sum_{j \geq 0} c_j H_j(x),$$

where $c_j = \frac{1}{j!}\mathbb{E}(g(\zeta_0 H_j(\zeta_0)))$. The Hermite rank of g is defined by

$$k = min\{j | c_j \neq 0\}.$$

Since $\mathbb{E}[g(\zeta_0)] = 0$, we have $k \geq 1$. Consider $(\zeta_n)_{n \in \mathbb{Z}}$ a stationary Gaussian sequence with mean zero and variance 1 which exhibits long range dependence in the sense that the correlation function satisfies

$$r(n) = \mathbb{E}(\zeta_0 \zeta_n) = n^{\frac{2H-2}{k}} L(n),$$

with $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity. Then the following family of stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\zeta_j)$$

converges as $n \longrightarrow \infty$, in the sense of finite dimensional distributions, to the self-similar stochastic process with stationary increments

$$Z_H^k(t) = c(H,k) \int_{\mathbb{R}^k} \left(\int_0^t \prod_{j=1}^k (s-y_j)_+^{-(\frac{1}{2}+\frac{1-H}{k})} ds \right) dB(y_1)...dB(y_k), \tag{2.1}$$

where $x_+ = max(x,0)$. The above integral is a Wiener-Itô multiple integral of order k with respect to the standard Brownian motion $(B(y))_{y\in\mathbb{R}}$ and the constant $c(H,k)$ is a normalizing constant that ensures $\mathbb{E}(Z_H^k(1))^2 = 1$.

The process $(Z_H^k(t))_{t\geq 0}$ is called the Hermite process. When $k=1$ the process given by (2.1) is nothing else that the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$. For $k=2$ the process is not Gaussian. If $k=2$ then the process (2.1) is known as the Rosenblatt process. It was introduced by Rosenblatt in [17] and was given its name by Taqqu in [20]. The fractional Brownian motion is of course the most studied process in the class of Hermite processes due to its significant importance in modeling. A stochastic calculus with respect to it has been intensively developed in the last decade. The Rosenblatt process is, after fBm, the most well known Hermite process.

We also recall the following properties of the Rorenblatt process:

- The process Z_H^k is H-self-similar in the sense that for any $c>0$,

$$(Z_H^k(ct)) =^{(d)} (c^H Z_H^k(t)), \tag{2.2}$$

  where " $=^{(d)}$ " means equivalence of all finite dimensional distributions. It has stationary increments and all moments are finite.
- From the stationarity of increments and the self-similarity, it follows that, for any $p \geq 1$

$$\mathbb{E}|Z_H(t) - Z_H(s)|^p \leq |\mathbb{E}(Z_H(1))|^p |t-s|^{pH}.$$

  As a consequence the Rosenblatt process has Hölder continuous paths of order γ with $0<\gamma<H$.

Self-similarity and long-range dependence make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{Z_H(t), t \in [0,T]\}$ the one-dimensional Rosenblatt process with parameter $H \in (1/2,1)$. By Tudor [22], it is well known that Z_H has the following integral representation:

$$Z_H(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1) \frac{\partial K^{H'}}{\partial u}(u,y_2) du \right] dB(y_1) dB(y_2), \tag{2.3}$$

where $B = \{B(t) : t \in [0, T]\}$ is a Wiener process, $H' = \frac{H+1}{2}$ and $K^H(t, s)$ is the kernel given by

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K^H(t, s) = 0$ if $t \leq s$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant.

The covariance of the Rosenblatt process $\{Z_H(t), t \in [0, T]\}$ satisfies, for every $s, t \geq 0$,

$$R_H(s,t) := \mathbb{E}(Z_H(t) Z_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The basic observation is the fact that the covariance structure of the Rosenblatt process is similar to the one of the fractional Brownian motion and this allows the use of the same classes of deterministic integrands as in the fractional Brownian motion case whose properties are known.

Now, we introduce Wiener integrals with respect to the Rosenblatt process. We refer to [22] for additional details on the Rosenblatt process .
By formula (2.3) we can write

$$Z_H(t) = \int_0^t \int_0^t I(\mathbf{1}_{[0,t]})(y_1, y_2) dB(y_1) dB(y_2),$$

where by I we denote the mapping on the set of functions $f : [0, T] \longrightarrow \mathbb{R}$ to the set of functions $f : [0, T]^2 \longrightarrow \mathbb{R}$

$$I(f)(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Let us denote by $\mathcal{E}$ the class of elementary functions on $\mathbb{R}$ of the form

$$f(.) = \sum_{j=1}^n a_j \mathbf{1}_{(t_j, t_{j+1}]}(.), \qquad 0 \leq t_j < t_{j+1} \leq T, \quad a_j \in \mathbb{R}, \quad i = 1, ..., n.$$

For $f \in \mathcal{E}$ as above, it is natural to define its Wiener integral with respect to the Rosenblatt process Z_H by

$$\int_0^T f(s) dZ_H(s) := \sum_{j=1}^n a_j \left[Z_H(t_{j+1}) - Z_H(t_j)\right] = \int_0^T \int_0^T I(f)(y_1, y_2) dB(y_1) dB(y_2). \tag{2.4}$$

