

A COLLAPSING SANDPILE PROBLEM WITH NONLOCAL BOUNDARY CONDITION

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ABSTRACT. In this work, we continue our study on the sandpile model proposed by Igbida [12]. We are particularly interested in the study of avalanches that occur on the surface of a pile of sand following a drying out. The main novelty here is that we do a theoretical and numerical analysis in the case of local and non-local boundaries conditions. We also present some results of numerical simulations in 2D.

1. INTRODUCTION AND MAIN RESULTS

The study of granular materials is an interesting subject both for the mathematical difficulties which ensue from it ([21, 22, 13]) and for the practical applications: modeling of the movement of sand dunes [10], modeling of glacier avalanches, modeling of lakes and rivers [20]. In this work, we theoretically and numerically study the collapse of a sand heap due to successions of avalanches of various sizes. Indeed, assuming that the humidity of the material is not constant, then one can naturally witness avalanches when the humidity of the pile changes even without the intervention of an external source of sand.

The model we are studying is due to L. Prigozhin [22] it is a model which results from the combination of the continuity equation of fluid mechanics and a phenomenological equation combining the angle of stability and the fact that the flow is directed towards the steepest descent. Since we are interested in the collapse of a pile of sand we must therefore consider an angle of stability depending on time. In the works cited above, the authors are interested in cases where the boundary conditions are local: homogenous Dirichlet or Neumann. The main novelty in this work is that we put a control on one of the boundaries of the domain by applying a non-local boundary condition at this level.

Let Ω be an open bounded domain in $\mathbb{R}^s (s \geq 1)$ such that $\partial\Omega$ is a Lipschitz and $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \neq \emptyset$ and $\Gamma_N \neq \emptyset$, the height u of the

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sanpile satisfies the following nonlinear PDE

$$\left\{ \begin{array}{ll} u_t - \nabla \cdot (m \nabla u) = 0 & \text{in } Q = \Omega \times (0, T) \\ |\nabla u| \leq c(t), \ m \geq 0, \ m(|\nabla u| - c(t)) = 0 & \text{in } Q = \Omega \times (0, T) \\ u = 0 & \text{on } \Gamma_D \times (0, T) \\ u = k(t) \text{ (unknown function of } t) & \text{on } \Gamma_N \times (0, T) \\ \int_{\Gamma_N} m \frac{\partial u}{\partial \eta} ds = A_0(t) \text{ (given function on } t) & \text{on } (0, T) \\ u(0) = u_0 & \text{in } \Omega \end{array} \right. \quad (1.1)$$

with $A_0 : (0, T) \rightarrow \mathbb{R}_+^*$, $k : (0, T) \rightarrow \mathbb{R}_+^*$ a non increasing function, u is the height of the surface, f represents the source and $m = m(x, t)$ is an unknown scalar function.

It is well known that the sand has an angle limit, the so-called angle of repose. It corresponds to the steepest angle which the surface of a mass of particles in bulk make with the ground. In our study, we assume that the moisture of the material is changing in time, we can assume that the angle of repose is a given time dependent function $\theta : [0, T) \rightarrow \theta(t) \in \mathbb{R}_+^*$. That implies that the tangent of the repose angle is a given time dependent function $c : [0, T) \rightarrow c(t) \in \mathbb{R}_+^*$.

Recall that on Γ_N , Dirichlet boundary condition

$$u = k(t) \text{ (unknown function of } t) \quad (1.2)$$

and the non-local boundary condition

$$\int_{\Gamma_N} m \frac{\partial u}{\partial \eta} ds = A_0(t) \text{ (given function on } t) \text{ on } (0, T), \quad (1.3)$$

means that the total sand flux passing through Γ_N is given by A_0 .

Beside the mathematical interest of non-local conditions, it seems that this type of boundary condition are also encountered in other physical applications. For example, in petroleum engineering model for well modeling in a 3D stratified petroleum reservoir with arbitrary geometry; this kind of boundary condition also arises in petroleum engineering, in the simulation of wells performance, since a nonlinear relation exists between the performance pressure tangential gradient and the fluid velocity along the well (see [11, 14, 24] for details). Another applications of this type of the boundary condition is in the study of the heat conduction within linear thermo-elasticity (see [3, 4]), and for the reaction-diffusion equation (see [18, 19]).

2. MAIN RESULTS

3. THEORETICAL STUDY OF THE PROBLEM

Let a, b in $[0, \infty)$, we will say that a sequence $(d_i)_{i=0}^n$, is an ε -discretization of the interval $[a, b]$ if :

$$\varepsilon = \varepsilon(n), \lim_{n \rightarrow \infty} \varepsilon = 0, |d_i - d_{i-1}| \leq \varepsilon, \text{ for any } i = 1, \dots, n \quad (3.1)$$

and $d_0 = a < d_1 < d_2 < \dots < d_n = b$ For $\varepsilon > 0$, we say that $(t_i)_{i=0}^n$ is an ε -discretization for the $[0, T)$ in the sense that $\varepsilon = \varepsilon(n)$, $\lim_{n \rightarrow \infty} \varepsilon(n) = 0$,

$t_0 = a < t_1 < t_2 < \dots < t_n = T$ and $t_i - t_{i-1} = \varepsilon$, for any $i = 0, \dots, n$. In the following, we assume that

$$c \in W^{1,\infty}(0, T) \text{ and } \min_{t \in (0, T)} c(t) =: \delta > 0.$$

Definition 3.1. We say that u_ε is an ε -approximate solution of (1.1) in $[0, T]$ if there exists $(t_i)_{i=1, \dots, n}$ and an ε -discretization of $[0, T]$ such that

$$u_\varepsilon(t) = u_i \text{ for } t \in [t_i, t_{i+1}], \quad i = 1, \dots, n-1 \quad (3.2)$$

and for $i = 1, \dots, n$, u_i solves the following Euler implicit time discretization of (1.1) :

$$\begin{cases} u_i - \varepsilon \nabla \cdot (m_i \nabla u_i) = u_{i-1} & \text{in } \Omega \\ |\nabla u_i| \leq c(t_i), \quad m_i \geq 0, \quad m_i(|\nabla u_i| - c(t_i)) = 0 & \text{in } \Omega \\ u_i = 0 & \text{on } \Gamma_D \\ u_i = k(t_i) & \text{on } \Gamma_N \\ \int_{\Gamma_N} m_i \frac{\partial u_i}{\partial \eta} ds = A_0(t_i). \end{cases} \quad (3.3)$$

In order to simplify the notations, we consider the following problem

$$\begin{cases} v - \nabla \cdot (m \nabla v) = g & \text{in } \Omega \\ |\nabla v| \leq d, \quad m \geq 0, \quad m(|\nabla v| - d) = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_D \\ v = \rho \text{ (unknown constant)} & \text{on } \Gamma_N \\ \int_{\Gamma_N} m \frac{\partial v}{\partial \eta} ds = \tilde{A} \end{cases} \quad (3.4)$$

with $d = c(t_i)$, $\rho = k(t_i)$ and $\tilde{A} = A_0(t_i)$.