Let $\mathcal{H}$ be the set of functions f such that

$$\mathcal{H} = \left\{ f : [0, T] \longrightarrow \mathbb{R} : \quad \|f\|_{\mathcal{H}} := \int_0^T \int_0^T (I(f)(y_1, y_2))^2 dy_1 dy_2 < \infty \right\}.$$

It hold that (see Maejima and Tudor [11])

$$\|f\|_{\mathcal{H}} = H(2H-1)\int_0^T\int_0^T f(u)f(v)|u-v|^{2H-2}dudv,$$

and, the mapping

$$f \longrightarrow \int_0^T f(u)dZ_H(u) \tag{2.5}$$

provides an isometry from $\mathcal{E}$ to $L^2(\Omega)$. On the other hand, it has been proved in [14] that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{H}$. As a consequence the mapping (2.5) can be extended to an isometry from $\mathcal{H}$ to $L^2(\Omega)$. We call this extension as the Wiener integral of $f \in \mathcal{H}$ with respect to Z_H.

Let us consider the operator K_H^* from $\mathcal{E}$ to $\mathbb{L}^2([0,T])$ defined by

$$(K_H^*\varphi)(y_1,y_2) = \int_{y_1\vee y_2}^T \varphi(r)\frac{\partial K}{\partial r}(r,y_1,y_2)dr,$$

where $K(.,.,.)$ is the kernel of Rosenblatt process in representation (2.3)

$$K(r,y_1,y_2) = \mathbf{1}_{[0,t]}(y_1)\mathbf{1}_{[0,t]}(y_2)\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du.$$

We refer to [22] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$. It follows from [22] that $\mathcal{H}$ contains not only functions but its elements could be also distributions. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|_{|\mathcal{H}|}^2 := \alpha_H\int_0^T\int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2}dsdt < \infty,$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions (see [22]).

Lemma 2.1.

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\psi \in \mathbb{L}^2([0,T])$*, we have*

$$\|\psi\|_{|\mathcal{H}|}^2 \le 2HT^{2H-1}\int_0^T |\psi(s)|^2ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty}\lambda_n < \infty$. where $\lambda_n \ge 0$ $(n = 1,2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1,2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional $Q-$Rosenblatt process on Y as

$$Z_H(t) = Z_Q(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n z_n(t), \tag{2.6}$$

where $(z_n)_{n\ge 0}$ is a family of real independent Rosenblatt process.

Note that the series (2.6) is convergent in $L^2(\Omega)$ for every $t \in [0,T]$, since

$$\mathbb{E}|Z_Q(t)|^2 = \sum_{n=1}^{\infty} \lambda_n \mathbb{E}(z_n(t))^2 = t^{2H} \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Note also that Z_Q has covariance function in the sense that

$$E\langle Z_Q(t), x\rangle\langle Z_Q(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-Rosenblatt process, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty}\langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.
Now, let $\phi(s)$; $s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, such that

$$\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|_{\mathcal{L}_2^0}^2 < \infty.$$

The Wiener integral of ϕ with respect to Z_Q is defined by

$$\int_0^t \phi(s) dZ_Q(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s) e_n dz_n(s) = \sum_{n=1}^{\infty} \int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\phi e_n)(y_1, y_2) dB(y_1) dB(y_2). \tag{2.7}$$

Now, we end this subsection by stating the following result which is fundamental to prove our result.

Lemma 2.2. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds < \infty$ then the above sum in (2.7) is well defined as a X-valued random variable and we have*

$$\mathbb{E}\|\int_0^t \psi(s) dZ_H(s)\|^2 \le 2Ht^{2H-1}\int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\|\int_0^t \psi(s) dZ_H(s)\|^2 &= \sum_{n=1}^{\infty} \mathbb{E}\|\int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\psi e_n)(y_1, y_2) dB_n(y_1) dB_n(y_2)\|^2 \\
&\le \sum_{n=1}^{\infty} 2Ht^{2H-1}\int_0^t \lambda_n \|\psi(s) e_n\|^2 ds \\
&= 2Ht^{2H-1}\int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.
\end{aligned}$$

□

Now we turn to state some notations and basic facts about the theory of semi-groups and fractional power operators. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on X. For the theory of strongly continuous semigroup, we refer to [13] and [6]. If $(S(t))_{t\geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^{\alpha}$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, the subspace $D(-A)^{\alpha}$ is dense in X, and the expression $\|h\|_{\alpha} = \|(-A)^{\alpha}h\|$ defines a norm in $D(-A)^{\alpha}$. If X_{α} represents the space $D(-A)^{\alpha}$ endowed with the norm $\|.\|_{\alpha}$, then the following properties are well known (cf. [13], p. 74).

Lemma 2.3. *Suppose that the preceding conditions are satisfied.*
(1) Let $0 < \alpha \leq 1$. Then X_{α} is a Banach space.
(2) If $0 < \beta \leq \alpha$ then the injection $X_{\alpha} \hookrightarrow X_{\beta}$ is continuous.
(3) For every $0 < \alpha \leq 1$ there exists $M_{\alpha} > 0$ such that

$$\|(-A)^{\alpha}S(t)\| \leq M_{\alpha}t^{-\alpha}e^{-\lambda t}, \quad t > 0, \;\; \lambda > 0.$$

3. Controllability results

In this section we derive controllability conditions for impulsive neutral stochastic functional differential equations with variable delays driven by Rosenblatt process in a real separable Hilbert space. In order to define the concept of mild solution of the problem (1.1), we consider the following space:
$\mathcal{PC}([-\tau, T], \mathbb{L}^2(\Omega, X)) := \{[-\tau, T] \longrightarrow \mathbb{L}^2(\Omega, X)$ such that $x(.)$ is continuous except for a finite number of points t_i at which the left and right limits exist and $x(t_k^+) = x(t_k)\}$.
It is easy to verify that $\mathcal{PC}$ is a Banach space (see Lemma 2.6 in [23]) with the supremum norm

$$\|\xi\|_{\mathcal{PC}} = (\sup_{u\in[-\tau,T]} \mathbb{E}\|\xi(u)\|^2)^{1/2}.$$

Definition 3.1. An X-valued process $\{x(t),\ t \in [-\tau, T]\}$, is called a mild solution of equation (1.1) if

i) $x(.) \in \mathcal{PC}([-\tau, T], \mathbb{L}^2(\Omega, X))$,
ii) $x(t) = \varphi(t)$, $-\tau \leq t \leq 0$.
iii) For arbitrary $t \in [0, T]$, we have

$$\begin{aligned} x(t) &= S(t)(\varphi(0) + g(0, \varphi(-\theta(0)))) - g(t, x(t - \theta(t))) \\ &\quad - \textstyle\int_0^t AS(t-s)g(s, x(s - \theta(s)))ds + \int_0^t S(t-s)f(s, x(s - \rho(s)))ds \\ &\quad + \textstyle\int_0^t S(t-s)(Bu)(s)ds + \int_0^t S(t-s)\sigma(s)dZ_H(s) \\ &\quad + \textstyle\sum_{0<t_k<t} S(t-t_k)I_k x(t_k^-), \quad \mathbb{P}-a.s \end{aligned} \tag{3.1}$$

iv) $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k x(t_k^-), \quad t = t_k,\ k = 1, 2, ..., m.$

Definition 3.2. The impulsive stochastic neutral functional differential equation (1.1) is said to be controllable on the interval $[-\tau, T]$, if for every initial stochastic process φ defined $[-\tau, 0]$, there exists a stochastic control $u \in L^2([0,T], U)$ such that the mild solution $x(.)$ of (1.1) satisfies $x(T) = x_1$, where x_1 and T are the preassigned terminal state and time, respectively.

For the controllability of the system (1.1), we impose the following conditions on data of the problem:

($\mathcal{H}.1$) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on X. Further, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$, and that, see Lemma 2.3,

$$\|S(t)\| \leq M \quad \text{and} \quad \|(-A)^{1-\beta}S(t)\| \leq \frac{M_{1-\beta}}{t^{1-\beta}}$$

for some constants $M,\ M_{1-\beta}$ and every $t \in [0,T]$.

($\mathcal{H}.2$) The function $f : [0,+\infty) \times X \to X$ satisfies the following Lipschitz conditions: that is, there exist positive constants $C_i := C_i(T), i = 1,2$ such that, for all $t \in [0,T]$ and $x, y \in X$
 (i) $\|f(t,x) - f(t,y)\| \leq C_1\|x-y\|$.
 (ii) $\|f(t,x)\|^2 \leq C_2(1 + \|x\|^2)$.

($\mathcal{H}.3$) There exist constants $\frac{1}{2} < \beta < 1,\ C_i := C_i(T), i = 3,4$, such that the function g is X_β-valued and satisfies for all $t \in [0,T]$ and $x, y \in X$
 (i) $\|(-A)^\beta g(t,x) - (-A)^\beta g(t,y)\| \leq C_3\|x-y\|$.
 (ii) $\|(-A)^\beta g(t,x)\|^2 \leq C_4^2(1 + \|x\|^2)$.

($\mathcal{H}.4$) The function $(-A)^\beta g$ is continuous in the quadratic mean sense:
For all $x \in \mathcal{PC}([-\tau,T], \mathbb{L}^2(\Omega, X))$,

$$\lim_{t\to s} \mathbb{E}\|(-A)^\beta g(t, x(t-\theta(t))) - (-A)^\beta g(s, x(s-\theta(s)))\|^2 = 0.$$

($\mathcal{H}.5$) The function $\sigma : [0,+\infty) \to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \quad \forall T > 0.$$

($\mathcal{H}.6$) The linear operator W from U into X defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds.$$

has an inverse operator W^{-1} that takes values in $L^2([0,T], U) \setminus kerW$, where $kerW = \{x \in L^2([0,T], U),\ Wx = 0\}$ (see [15]), and there exists finite positive constants $M_b,\ M_w$ such that $\|B\| \leq M_b$ and $\|W^{-1}\| \leq M_w$.

($\mathcal{H}.7$) $I_k \in C(X,X)$ and there exist a finite positive constants $\alpha_k,\ \widetilde{\alpha}_k$ such that for all $t \in [0,T]$ and $x, y \in X$, we have:

$$\|I_k(x) - I_k(y)\| \leq \alpha_k\|x-y\| \qquad \text{and} \qquad \|I_k x(t)\|^2 \leq \widetilde{\alpha}_k(1+\|x\|^2),\ k = 1,2,...,m.$$

Moreover, we assume that $\varphi \in \mathcal{C}([-\tau,0], \mathbb{L}^2(\Omega, X))$.

The main result of this paper is given in the next theorem.