We introduce now, the space

$$V = \{z \in H^1(\Omega), \quad z|_{\Gamma_D} = 0, \quad z|_{\Gamma_N} = \text{constant}\}$$

and the convex set

$$K(d) = \{z \in W^{1,\infty}(\Omega) \cap V; \quad |\nabla z| \leq d \text{ a.e. in } \Omega\}.$$

We define the function $F_{\tilde{A}}$ in $L^2(\Omega)$ by

$$F_{\tilde{A}}(z) = \begin{cases} -\tilde{A}z|_{\Gamma_N} & \text{if } z \in K(d) \\ +\infty & \text{otherwise.} \end{cases} \quad (3.5)$$

The subdifferential of $F_{\tilde{A}}$ in $L^2(\Omega)$ is given by $g \in \partial F_{\tilde{A}}(v)$ if and only if

$$F_{\tilde{A}}(z) \geq F_{\tilde{A}}(v) + (g, z - v) \text{ for all } z \in L^2(\Omega),$$

where, $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$. It is not difficult to see that $\partial F_{\tilde{A}}$ is a maximal monotone graph in $L^2(\Omega)$.

We can now introduce our notion of solution for problems (3.4) and (1.1).

Definition 3.2. For a given $g \in L^2(\Omega)$, we say that v is variational solution of (3.4) if $v \in K(d)$ and

$$\int_{\Omega} g(z - v)dx + \tilde{A}(z|_{\Gamma_N} - v|_{\Gamma_N}) \leq 0 \text{ for any } z \in K(d). \quad (3.6)$$

Definition 3.3. For given $u_0 \in K(c(t_0))$, we say that u (resp. u_ε) is a variational solution (resp. ε -approximate solution) of (1.1) if $u \in W^{1,1}(0, T; L^2(\Omega))$, $u(0) = u_0$ and for any $t \in (0, T)$, $u(t) \in K(c(t))$, and

$$\int_{\Omega} -u_t(t)(z - u(t))dx + A_0(t)(z|_{\Gamma_N} - u(t)|_{\Gamma_N}) \leq 0 \text{ for any } z \in K(c(t)) \quad (3.7)$$

(resp. u_ε given by (3.2) and u_i is a variational solution of (3.3)).

From (3.7), we get that u is a solution of problem (1.1) if and only if u is a solution of the following problem

$$\begin{cases} \frac{d}{dt}u(t) + \partial F_{A_0}(u(t)) & \ni 0 \\ u(0) & = u_0. \end{cases} \quad (3.8)$$

We have the following result.

Proposition 3.4. For a given $g \in L^2(\Omega)$, there exists a unique $v \in K(d)$ solution of (3.4).

Proof. Since $\partial F_{\tilde{A}}$ is a maximal monotone graph, thanks to [2] we known that for any $g \in L^2(\Omega)$, there exists a unique v solution of

$$v + \partial F_{\tilde{A}}(v) \ni g.$$

Then, for any $z \in K(d)$, we have

$$F_{\tilde{A}}(z) \geq F_{\tilde{A}}(v) + (g - v, z - v),$$

which is equivalent to

$$-\tilde{A}z|_{\Gamma_N} \geq -\tilde{A}v|_{\Gamma_N} + \int_{\Omega} (g - v)(z - v)dx$$

so,

$$\int_{\Omega} (g - v)(z - v)dx + \tilde{A}(z|_{\Gamma_N} - v|_{\Gamma_N}) \leq 0$$

□

To prove the existence and uniqueness of the solution for the (1.1), we need the following lemma

Lemma 3.5. Let $c : t \in [0, T) \rightarrow c(t) \in \mathbb{R}_+^*$ and $u_0 \in K(c(0))$. Then, u is a solution of (1.1) if and only if $v(t, \cdot) = \frac{u(t, \cdot)}{c(t)}$ is a solution of

$$\begin{cases} v_t + \partial F_{\frac{\tilde{A}}{c}}(v) \ni h \\ v(0) = \frac{u(0)}{c(0)}, \end{cases} \quad (3.9)$$

with $h = -\frac{c'(t)}{c(t)}v(t)$, for any $t \in [0, T)$ and

$$F_{\frac{\tilde{A}}{c}}(z) := \begin{cases} -\frac{\tilde{A}}{c(t)}z_{|\Gamma_N} & \text{if } z \in K(1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let u a solution of (1.1). We have

$$\int_{\Omega} (-u_t)(z - u)dx + \tilde{A}(z_{|\Gamma_N} - u_{|\Gamma_N}) \leq 0. \quad (3.10)$$

Denoting for any $t \in [0, T)$,

$$A := \int_{\Omega} (-u_t)(z - u)dx + \tilde{A}(z_{|\Gamma_N} - u_{|\Gamma_N}).$$

We take for $t \in [0, T)$, $v(t, \cdot) = \frac{u(t, \cdot)}{c(t)}$ to obtain

$$\begin{aligned} A &= \int_{\Omega} [-(v_t c(t) + v(t) c'(t))](z - v(t) c(t))dx + \tilde{A}(z_{|\Gamma_N} - (v c(t))_{|\Gamma_N}) \\ &= \int_{\Omega} [-v_t c(t) - v(t) c'(t)](z - v(t) c(t))dx + \tilde{A}(z_{|\Gamma_N} - (v c(t))_{|\Gamma_N}) \\ &= \int_{\Omega} (c(t))^2 \left(-v_t - \frac{v(t) c'(t)}{c(t)} \right) \left(\frac{z}{c(t)} - v(t) \right) dx + (c(t))^2 \left(\frac{\tilde{A}}{c(t)} \left(\frac{z_{|\Gamma_N}}{c(t)} - v_{|\Gamma_N} \right) \right). \end{aligned} \quad (3.11)$$

Since $A \leq 0$ for any $t \in [0, T)$, then,

$$\int_{\Omega} \left(-v_t - \frac{v(t) c'(t)}{c(t)} \right) \left(\frac{z}{c(t)} - v(t) \right) dx + \frac{\tilde{A}}{c(t)} (z_{|\Gamma_N} - v_{|\Gamma_N}) \leq 0, \text{ for any } t \in [0, T)$$

Since $z \in K(c(t))$, then $w = \frac{z}{c(t)} \in K(1)$ for any $t \in [0, T)$ and, it follows that

$$\int_{\Omega} (h - v_t) (w - v(t)) dx + \frac{\tilde{A}}{c(t)} (w_{|\Gamma_N} - v_{|\Gamma_N}) \leq 0;$$

i.e v is a solution of (3.9).

Now we suppose that $v(\cdot, t) = \frac{u(\cdot, t)}{c(t)}$ is a solution of (3.9), then

$$\int_{\Omega} (h - v_t) (w - v(t)) dx + \frac{\tilde{A}}{c(t)} (w_{|\Gamma_N} - v_{|\Gamma_N}) \leq 0, \text{ for any } w \in K(1).$$

We set

$$B = \int_{\Omega} (h - v_t) (w - v(t)) dx + \frac{\tilde{A}}{c(t)} (w_{|\Gamma_N} - v_{|\Gamma_N}),$$

we get

$$\begin{aligned}
B &= \int_{\Omega} (h - v_t) (w - v(t)) dx + \frac{\tilde{A}}{c(t)} (w|_{\Gamma_N} - v|_{\Gamma_N}) \\
&= \int_{\Omega} \left(-\frac{c'(t)}{c(t)^2} u(t) - \frac{u_t c(t) - u(t) c'(t)}{c(t)^2} \right) \left(w - \frac{u(t)}{c(t)} \right) dx + \frac{\tilde{A}}{c(t)} (w|_{\Gamma_N} - v|_{\Gamma_N}) \\
&= \int_{\Omega} \left(-\frac{u_t}{c(t)} \right) \left(w - \frac{u(t)}{c(t)} \right) dx + \frac{\tilde{A}}{c(t)} (w|_{\Gamma_N} - v|_{\Gamma_N}). \tag{3.12}
\end{aligned}$$

Since $B \leq 0$ and $(c(t))^2 B \leq 0$, we get

$$\int_{\Omega} (-u_t) (c(t)w - u(t)) dx + \tilde{A}(w|_{\Gamma_N} \cdot c(t) - u|_{\Gamma_N}) \leq 0, \text{ for any } w \in K(1).$$

Since $w \in K(1)$, then $c(t)w \in K(c(t))$, therefore

$$\int_{\Omega} (-u_t) (z - u(t)) dx + \tilde{A}(z|_{\Gamma_N} - u|_{\Gamma_N}) \leq 0, \text{ for any } z \in K(c(t)).$$

So, u is a solution of (3.4) □

We therefore have the tools at our disposal to introduce the main theoretical result of this paper.

Theorem 3.6. *Let $u_0 \in K(c(0))$ and $T > 0$. Then,*

- 1- *The problem (1.1) has a unique variational solution $u \in W^{1,1}(0, T; L^2(\Omega))$.*
- 2- *For any subsequence $\varepsilon \rightarrow 0$, there exists a unique ε -approximate solution of (1.1) and*

$$u_{\varepsilon} \rightarrow u \in \mathcal{C}([0, T]; L^2(\Omega)) \text{ as } \varepsilon \rightarrow 0.$$

In particular, if $A_0 \geq 0$, then $u \geq 0$ a.e. in Ω .

Proof. 1) Using [2, Proposition 3.13], we deduce that for any $v_0 \in K(1)$, (3.9) has a unique solution $v \in W^{1,1}(0, T; L^2(\Omega))$. This ends up the proof of the existence of a solution of (1.1).

2) To prove the convergence of the ε -approximate solution, let us consider, for $i = 1, \dots, n$,

$$u_i + \partial F_{c(t_i)}(u_i) \ni \varepsilon u_{i-1}, \tag{3.13}$$

where

$$F_{c(t_i)}(z) = \begin{cases} -A(t_i)z|_{\Gamma_N} & \text{if } z \in K(c(t_i)) \\ +\infty & \text{otherwise.} \end{cases}$$

For $i = 0, 1, \dots, n$, setting $z_i = u_i/c(t_i)$, we get

$$\int_{\Omega} \left(\frac{c(t_{i-1})}{c(t_i)} z_{i-1} - z_i \right) (w - z_i) dx + \frac{A(t_i)}{c(t_i)} (w|_{\Gamma_N} - z_i|_{\Gamma_N}) \leq 0, \tag{3.14}$$

for all $w \in K(1)$. Which is equivalent to saying

$$z_i + \partial F_{\frac{A}{c}}(z_i) \ni \frac{c(t_{i-1})}{c(t_i)} z_{i-1}. \tag{3.15}$$

Now, we introduce the Euler implicit discretization in time associated with (3.9),

$$v_i + \partial F_{\frac{A}{c}}(v_i) \ni \varepsilon h_i + v_{i-1}, \quad (3.16)$$

Let v_ε , defining by

$$v_\varepsilon := \begin{cases} v_0 & \text{for } t \in (0, t_1] \\ v_i & \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \dots, n. \end{cases}$$

Thanks to the nonlinear semi-group theory (see for instance [2, Theorem 4.6]), it follows that

$$v_\varepsilon \rightarrow v \in \mathcal{C}([0, T]; L^2(\Omega)) \quad \text{as } \varepsilon \rightarrow 0,$$

where v is a solution of (3.9). Problem (3.16) is equivalent to

$$v_i + \partial F_{c(t_i)}(v_i) \ni -\frac{c(t_i) - c(t_{i-1})}{c(t_i)}(v(t_{i-1}) - v_{i-1}) + \frac{c(t_{i-1})}{c(t_i)}v_{i-1}, \quad (3.17)$$

for $i = 1, \dots, n$. We take z_i as a test function in (3.17) to obtain

$$\begin{aligned} & \int_{\Omega} \left(\frac{\varepsilon f_i}{c(t_i)} - \frac{c(t_i) - c(t_{i-1})}{c(t_i)}(v(t_{i-1}) - v_{i-1}) + \frac{c(t_{i-1})}{c(t_i)}v_{i-1} - v_i \right) (z_i - v_i) dx \\ & + \frac{A(t_i)}{c(t_i)}(z_i|_{\Gamma_N} - v_i|_{\Gamma_N}) \leq 0. \end{aligned} \quad (3.18)$$

Now, we take v_i as a test function in (3.15) to get

$$\int_{\Omega} \left(\frac{c(t_{i-1})}{c(t_i)}z_{i-1} - z_i \right) (v_i - z_i) dx + \frac{A(t_i)}{c(t_i)}(v_i|_{\Gamma_N} - z_i|_{\Gamma_N}) \leq 0. \quad (3.19)$$

Combining (3.18) and (3.19), we obtain

$$\int_{\Omega} \left(-\frac{c(t_i) - c(t_{i-1})}{c(t_i)}(v(t_{i-1}) - v_{i-1}) + \frac{c(t_{i-1})}{c(t_i)}(v_{i-1} - z_{i-1}) + (z_i - v_i) \right) (z_i - v_i) dx \leq 0,$$

which implies

$$\begin{aligned} & \int_{\Omega} \left(-\frac{c(t_i) - c(t_{i-1})}{c(t_i)}(v(t_{i-1}) - v_{i-1}) + \frac{c(t_{i-1})}{c(t_i)}(v_{i-1} - z_{i-1}) \right) (z_i - v_i) dx \\ & + \int_{\Omega} (z_i - v_i)^2 dx \leq 0. \end{aligned} \quad (3.20)$$