Theorem 3.3. *Suppose that* $(\mathcal{H}.1)-(\mathcal{H}.7)$ *hold. Then, the system (1.1) is controllable on* $[-\tau, T]$ *provided*

$$\|(-A)^{-\beta}\|^2 C_3^2 + M^2 m \sum_{k=1}^{m} \alpha_k < \frac{1}{8}. \tag{3.2}$$

Proof. Fix $T > 0$ and let $\mathcal{PC} := \mathcal{PC}([-\tau, T], \mathbb{L}^2(\Omega, X)$ be the Banach space of piecewise continuous maps from $[-\tau, T]$ into $\mathbb{L}^2(\Omega, X)$, equipped with the supremum norm $\|\xi\|_{\mathcal{PC}} = (\sup_{u\in[-\tau,T]} \mathbb{E}\|\xi(u)\|^2)^{1/2}$ and let us consider the set

$$S_T = \{x \in \mathcal{PC} : x(s) = \varphi(s), \text{ for } s \in [-\tau, 0]\}.$$

S_T is a closed subset of $\mathcal{PC}$ provided with the norm $\|.\|_{\mathcal{PC}}$.

Using the hypothesis (H6) for an arbitrary function $x(.)$, define the stochastic control

$$\begin{aligned} u(t) &= W^{-1}\{x_1 - S(T)(\varphi(0) + g(0, \varphi(-\theta(0)))) + g(T, x(T-\theta(T))) \\ &+ \int_0^T AS(T-s)g(s, x(s-\theta(s)))ds - \int_0^T S(T-s)f(s, x(s-\rho(s))ds \\ &- \int_0^T S(T-s)\sigma(s)dZ_H(s) + \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)\}(t). \end{aligned}$$

To formulate the controllability problem in the form suitable for application of the Banach fixed point theorem, put the control $u(.)$ into the stochastic control system (3.1) and obtain a non linear operator ψ on S_T given by $\psi(x)(t) = \varphi(t)$ for $t \in [-\tau, 0]$ and for $t \in [0, T]$

$$\begin{aligned} \psi(x)(t) &= S(t)(\varphi(0) + g(0, \varphi(-\theta(0)))) - g(t, x(t-\theta(t))) - \int_0^t AS(t-s)g(s, x(s-\theta(s)))ds \\ &+ \int_0^t S(t-s)f(s, x(s-\rho(s)))ds + \int_0^t S(t-\nu)\sigma(s)dZ_H(s) + \sum_{0<t_k<t} S(t-t_k)I_k x(t_k^-) \\ &+ \int_0^t S(t-\nu)BW^{-1}\{x_1 - S(T)(\varphi(0) + g(0, \varphi(-\theta(0)))) + g(T, x(T-\theta(T))) \\ &+ \int_0^T AS(T-s)g(s, x(s-\theta(s)))ds - \int_0^T S(T-s)f(s, x(s-\rho(s))ds \\ &- \int_0^T S(T-s)\sigma(s)dZ_H(s) + \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)\}d\nu. \end{aligned}$$

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator ψ. Clearly, $\psi x(T) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time

T, provided we can obtain a fixed point of the operator ψ which implies that the system in controllable.

Next we will show by using Banach fixed point theorem that ψ has a unique fixed point. We divide the subsequent proof into two steps.

Step 1: For arbitrary $x \in S_T$, let us prove that $t \to \psi(x)(t)$ is continuous on the interval $[0,T]$ in the $\mathbb{L}^2(\Omega, X)$-sense.
Let $0 < t < T$ and $|h|$ be sufficiently small. Then for any fixed $x \in S_T$, we have

$$\begin{aligned}
\mathbb{E}\|\psi(x)(t &+ h) - \psi(x)(t)\|^2 \leq 7\mathbb{E}\|(S(t+h) - S(t))(\varphi(0) + g(0, \varphi(-\theta(0))))\|^2 \\
&+ 7\mathbb{E}\|g(t+h, x(t+h-\theta(t+h))) - g(t, x(t-\theta(t)))\|^2 \\
&+ 7\mathbb{E}\|\int_0^{t+h} AS(t+h-s)g(s, x(s-\theta(s))ds - \int_0^t AS(t-s)g(s, x(s-\theta(s))ds\|^2 \\
&+ 7\mathbb{E}\|\int_0^{t+h} S(t+h-s)f(s, x(s-\rho(s)))ds - \int_0^t S(t-s)f(s, x(s-\rho(s)))ds\|^2 \\
&+ 7\mathbb{E}\|\int_0^{t+h} S(t+h-s)\sigma(s)dZ_H(s) - \int_0^t S(t-s)\sigma(s)dZ_H(s)\|^2 \\
&+ 7\mathbb{E}\|\sum_{0<t_k<t+h} S(t+h-t_k)I_k x(t_k^-) - \sum_{0<t_k<t} S(t-t_k)I_k x(t_k^-)\|^2 \\
&+ 7\mathbb{E}\|\int_0^{t+h} S(t+h-\nu)BW^{-1}\{x_1 - S(T)(\varphi(0) + g(0, \varphi(-\theta(0)))) + g(T, x(T-\theta(T))) \\
&+ \int_0^T AS(T-s)g(s, x(s-\theta(s)))ds - \int_0^T S(T-s)f(s, x(s-\rho(s))ds \\
&- \int_0^T S(T-s)\sigma(s)dZ_H(s) + \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)\}d\nu \\
&- \int_0^t S(t-\nu)BW^{-1}\{x_1 - S(T)(\varphi(0) + g(0, \varphi(-\theta(0)))) + g(T, x(T-\theta(T))) \\
&+ \int_0^T AS(T-s)g(s, x(s-\theta(s)))ds - \int_0^T S(T-s)f(s, x(s-\rho(s))ds \\
&- \int_0^T S(T-s)\sigma(s)dZ_H(s) + \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)\}d\nu \\
&= 7\sum_{1\leq i\leq 7} \mathbb{E}\|I_i(h)\|^2.
\end{aligned}$$