So, using the Hölder inequality in (3.20), we deduce that

$$\begin{aligned} & \|z_i - v_i\|_{L^2(\Omega)}^2 \\ & \leq \int_{\Omega} \left(\frac{c(t_i) - c(t_{i-1})}{c(t_i)}(v(t_{i-1}) - v_{i-1}) - \frac{c(t_{i-1})}{c(t_i)}(v_{i-1} - z_{i-1}) \right) (z_i - v_i) dx \\ & \leq \left\| \frac{c(t_i) - c(t_{i-1})}{c(t_i)}(v(t_{i-1}) - v_{i-1}) - \frac{c(t_{i-1})}{c(t_i)}(v_{i-1} - z_{i-1}) \right\|_{L^2(\Omega)} \|z_i - v_i\|_{L^2(\Omega)}. \end{aligned}$$

It follows that

$$\begin{aligned} \|z_i - v_i\|_{L^2(\Omega)} & \leq \left| \frac{c(t_{i-1})}{c(t_i)} \right| \|v_{i-1} - z_{i-1}\|_{L^2(\Omega)} + \left| \frac{c(t_i) - c(t_{i-1})}{c(t_i)} \right| \\ & \quad \times \|v(t_{i-1}) - v_{i-1}\|_{L^2(\Omega)}, \end{aligned} \quad (3.21)$$

for $i = 1, \dots, n$. Since $v_0 = z_0$, then iterating (3.21) for $i = k, \dots, 1$, we obtain

$$\begin{aligned} \|v_k - z_k\|_{L^2(\Omega)} &\leq \sum_{i=1}^k \left| \frac{c(t_{k-i+1}) - c(t_{k-i})}{c(t_k)} \right| \|v(t_{k-i}) - v_{k-i}\|_{L^2(\Omega)} \\ &\leq \frac{1}{\delta} \sum_{i=1}^k |c(t_{k-i+1}) - c(t_{k-i})| \|v(t_{k-i}) - v_{k-i}\|_{L^2(\Omega)} \\ &\leq \frac{\|c'\|_\infty}{\delta} \sum_{i=1}^{k-1} (t_{k-i+1} - t_{k-i}) \|v(t_{k-i}) - v_{k-i}\|_{L^2(\Omega)}. \end{aligned}$$

Setting $\tilde{v}_\varepsilon(t) = v(t_i)$ if $t \in [t_i, t_{i+1})$ for $i = 0, 1, \dots, n-1$, we get

$$\begin{aligned} \|v_k - z_k\|_{L^2(\Omega)} &\leq \frac{\|c'\|_\infty}{\delta} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|\tilde{v}_\varepsilon(t) - v_\varepsilon(t)\|_{L^2(\Omega)} dt \\ &\leq \frac{\|c'\|_\infty}{\delta} \int_0^T \|\tilde{v}_\varepsilon(t) - v_\varepsilon(t)\|_{L^2(\Omega)} dt \end{aligned}$$

and

$$\|v_\varepsilon(t) - z_\varepsilon(t)\|_{L^2(\Omega)} \leq \frac{\|c'\|_\infty}{\delta} \int_0^T \|\tilde{v}_\varepsilon(t) - v_\varepsilon(t)\|_{L^2(\Omega)} dt, \quad \text{for any } t \in [0, T].$$

Since $\varepsilon \rightarrow 0$ and $\tilde{v}_\varepsilon \rightarrow v$ in $\mathcal{C}([0, T]; L^2(\Omega))$, we have

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|v_\varepsilon(t) - z_\varepsilon(t)\|_{L^2(\Omega)} = 0. \quad (3.22)$$

We define c_ε by $c_\varepsilon(t) = c(t_i)$ for $t \in [t_i, t_{i+1})$ and $i = 0, 1, \dots, n$. We have

$$\begin{aligned} &\|u(t) - u_\varepsilon(t)\|_{L^2(\Omega)} \\ &\leq c_\varepsilon(t) \left\| \frac{u(t)}{c_\varepsilon(t)} - z_\varepsilon(t) \right\|_{L^2(\Omega)} \\ &\leq \|c\|_\infty \left(\left\| \frac{u(t)}{c_\varepsilon(t)} - v(t) \right\|_{L^2(\Omega)} + \|v(t) - v_\varepsilon(t)\|_{L^2(\Omega)} + \|v_\varepsilon(t) - z_\varepsilon(t)\|_{L^2(\Omega)} \right). \end{aligned}$$

Combining (3.22), with the fact that $v_\varepsilon \rightarrow v$ in $\mathcal{C}([0, T]; L^2(\Omega))$ and $c_\varepsilon \rightarrow c$ in $\mathcal{C}([0, T])$, and as $\varepsilon \rightarrow 0$, we deduce that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|u(t) - u_\varepsilon(t)\|_{L^2(\Omega)} = 0. \quad (3.23)$$

So, $u_\varepsilon \rightarrow u$ in $\mathcal{C}([0, T]; L^2(\Omega))$ as $\varepsilon \rightarrow 0$.

Our aim is to show that if $A \geq 0$ et $u_0 \geq 0$, alors, $u \geq 0$ p.p. in $]0, T[\times \Omega$.

From (3.7), we get that

$$\int_{\Omega} (u_{i-1} - u_i)(w - u_i) dx + A(t_i)(w|_{\Gamma_N} - (u_i)|_{\Gamma_N}) \leq 0. \quad (3.24)$$

For $i = 1$, we have

$$\int_{\Omega} (u_0 - u_1)(w - u_1) dx + A_1(w|_{\Gamma_N} - (u_1)|_{\Gamma_N}) \leq 0, \quad w \in K(c(t_1)). \quad (3.25)$$

We take $w = u_1 + (\frac{c(t_1)}{c(t_0)}u_0 - u_1)^+$ as test function. We have that $w \in K(c(t_1))$ and

$$w = \begin{cases} u_1 & \text{if } \frac{c(t_1)}{c(t_0)}u_0 \leq u_1 \\ \frac{c(t_1)}{c(t_0)}u_0 & \text{if } \frac{c(t_1)}{c(t_0)}u_0 > u_1 \end{cases}$$

From above we get that

$$\int_{\Omega} (u_0 - u_1) \left(\frac{c(t_1)}{c(t_0)}u_0 - u_1 \right)^+ dx + A_1 \left(\frac{c(t_1)}{c(t_0)}(u_0)_{|\Gamma_N} - (u_1)_{|\Gamma_N} \right)^+ \leq 0.$$

Since c is a non decreasing function, we get that

$$\int_{\Omega} \left(\frac{c(t_1)}{c(t_0)}u_0 - u_1 \right)^+ dx + A_1 \left(\frac{c(t_1)}{c(t_0)}(u_0)_{|\Gamma_N} - (u_1)_{|\Gamma_N} \right)^+ \leq 0.$$

That implies

$$\begin{cases} \left(\frac{c(t_1)}{c(t_0)}u_0 - u_1 \right)^+ = 0 & a.e. \quad in \Omega \\ \left(\frac{c(t_1)}{c(t_0)}(u_0)_{|\Gamma_N} - (u_1)_{|\Gamma_N} \right)^+ = 0 & a.e. \quad on \Gamma_N \end{cases}$$

i.e.