By the strong continuity of $S(t)$, we have

$$\lim_{h\to 0}(S(t+h)-S(t))(\varphi(0)+g(0,\varphi(-\theta(0))))=0.$$

The condition $(\mathcal{H}.1)$ assures that

$$\|(S(t+h)-S(t))(\varphi(0)+g(0,\varphi(-\theta(0))))\|\leq 2M\|\varphi(0)+g(0,\varphi(-\theta(0))))\|\in\mathbb{L}^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\to 0}\mathbb{E}\|I_1(h)\|^2=0$$

By using the fact that the operator $(-A)^{-\beta}$ is bounded, we obtain that

$$\mathbb{E}\|I_2(h)\|^2\leq\|(-A)^{-\beta}\|^2\mathbb{E}\|(-A)^{\beta}g(t+h,x(t+h-\theta(t+h)))-(-A)^{\beta}g(t,x(t-\theta(t)))\|^2$$

Then, we conclude by condition $(\mathcal{H}.4)$ that

$$\lim_{h\to 0}\mathbb{E}\|I_2(h)\|^2=0.$$

For the third term $I_3(h)$, we suppose that $h>0$ (Similar estimates hold for $h<0$), then we have

$$\begin{aligned}\|I_3(h)\| &\leq \|\int_0^t(S(h)-I)(-A)^{1-\beta}S(t-s)(-A)^{\beta}g(s,x(s-\theta(s)))ds\|\\ &\quad+\|\int_t^{t+h}(-A)^{1-\beta}S(t+h-s)(-A)^{\beta}g(s,x(s-\theta(s)))ds\|\\ &\leq I_{31}(h)+I_{32}(h).\end{aligned}$$

By Hölder's inequality, one has that

$$\mathbb{E}\|I_{31}(h)\|^2\leq t\mathbb{E}\int_0^t\|(S(h)-I)(-A)^{1-\beta}S(t-s)(-A)^{\beta}g(s,x(s-\theta(s)))\|^2ds$$

By using the strong continuity of $S(t)$, we have for each $s\in[0,t]$,

$$\lim_{h\to 0}(S(h)-I)(-A)^{1-\beta}S(t-s)(-A)^{\beta}g(s,x(s-\theta(s)))=0.$$

By using condition $(\mathcal{H}.1)$, condition (ii) in $(\mathcal{H}.3)$ and the fact that $\frac{1}{2}<\beta<1$, we obtain

$$\begin{aligned}&\|(S(h)-I)(-A)^{1-\beta}S(t-s)(-A)^{\beta}g(s,x(s-\theta(s)))\|\\ &\leq\frac{(M+1)M_{1-\beta}}{(t-s)^{1-\beta}}\|(-A)^{\beta}g(s,x(s-\theta(s)))\|\in\mathbb{L}^2([0,T]\times\Omega),\end{aligned}$$

then, we conclude by the Lebesgue dominated theorem that $\lim_{h\to 0}\mathbb{E}\|I_{31}(h)\|^2=0$. By conditions $(\mathcal{H}.1)$, $(\mathcal{H}.3)$ and Hölder's inequality, we get

$$\mathbb{E}\|I_{32}(h)\|^2\leq\frac{M^2_{1-\beta}}{2\beta-1}.h^{2\beta-1}\int_0^T C_4^2(\mathbb{E}\|x(s-\theta(s))\|^2+1)ds,$$

then

$$\lim_{h\to 0}\mathbb{E}\|I_3(h)\|^2=0.$$

Similar computations can be used to show that $\lim_{h\to 0}\mathbb{E}\|I_4(h)\|^2=0$.

For the term $I_5(h)$, we have

$$\begin{aligned}
\|I_5(h)\| &\leq \|\int_0^t (S(h)-I)S(t-s)\sigma(s)dZ_H(s)\| \\
&+ \|\int_t^{t+h} S(t+h-s)\sigma(s)dZ_H(s)\| \\
&\leq I_{51}(h)+I_{52}(h).
\end{aligned}$$

By condition $(\mathcal{H}.1)$ and Lemma 2.2, we get that

$$\begin{aligned}
\mathbb{E}\|I_{51}(h)\|^2 &\leq 2Ht^{2H-1}\int_0^t \|(S(h)-I)S(t-s)\sigma(s)\|^2_{\mathcal{L}_2^0}ds \\
&\leq 2HT^{2H-1}M^2\int_0^T \|(S(h)-I)\sigma(s)\|^2_{\mathcal{L}_2^0}ds.
\end{aligned}$$

Since $\lim\limits_{h\to 0}\|(S(h)-I)\sigma(s)\|^2_{\mathcal{L}_2^0}=0$ and

$$\|(S(h)-I)\sigma(s)\|^2_{\mathcal{L}_2^0}\leq 4M^2\|\sigma(s)\|^2_{\mathcal{L}_2^0}\in \mathbb{L}^1([0,T],ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}\|I_{51}(h)\|^2=0.$$

Again by Lemma 2.2, we get that

$$\mathbb{E}\|I_{52}(h)\|^2\leq 2Hh^{2H-1}M^2\int_t^{t+h} \|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\to 0.$$

For the estimation of term $I_6(h)$, we make use condition $(\mathcal{H}.7)$,

$$\begin{aligned}
\mathbb{E}\|I_6(h)\|^2 &= \leq \sum\nolimits_{0<t_k<T}\mathbb{E}\|I_kx(t_k^-)\|^2\|S(t-t_k)\|^2\|S(h)-I\|^2 \\
&\leq M^2m\sum_{k=1}^m \widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathbb{E}\|x(s)\|^2)\|S(h)-I\|^2,
\end{aligned}$$

then $\lim_{h\to 0}\mathbb{E}\|I_6(h)\|^2=0$.
Now let's observe that

$$\begin{aligned}
\mathbb{E}\|I_7(h)\|^2 &\leq 2\mathbb{E}\|\int_t^{t+h} S(t+h-\nu)BW^{-1}\{x_1-S(T)(\varphi(0)+g(0,\varphi(-\theta(0))))\\
&+ g(T,x(T-\theta(T)))+\int_0^T AS(T-s)g(s,x(s-\theta(s)))ds \\
&- \int_0^T S(T-s)f(s,x(s-\rho(s)))ds-\int_0^T S(T-s)\sigma(s)dZ_H(s) \\
&+ \sum_{0<t_k<T} S(T-t_k)I_kx(t_k^-)\}d\nu\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\mathbb{E}\| \int_0^t (S(h)-I)S(t-\nu)BW^{-1}\{x_1 - S(T)(\varphi(0)+g(0,\varphi(-\theta(0))))\\
&+ \quad g(T,x(T-\theta(T))) + \int_0^T AS(T-s)g(s,x(s-\theta(s)))ds\\
&- \quad \int_0^T S(T-s)f(s,x(s-\rho(s))ds - \int_0^T S(T-s)\sigma(s)dZ_H(s)\\
&+ \quad \sum_{0<t_k<T} S(T-t_k)I_k x(t_k^-)\}d\nu\|^2\\
&\leq \quad 2[\mathbb{E}\|I_{7,1}(h)\|^2 + \mathbb{E}\|I_{7,2}(h)\|^2].
\end{aligned}
$$

Let's first deal with $I_{7,1}(h)$, it follows from the conditions $(\mathcal{H}.1)-(\mathcal{H}.7)$ that

$$
\begin{aligned}
\mathbb{E}\|I_{7,1}(h)\|^2 \quad &\leq 7M^2M_b^2M_w^2 \textstyle\int_t^{t+h}\{\mathbb{E}\|x_1\|^2 + M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-\theta(0)))\|^2\\
&+\|(-A)^{-\beta}\|^2C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+M_{1-\beta}^2T^{2\beta}C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + M^2TC_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+2M^2HT^{2H-1}\textstyle\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0}ds + M^2m\sum_{k=1}^m \widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathbb{E}\|x(s)\|^2)\}d\nu.
\end{aligned}
$$

It results that

$$
\lim_{h\to 0}\mathbb{E}||I_{7,1}(h)||^2 = 0.
$$

In a similar way, we have

$$
\begin{aligned}
\mathbb{E}\|I_{7,2}(h)\|^2 \quad &\leq 7M^2M_b^2M_w^2 \textstyle\int_0^t \|S(h)-I\|^2\{\mathbb{E}\|x_1\|^2\\
&+M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-\theta(0)))\|^2 + \|(-A)^{-\beta}\|^2C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+\tfrac{M_{1-\beta}^2}{2\beta-1}T^{2\beta}C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+2M^2HT^{2H-1}\textstyle\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0}ds + M^2m\sum_{k=1}^m \widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathbb{E}\|x(s)\|^2)\}d\nu.
\end{aligned}
$$

Since

$$\|S(h)-I\|^2\{\mathbb{E}\|x_1\|^2+M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-\theta(0)))\|^2+\|(-A)^{-\beta}\|^2C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)$$

$$+\tfrac{M_{1-\beta}^2}{2\beta-1}T^{2\beta}C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)$$

$$+2M^2HT^{2H-1}\textstyle\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+M^2m\sum_{k=1}^m\widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathbb{E}\|x(s)\|^2)\}$$

$$\leq 4M^2\{\mathbb{E}\|x_1\|^2+M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-\theta(0)))\|^2$$

$$+\|(-A)^{-\beta}\|^2C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+\tfrac{M_{1-\beta}^2}{2\beta-1}T^{2\beta}C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)$$

$$+M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+2M^2HT^{2H-1}\textstyle\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds$$

$$+M^2m\textstyle\sum_{k=1}^m\widetilde{\alpha}_k(1+\sup_{s\in[0,T]}\mathbb{E}\|x(s)\|^2)\}\in L^1([0,T],ds]),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}||I_{7,2}(h)||^2=0.$$

The above arguments show that $\lim_{h\to 0}\mathbb{E}\|\psi(x)(t+h)-\psi(x)(t)\|^2=0$. Hence, we conclude that the function $t\to\psi(x)(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2$-sense.