$$\begin{cases} \frac{c(t_1)}{c(t_0)}u_0 \leq u_1 & a.e. \quad in \Omega \\ \frac{c(t_1)}{c(t_0)}u_0 \leq u_1 & a.e. \quad on \Gamma_N. \end{cases}$$

We deduce that

$$\begin{cases} u_1 \geq 0 & a.e. \quad in \Omega \\ u_1 \geq 0 & a.e. \quad on \Gamma_N \end{cases}$$

For $i=2$, we take $w = u_2 + (\frac{c(t_1)}{c(t_0)}u_0 - u_2)^+$ as test function in (3.24) and using same argument as above, we show that

$$\begin{cases} u_2 \geq 0 & a.e. \quad in \Omega \\ u_2 \geq 0 & a.e. \quad on \Gamma_N. \end{cases}$$

By induction for $i = 3, \dots, N$, We show that

$$\begin{cases} u_i \geq 0 & a.e. \quad in \Omega \\ u_i \geq 0 & a.e. \quad on \Gamma_N. \end{cases}$$

We get that u_ε is positive. Letting ε to 0, we get that the solution u is positive. $\square$

4. THE NUMERICAL PROBLEM

Throughout this section, we assume that $g \in W^{1,\infty}(\mathbb{R}^N)$ is compactly supported and Ω is a square including the support of g and satisfying

$$0 \leq g(x) \leq \text{dist}(x, \partial\Omega) \quad (4.1)$$

We introduce the following result.

Proposition 4.1. *Let $v \in K(d)$, then, v is solution of (3.4) if and only if v is a solution of the following minimization problem.*

$$\min_{z \in K(d)} \left\{ \frac{1}{2} \int_{\Omega} |z - g|^2 dx - \tilde{A}z|_{\Gamma_N} \right\}. \quad (4.2)$$

Proof. Let $v \in K(d)$ be a solution of (3.4). Then, we have

$$\int_{\Omega} (g - v)(z - v) dx + \tilde{A}(z|_{\Gamma_N} - v|_{\Gamma_N}) \leq 0, \quad \forall z \in K(d). \quad (4.3)$$

Since

$$\begin{aligned} \|v - g\|_{L^2(\Omega)}^2 - \|z - g\|_{L^2(\Omega)}^2 &= \|v - g\|_{L^2(\Omega)}^2 - \|z - v + v - g\|^2 \\ &= \|v - g\|_{L^2(\Omega)}^2 - \|z - v\|_{L^2(\Omega)}^2 - \int_{\Omega} (z - v)(v - g) dx \\ &\quad - \int_{\Omega} (v - g)(z - v) dx + \|v - g\|_{L^2(\Omega)}^2 \\ &= 2 \int_{\Omega} (g - v)(z - v) dx - \|z - v\|_{L^2(\Omega)}^2, \end{aligned} \quad (4.4)$$

then,

$$\frac{1}{2} \int_{\Omega} |v - g|^2 dx - \frac{1}{2} \int_{\Omega} |z - g|^2 dx = \int_{\Omega} (g - v, z - v) dx - \frac{1}{2} \int_{\Omega} |v - z|^2 dx.$$

Thus,

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} |v - g|^2 dx - \frac{1}{2} \int_{\Omega} |z - g|^2 dx + \tilde{A}(z|_{\Gamma_N} - v|_{\Gamma_N}) \\ &= \int_{\Omega} (g - v, z - v) dx + \tilde{A}(z|_{\Gamma_N} - v|_{\Gamma_N}) - \frac{1}{2} \int_{\Omega} |v - z|^2 dx. \end{aligned}$$

Therefore, by using (4.3), we deduce that

$$\frac{1}{2} \int_{\Omega} |v - g|^2 dx - \frac{1}{2} \int_{\Omega} |z - g|^2 dx - \tilde{A}(z|_{\Gamma_N} - v|_{\Gamma_N}) \leq 0;$$

which is equivalent to saying

$$\frac{1}{2} \int_{\Omega} |v - g|^2 dx - \tilde{A}v|_{\Gamma_N} \leq \frac{1}{2} \int_{\Omega} |z - g|^2 dx - \tilde{A}z|_{\Gamma_N}, \quad \text{for any } z \in K(d).$$

So, $v \in K$ is a solution of (4.2).

Now, we suppose that $v \in K(d)$ is a solution of the minimization problem (4.2).

Let $z_0 \in K(d)$, since $K(d)$ is convex, then for any $t \in [0, 1]$, $z = (1 - t)v + tz_0 \in$

$K(d)$.

We have

$$\begin{aligned}
\frac{1}{2} \int_{\Omega} |v - g|^2 dx - \tilde{A}v|_{\Gamma_N} &\leq \frac{1}{2} \int_{\Omega} |g - ((1-t)v + tz_0)|^2 dx \\
&\quad - \tilde{A}((1-t)v + tz_0)|_{\Gamma_N} \\
&\leq \frac{1}{2} \int_{\Omega} |(g-v) - t(z_0-v)|^2 dx \\
&\quad - \tilde{A}((1-t)v)|_{\Gamma_N} - \tilde{A}(tz_0)|_{\Gamma_N} \\
&\leq \frac{1}{2} \|(g-v) - t(z_0-v)\|_{L^2(\Omega)}^2 \\
&\quad - (1-t)\tilde{A}v|_{\Gamma_N} - t\tilde{A}z_0|_{\Gamma_N}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2} \int_{\Omega} |v - g|^2 dx - \tilde{A}v|_{\Gamma_N} &\leq \frac{1}{2} \int_{\Omega} |v - g|^2 dx - t(g-v, z_0-v) + \frac{t^2}{2} \int_{\Omega} |z_0-v|^2 dx \\
&\quad - \tilde{A}v|_{\Gamma_N} - t\tilde{A}(z_0-v)|_{\Gamma_N}.
\end{aligned}$$

So,

$$(g-v, z_0-v) + \tilde{A}(z_0-v)|_{\Gamma_N} \leq \frac{t}{2} \int_{\Omega} |z_0-v|^2 dx. \quad (4.5)$$

Hence, by letting $t \rightarrow 0$ in (4.5), we obtain

$$\int_{\Omega} (g-v)(z_0-v) dx + \tilde{A}(z_0-v)|_{\Gamma_N} \leq 0, \quad \text{for any } z_0 \in K(d),$$

which implies that

$$\int_{\Omega} (g-v)z_0 dx + \tilde{A}z_0|_{\Gamma_N} \leq \int_{\Omega} (g-v)v dx + \tilde{A}v|_{\Gamma_N} \quad \text{for any } z_0 \in K(d). \quad (4.6)$$