Step 2. Now, we are going to show that ψ is a contraction mapping in S_{T_1} with some $T_1\leq T$ to be specified later. Let $x,y\in S_T$, we obtain for any fixed $t\in[0,T]$

$$\begin{aligned}
\mathbb{E}\|\psi(x)(t) &- \psi(y)(t)\|^2\leq 8\|(-A)^{-\beta}\|^2\mathbb{E}\|(-A)^{\beta}g(t,x(t-\theta(t)))-(-A)^{\beta}g(t,y(t-\theta(t)))\|^2\\
&+ 8\mathbb{E}\|\int_0^t(-A)^{1-\beta}S(t-s)(-A)^{\beta}(g(s,x(s-\theta(s)))-g(s,y(s-\theta(s))))ds\|^2\\
&+ 8\mathbb{E}\|\int_0^t S(t-s)(f(s,x(s-\rho(s)))-f(s,y(s-\rho(s))))ds\|^2\\
&+ 8\mathbb{E}\|\sum_{0<t_k<T}S(T-t_k)(I_kx(t_k^-)-I_ky(t_k))\|^2\\
&+ 8\mathbb{E}\|\int_0^t S(t-\nu)BW^{-1}[g(T,x(T-\theta(T)))-g(T,y(T-\theta(T)))]d\nu\|^2\\
&+ 8\mathbb{E}\|\int_0^t S(t-\nu)BW^{-1}\int_0^T AS(T-s)[g(s,x(s-\theta(s)))-g(s,y(s-\theta(s)))]ds]d\nu\|^2\\
&+ 8\mathbb{E}\|\int_0^t S(t-\nu)BW^{-1}\int_0^T S(T-s)[f(s,x(s-\rho(s)))-f(s,y(s-\rho(s)))]dsd\nu\|^2
\end{aligned}$$

$$+\quad 8\mathbb{E}\|\int_0^t S(t-\nu)BW^{-1}\sum_{0<t_k<T} S(T-t_k)[I_k x(t_k^-)-I_k y(t_k^-)]d\nu\|^2.$$

By Lipschitz property of $(-A)^{\beta}g$ and f combined with Hölder's inequality, we obtain

$$\begin{aligned}
\mathbb{E}\|\psi(x)(t)-\psi(y)(t)\|^2 \quad & \leq 8\|(-A)^{-\beta}\|^2 C_3^2\mathbb{E}\|x(t-\theta(t))-y(t-\theta(t))\|^2 \\
& +8C_3^2M_{1-\beta}^2\left(\tfrac{t^{2\beta-1}}{2\beta-1}\right)\textstyle\int_0^t \mathbb{E}\|x(s-r)-y(s-r)\|^2ds \\
& +8tM^2C_1^2\textstyle\int_0^t \mathbb{E}\|x(s-\rho(s))-y(s-\rho(s))\|^2ds \\
& +8M^2m\textstyle\sum_{k=1}^m \alpha_k \sup_{s\in[0,T]}\mathbb{E}\|x(s)-y(s)\|^2 \\
& +8\mathbf{t}M^2M_b^2M_w^2[\mathbb{E}\|x(T-\theta(T))-y(T-\theta(T))\|^2 \\
& +C_3^2M_{1-\beta}^2\tfrac{T^{2\beta-1}}{2\beta-1}\textstyle\int_0^T \mathbb{E}\|x(s-\theta(s))-y(s-\theta(s))\|^2ds \\
& +TC_1M^2\textstyle\int_0^T \mathbb{E}\|x(s-\rho(s))-y(s-\rho(s))\|^2ds \\
& +M^2m\textstyle\sum_{k=1}^m \alpha_k \sup_{s\in[0,T]}\mathbb{E}\|x(s)-y(s)\|^2].
\end{aligned}$$

Hence

$$\sup_{s\in[-\tau,T]}\mathbb{E}\|\psi(x)(s)-\psi(y)(s)\|^2 \leq \gamma(t)\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)-y(s)\|^2,$$

where

$$\begin{aligned}
\gamma(t) = \quad & 8[\|(-A)^{-\beta}\|^2C_3^2 + M^2m\textstyle\sum_{k=1}^m\alpha_k + \tfrac{C_3^2M_{1-\beta}^2}{(2\beta-1)}\mathbf{t}^{2\beta} + M^2C_1^2\mathbf{t}^2 \\
& +\mathbf{t}M^2M_b^2M_w^2(1+C_3^2M_{1-\beta}^2\tfrac{T^{2\beta}}{2\beta-1}+T^2C_1M^2+M^2m\textstyle\sum_{k=1}^m\alpha_k)].
\end{aligned}$$

By condition (3.2), we have $\gamma(0) = 8(\|(-A)^{-\beta}\|^2C_3^2 + M^2m\sum_{k=1}^m\alpha_k) < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \gamma(T_1) < 1$ and ψ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $[-\tau, T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau, T]$ in finitely many steps. Clearly, $(\psi x)(T) = x_1$ which implies that the system (1.1) is controllable on $[-\tau, T]$. This completes the proof. □