From (4.6) we deduce that v is a solution of (3.4) $\square$

For the numerical analysis of the collapse of an unstable sandpile problem we use a dual formulation. The minimization problem (4.1) is equivalent to

$$\min \{F(z) + H_d(\Lambda z); z \in C^1(\Omega)\}, \quad (4.7)$$

where $\Lambda z := \nabla z$ is a linear operator from $C^1(\Omega)$ to $(C(\Omega))^N$, $F : C^1(\Omega) \rightarrow \mathbb{R}$ and $H : (C(\Omega))^N \rightarrow \overline{\mathbb{R}}$ are convex functions defined by

$$F(z) = \frac{1}{2} \int_{\Omega} |z - f|^2 dx + az|_{\Gamma_N}$$

and

$$H_d(\sigma) = \begin{cases} 0 & \text{if } |\sigma(x)| \leq d \quad \forall x \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Thanks to [9], the associated dual problem is given by

$$\sup \left\{ -F^*(\Lambda^* \sigma) - H_d^*(-\sigma); \sigma \in (C(\Omega))^N \right\}, \quad (4.8)$$

where F^* and H_d^* are the Legendre transforms of F and H_d , Λ^* is the conjugate operator of Λ and $(C(\Omega)^N)^*$ is the dual space of $(C(\Omega))^N$. This problem leads to

$$\sup \left\{ -G(\sigma) ; \sigma \in (C(\Omega)^N)^* \right\}, \quad (4.9)$$

where

$$G(\sigma) = \frac{1}{2} \int_{\Omega} (\operatorname{div}(\sigma))^2 dx + \int_{\Omega} f \operatorname{div}(\sigma) dx + d \int_{\Omega} |\sigma| dx$$

In order to simplify the analysis, we reduce the study of the dual problem to a subspace of the $H_{div}(\Omega)$ space as we suggested in the previous works (see [7, 8, 13, 17]). And we will show how this approach allows to efficiently approximate the solution of the minimization problem. More precisely we are going to look at the problem 4.9 in the space:

$$H_{div, \tilde{A}}(\Omega) = \left\{ \sigma \in H_{div}(\Omega) ; \int_{\Omega} -\operatorname{div}(\sigma) \xi dx = \int_{\Omega} \sigma \cdot \nabla \xi dx - a \xi|_{\Gamma_N}, \quad \forall \xi \in V \right\}.$$

with

$$H_{div}(\Omega) := \left\{ w \in (L^2(\Omega))^N ; \operatorname{div}(w) \in L^2(\Omega) \right\}.$$

It should be noted that the analysis of optimization problems in the space $H_{div}(\Omega)$ appears in many works in the literature (see [16]). In the rest of the paper, we establish the connection between the dual problem (4.2) and the primal problem (4.9). We start by proving the following result.

Lemma 4.2. *For any $g \in L^2(\Omega)$, $w \in H_{div, \tilde{A}}(\Omega)$ and $z \in K(d)$, we have*

$$-G(w) \leq J(z), \quad (4.10)$$

$$\text{where } J(z) = \frac{1}{2} \int_{\Omega} |z - g|^2 dx - \tilde{A}z|_{\Gamma_N}.$$

Proof. Let $w \in H_{div, \tilde{A}}(\Omega)$ and $z \in K(d)$ be fixed. Since

$$\frac{1}{2} (\operatorname{div}(w) - (z - g))^2 \geq 0,$$

we deduce that

$$-\frac{1}{2} \int_{\Omega} (\operatorname{div}(w))^2 dx - \int_{\Omega} \operatorname{div}(w) g dx + \int_{\Omega} \operatorname{div}(w) z dx \leq \frac{1}{2} \int_{\Omega} (z - g)^2 dx,$$

which implies that

$$-\frac{1}{2} \int_{\Omega} (\operatorname{div}(w))^2 dx - \int_{\Omega} \operatorname{div}(w) g dx \leq \frac{1}{2} \int_{\Omega} (z - g)^2 dx - \int_{\Omega} w \cdot \nabla z - \int_{\Gamma_N} \frac{\partial w}{\partial \eta} z ds.$$

Using the fact that $w \in H_{div, \tilde{A}}(\Omega)$, we get

$$-\frac{1}{2} \int_{\Omega} (\operatorname{div}(w))^2 dx - \int_{\Omega} \operatorname{div}(w) g dx \leq \frac{1}{2} \int_{\Omega} (z - g)^2 dx + \int_{\Omega} w \cdot \nabla z dx - \tilde{A}z|_{\Gamma_N}.$$

Therefore,

$$-G(w) \leq \frac{1}{2} \int_{\Omega} (z - g)^2 dx - \tilde{A}z|_{\Gamma_N} + \int_{\Omega} w \cdot \nabla z dx - \int_{\Omega} |w| dx.$$

Since $z \in K(d)$, we have

$$\int_{\Omega} w \cdot \nabla z dx - \int_{\Omega} |w| dx \leq 0.$$

Thus,

$$-G(w) \leq \frac{1}{2} \int_{\Omega} |z - g|^2 dx - \tilde{A}z|_{\Gamma_N} \square$$

□

Even if it is not clear at our level that there exists $(w, v) \in H_{div, \tilde{A}}(\Omega) \times K(d)$ such that $-G(w) = J(v)$. However, we establish some kind of connection between the dual primal and the primal problems. In order to prove this result, we introduce the following elliptic problem

$$(S_{\varepsilon}) \begin{cases} v_{\varepsilon} - \nabla \cdot \omega_{\varepsilon} = g & \text{in } \Omega \\ \omega_{\varepsilon} = \phi_{\varepsilon}(\nabla v_{\varepsilon}) & \text{in } \Omega \\ v_{\varepsilon} = 0 & \text{on } \Gamma_D \\ v_{\varepsilon} = h & \text{on } \Gamma_N \\ \int_{\Gamma_N} w_{\varepsilon} \cdot \eta ds = \tilde{A}, \end{cases}$$

where for any $\varepsilon > 0$, $\phi_{\varepsilon} : E \rightarrow \mathbb{R}^s$ is given by

$$\phi_{\varepsilon}(r) = \frac{1}{\varepsilon}(|r| - d)^+ \frac{r}{|r|} \text{ for all } r \in E$$

with $E = \mathbb{R}^s \setminus \{0_{\mathbb{R}^s}\}$.

ϕ_{ε} satisfies the following properties.

- (i) for any $r_1, r_2 \in E$, $(\phi_{\varepsilon}(r_1) - \phi_{\varepsilon}(r_2)) \cdot (r_1 - r_2) \geq 0$,
- (ii) there exists $\varepsilon_0 > 0$ and $M > 1$ such that $\phi_{\varepsilon}(r) \cdot r \geq |r|^2$ for any $|r| > M$ and $\varepsilon < \varepsilon_0$,
- (iii) for any $\varepsilon > 0$ and $r \in E$, $|\phi_{\varepsilon}(r)| \leq \phi_{\varepsilon}(r) \cdot r$.

Lemma 4.3. *There exists a unique weak solution v_{ε} for problem $(S_{\varepsilon})_{0 < \varepsilon < \varepsilon_0}$ in the sense that $v_{\varepsilon} \in V$, $w_{\varepsilon} = \phi_{\varepsilon}(\nabla v_{\varepsilon}) \in (L^2(\Omega))^s$ and $\forall z \in V$,*

$$\int_{\Omega} v_{\varepsilon} z dx + \int_{\Omega} \phi_{\varepsilon}(\nabla v_{\varepsilon}) \nabla z dx = \int_{\Omega} g z dx + \tilde{A}z|_{\Gamma_N}. \quad (4.11)$$

Moreover, $(v_{\varepsilon})_{0 < \varepsilon < \varepsilon_0}$ is bounded in V , $(w_{\varepsilon})_{0 < \varepsilon < \varepsilon_0}$ is bounded in $(L^1(\Omega))^s$ and for any Borel set $B \subset \Omega$, we have

$$\liminf_{\varepsilon \rightarrow 0} \int_B |\nabla v_{\varepsilon}| dx \leq |B|. \quad (4.12)$$

Proof. The first aims is to show that problem (S_{ε}) has a unique weak solution v_{ε} in the sense

$$\int_{\Omega} v_{\varepsilon} z dx + \int_{\Omega} \phi_{\varepsilon}(\nabla v_{\varepsilon}) \nabla z dx = \int_{\Omega} g z dx + \tilde{A}z|_{\Gamma_N}.$$

For that, we define $J : V \rightarrow V^*$ with

$$\langle Jv, z \rangle = \int_{\Omega} vz dx + \int_{\Omega} \phi_{\varepsilon}(\nabla v) \nabla z dx$$

and $C : V \rightarrow \mathbb{R}$ with

$$\langle C, z \rangle = \int_{\Omega} gz dx + \tilde{A}z|_{\Gamma_N}.$$

Thanks to (i), (ii) and (iii), it follows that operator J is monotone, coercive, hemicontinuous and bounded.

Moreover, since C is a linear form in V^* , thanks to [15], one deduces that there exists $v_{\varepsilon} \in V$ such that

$$\langle Jv_{\varepsilon}, z \rangle = \langle C, z \rangle \text{ for any } z \text{ in } V.$$

Now one shows that v_{ε} is unique.

To do that, we suppose that problem $(S_{\varepsilon})_{0 < \varepsilon < \varepsilon_0}$ admit two solutions u and v such that $u \neq v$ and we subtract the two equations obtained by replacing respectively v_{ε} by u and v in (4.11) to get

$$\int_{\Omega} (u - v)z dx + \int_{\Omega} (\phi_{\varepsilon}(\nabla u) - \phi_{\varepsilon}(\nabla v)) \nabla z dx = 0, \quad \forall z \in V. \quad (4.13)$$

Now, we take $z = u - v$ as a test function in (4.13) to obtain

$$\int_{\Omega} (u - v)^2 dx + \int_{\Omega} (\phi_{\varepsilon}(\nabla u) - \phi_{\varepsilon}(\nabla v)) (\nabla u - \nabla v) dx = 0, \quad \forall z \in V. \quad (4.14)$$

From the property (i) of ϕ_{ε} , it follows that $u = v$ a.e. in Ω .

So, problem (S_{ε}) has a unique solution $v_{\varepsilon} \in V$.

Let us show that

$$\liminf_{\varepsilon \rightarrow 0} \int_B |\nabla v_{\varepsilon}| dx \leq |B|,$$

Notice that $|\omega_{\varepsilon}| = |\phi_{\varepsilon}(\nabla v_{\varepsilon})| \leq \frac{1}{\varepsilon} |||\nabla v_{\varepsilon}|| - 1|$, which implies that $\omega_{\varepsilon} \in (L^2(\Omega))^s$.

Taking v_{ε} as a test function in (4.11), we get

$$\begin{aligned} \int_{\Omega} v_{\varepsilon}^2 dx + \frac{1}{\varepsilon} \int_{\Omega} (|\nabla v_{\varepsilon}| - d)^+ |\nabla v_{\varepsilon}| dx &= \int_{\Omega} gv_{\varepsilon} dx + \tilde{A}v_{\varepsilon}|_{\Gamma_N} \\ &= \int_{\Omega} gv_{\varepsilon} dx + \tilde{A}h \\ &\leq \int_{\Omega} gv_{\varepsilon} dx + \tilde{A}h \\ &\leq \|g\|_{L^2(\Omega)} \|v_{\varepsilon}\|_{H^1(\Omega)} + |\tilde{A}h|. \end{aligned} \quad (4.15)$$

Therefore,

$$\|v_{\varepsilon}\|_{L^2(\Omega)}^2 \leq \|g\|_{L^2(\Omega)} \|v_{\varepsilon}\|_{H^1(\Omega)} + |h\tilde{A}|. \quad (4.16)$$

From (4.15) and property (ii) of ϕ_ε , one deduces that

$$\begin{aligned}
\int_{\Omega} |\nabla v_\varepsilon|^2 dx &\leq \int_{\|\nabla v_\varepsilon\| \leq M} |\nabla v_\varepsilon|^2 dx + \int_{\|\nabla v_\varepsilon\| > M} |\nabla v_\varepsilon|^2 dx \\
&\leq \int_{\|\nabla v_\varepsilon\| \leq M} |\nabla v_\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} (|\nabla v_\varepsilon| - d)^+ |\nabla v_\varepsilon| dx \\
&\leq \int_{\|\nabla v_\varepsilon\| \leq M} M^2 dx + \|g\|_{L^2(\Omega)} \|v_\varepsilon\|_{H^1(\Omega)} + |h\tilde{A}| \\
&\leq |M|^2 |\Omega| + \|g\|_{L^2(\Omega)} \|v_\varepsilon\|_{H^1(\Omega)} + |h\tilde{A}|.
\end{aligned} \tag{4.17}$$

Now, adding (4.16) and (4.17), it follows that

$$\|v_\varepsilon\|_{H^1(\Omega)}^2 \leq |M|^2 |\Omega| + 2\|g\|_{L^2(\Omega)} \|v_\varepsilon\|_{H^1(\Omega)} + 2|h\tilde{A}|, \tag{4.18}$$

and by Young's inequality we get

$$\|v_\varepsilon\|_{H^1(\Omega)}^2 \leq |M|^2 |\Omega| + 2\|g\|_{L^2(\Omega)}^2 + \frac{1}{2}\|v_\varepsilon\|_{H^1(\Omega)}^2 + 2|h\tilde{A}|.$$

Therefore

$$\|v_\varepsilon\|_{H^1(\Omega)}^2 \leq 2|M|^2 |\Omega| + 4\left(\|g\|_{L^2(\Omega)}^2 + |h\tilde{A}|\right). \tag{4.19}$$

From (4.19), we get that v_ε is bounded in V .

From (iii) and (4.15) we get that $(w_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ is uniformly bounded in $(L^1(\Omega))^s$.