4. Example

We consider the following impulsive neutral stochastic partial differential equation with finite delays τ_1 and τ_2 $(0 \leq \tau_i \leq \tau < \infty,\ i = 1, 2)$, driven by a Rosenblatt process of the form

$$\begin{cases} d[x(t,\xi)+\frac{\gamma_1}{M_{\frac{1}{4}}\|(-A)^{\frac{3}{4}}\|}x(t-\tau_1,\xi)]=[\frac{\partial^2}{\partial^2\xi}x(t,\xi)+\gamma_2 x(t-\tau_2,\xi)+v(t,\xi)]dt+\sigma(t)dZ_H(t), \\ t\in[0,T],\ \ t\neq t_k,\ 0\leq\xi\leq\pi, \\ \Delta x(t_k,\xi)=x(t_k^+,\xi)-x(t_k^-,\xi)=\gamma_3 x(t_k,\xi), \quad t=t_k,\ k=1,2,....,m, \\ x(t,0)=x(t,\pi)=0, \quad t\in[0,T], \quad \gamma_i>0,\ i=1,2,3, \\ x(s,\xi)=\varphi(s,\xi),\ \varphi(s,.)\in L^2([0,\pi])\ ; -\tau\leq s\leq 0 \quad a.s., \end{cases}$$

where $M_{\frac{1}{4}}$ is the corresponding constant in Lemma 2.3, $Z_H(t)$ is one-dimensional Rosenblatt process. For the convenience of writing, in the following, the variable ξ of $x(t,\xi)$ is omitted.

We rewrite (4.1) into abstract form of (1.1). Let $X=L^2([0,\pi])$ and $Y=\mathbb{R}$. Define the operator $A:D(A)\subset X\longrightarrow X$ given by $A=\frac{\partial^2}{\partial^2\xi}$ with

$$D(A)=\{y\in X:\ y' \text{ is absolutely continuous}, y''\in X, \quad y(0)=y(\pi)=0\},$$

then we get

$$Ax=\sum_{n=1}^{\infty}n^2<x,e_n>_X e_n, \quad x\in D(A),$$

where $e_n:=\sqrt{\frac{2}{\pi}}\sin nx,\ n=1,2,....$ is an orthogonal set of eigenvector of $-A$.

The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}}x=\sum_{n=1}^{\infty}n^{\frac{3}{2}}<x,e_n>_X e_n,$$

with domain

$$D((-A)^{\frac{3}{4}})=X_{\frac{3}{4}}=\{x\in X,\sum_{n=1}^{\infty}n^{\frac{3}{2}}<x,e_n>_X e_n\in X\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ in X, and is given by (see [13])

$$S(t)x=\sum_{n=1}^{\infty}e^{-n^2t}<x,e_n>e_n,$$

for $x\in X$ and $t\geq 0$. Furthermore, $\|S(t)\|\leq e^{\pi^2 t}$, thus satisfying $(\mathcal{H}.1)$. We assume that the following conditions hold:

- $B:U\longrightarrow X$ is a bounded linear operator defined by
$$Bu(t)(\xi)=v(t,\xi),\ 0\leq\xi\leq\pi,\ u\in L^2([0,T],U).$$
- The operator $W:L^2([0,T],U)\longrightarrow X$ defined by
$$Wu=\int_0^T S(T-s)v(t,\xi)ds$$
has an inverse W^{-1} and satisfies condition $(\mathcal{H}.6)$. For the construction of the operator W and its inverse, see [15].

To rewrite the initial-boundary value problem (4.1) in the abstract form we assume the following:

$$g(t, x(t-\theta(t))) = \frac{\gamma_1}{M_{\frac{1}{4}}\|(-A)^{\frac{3}{4}}\|} x(t-\tau_1),$$

$$f(t, x(t-\rho(t))) = \gamma_2 x(t-\tau_2),$$

$$I_k x(t_k^-) = \gamma_3 x(t_k), \quad k = 1, 2, ..., m.$$

It is obvious that all the assumptions are satisfied with
$\beta = \frac{3}{4}$, $M = 1$, $C_1 = C_2 = \gamma_2$, $C_3 = C_4 = \frac{\gamma_1}{M_{\frac{1}{4}}}$, $\alpha_k = \widetilde{\alpha}_k = \gamma_3$, $k = 1, 2, ..., m$.
$\|(-A)^{\frac{3}{4}}\| = 1$, and from the definition of $(-A)^{-\frac{3}{4}}$, we have

$$\|(-A)^{-\frac{3}{4}}\| \le \frac{1}{\Gamma(\frac{3}{4})} \int_0^\infty t^{\frac{-1}{4}} \|S(t)\| dt \le \frac{1}{\pi^{\frac{3}{2}}}.$$

Thus, under appropriate condition on operator σ, it is possible to choose γ_1, γ_3 in such way that constant $\frac{\gamma_1^2}{\pi^3 M_{\frac{1}{4}}^2} + m^2\gamma_3 < \frac{1}{8}$. Hence by Theorem 3.3, the system (4.1) is controllable on $[-\tau, T]$.

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[1] Department of Mathematics, Regional Center for the Professions of Education and Training, Marrakesh, Morocco

Email address: hajjisalahe@gmail.com

[2] Cadi Ayyad University, National School of Applied Sciences, 46000 Safi, Morocco

Email address: e.lakhel@uca.ma