We consider now a fixed Borel set denoted by $B \subset \Omega$. One has

$$\|\nabla v_\varepsilon\|_{L^1(B)} \leq \|(|\nabla v_\varepsilon| - d)^+ + 1\|_{L^1(B)} \tag{4.20}$$

$$\begin{aligned}
&\leq \|(|\nabla v_\varepsilon| - d)^+\|_{L^1(B)} + |B| \\
&\leq \int_B (|\nabla v_\varepsilon| - d)^+ dx + |B| \\
&\leq \frac{1}{d} \int_B d(|\nabla v_\varepsilon| - d)^+ dx + |B| \\
&\leq \frac{1}{d} \int_B d(|\nabla v_\varepsilon| - d)^+ dx + |B| \\
&\leq \frac{\varepsilon}{d} (\|g\|_{L^2(\Omega)} \|v_\varepsilon\|_{H^1(\Omega)}) + |\tilde{A}h| + |B|.
\end{aligned} \tag{4.21}$$

Since v_ε is bounded in $H^1(\Omega)$, by letting $\varepsilon \rightarrow 0$ in (4.20) it follows that

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla v_\varepsilon| dx \leq |B|$$

□

Theorem 4.4. *Let $g \in L^2(\Omega)$ and v a solution of (3.4). Then, there exists a sequence $(w_\varepsilon)_{\varepsilon > 0}$ in $H_{div, \tilde{A}}(\Omega)$ such that, as $\varepsilon \rightarrow 0$,*

$$\int_{\Omega} |w_\varepsilon| dx \rightarrow \int_{\Omega} v(g - v) dx + \tilde{A}v|_{\Gamma_N}, \tag{4.22}$$

$$div(w_\varepsilon) \rightarrow v - g \text{ in } L^2(\Omega) \tag{4.23}$$

and

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} G(w_\varepsilon) &= \inf_{w \in H_{div, \tilde{A}}} G(w) \\
&= -\min_{z \in K} J(z) \\
&= \frac{1}{2} \int_{\Omega} |v - g|^2 dx - \tilde{A}v|_{\Gamma_N}.
\end{aligned} \tag{4.24}$$

Proof. Thanks to lem 4.3, one has that the sequence $(v_\varepsilon)_{\varepsilon > 0}$ is bounded in V . Therefore, we can extract a subsequence (still denoted by $(v_\varepsilon)_{\varepsilon > 0}$) such that

$$v_\varepsilon \rightarrow \tilde{v} \text{ in } V - \text{weak and in } L^2(\Omega).$$

Hence, by problem (S_ε) , we deduce that

$$div(\omega_\varepsilon) \rightarrow \tilde{v} - g \text{ in } L^2(\Omega).$$

We define the set A_δ by $A_\delta = [|\nabla \tilde{v}| \geq 1 + \delta]$, with $\delta > 0$.

Using the fact that $\nabla v_\varepsilon \rightarrow \nabla \tilde{v}$ in $(L^1(\Omega))^s$ -weak as $\varepsilon \rightarrow 0$, it follows that

$$\begin{aligned}
(1 + \delta)|A_\delta| &\leq \int_{A_\delta} |\nabla \tilde{v}| dx \\
&\leq \liminf_{\varepsilon \rightarrow 0} \int_{A_\delta} |\nabla v_\varepsilon| dx.
\end{aligned}$$

From lemma 4.3, one has $B = A_\delta$ and then

$$(1 + \delta)|A_\delta| \leq |A_\delta|.$$

Therefore $|A_\delta| = 0$ since $\delta > 0$.

So, $|\nabla \tilde{v}| \leq 1$ a.e. in Ω and $\tilde{v} \in K(d)$.

Now we must show that the solution $\tilde{v}$ is also a solution of the problem (4.2). For any $z \in K(d)$, one has

$$\begin{aligned}
&\int_{\Omega} (g - \tilde{v})(z - \tilde{v}) dx + \tilde{A}(z|_{\Gamma_N} - \tilde{v}|_{\Gamma_N}) \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} -\nabla \phi_\varepsilon(\nabla v_\varepsilon)(z - \tilde{v}) dx + \tilde{A}(z|_{\Gamma_N} - \tilde{v}|_{\Gamma_N}) \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_\varepsilon(\nabla v_\varepsilon) \cdot \nabla(z - \tilde{v}) dx + \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_N} \phi_\varepsilon(\nabla v_\varepsilon) \eta(z - \tilde{v}) ds + \tilde{A}(z|_{\Gamma_N} - \tilde{v}|_{\Gamma_N}) \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_\varepsilon(\nabla v_\varepsilon) \nabla(z - \tilde{v}) dx - \tilde{A}(z|_{\Gamma_N} - \tilde{v}|_{\Gamma_N}) + \tilde{A}(z|_{\Gamma_N} - \tilde{v}|_{\Gamma_N}) \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_\varepsilon(\nabla v_\varepsilon) \nabla(z - \tilde{v}) dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\phi_\varepsilon(\nabla v_\varepsilon) - \phi_\varepsilon(\nabla z)) \nabla(z - \tilde{v}) dx \leq 0.
\end{aligned}$$

i.e $\tilde{v}$ is also a solution of (4.2), therefore $\tilde{v} = v$.

Now, we must show that ω_ε satisfies (4.22).

From property (iii) of ϕ_ε , it comes that

$$\begin{aligned}
\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} |\omega_\varepsilon| dx &= \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} |\phi_\varepsilon(\nabla v_\varepsilon)| dx \\
&\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \phi_\varepsilon(\nabla v_\varepsilon) \nabla v_\varepsilon dx \\
&\leq \limsup_{\varepsilon \rightarrow 0} \left(\int_{\Gamma_N} \phi_\varepsilon(\nabla v_\varepsilon) \cdot \eta v_\varepsilon dx - \int_{\Omega} \nabla \phi_\varepsilon(\nabla v_\varepsilon) v_\varepsilon dx \right) \\
&\leq \limsup_{\varepsilon \rightarrow 0} \left(\int_{\Omega} (g - v) v_\varepsilon dx + \tilde{A} v|_{\Gamma_N} \right) \\
&\leq \int_{\Omega} (g - v) \tilde{v} dx + \tilde{A} v|_{\Gamma_N}. \tag{4.25}
\end{aligned}$$

Using the fact that $\tilde{v}$ is a variational solution of problem (4.2) it follows that

$$\begin{aligned}
\int_{\Omega} (g - v) v dx + \tilde{A} v|_{\Gamma_N} &\leq \int_{\Omega} (g - \tilde{v}) \tilde{v} dx + \tilde{A} \tilde{v}|_{\Gamma_N} \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (g - v_\varepsilon) \tilde{v} dx + \tilde{A} \tilde{v}|_{\Gamma_N} \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} -\nabla \omega_\varepsilon \cdot \tilde{v} dx + \tilde{A} \tilde{v}|_{\Gamma_N} \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \omega_\varepsilon \cdot \nabla \tilde{v} dx \\
&\leq \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\omega_\varepsilon| dx. \tag{4.26}
\end{aligned}$$

From (4.25) and (4.26), it follows that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\omega_\varepsilon| dx = \int_{\Omega} (g - v) v dx + \tilde{A} v|_{\Gamma_N}. \tag{4.27}$$

To end the proof of Theorem 4.4, it remains to prove (4.24). For that, we use (4.22), (4.23) and (4.27) to get

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} (-G(\omega_\varepsilon)) &= \lim_{\varepsilon \rightarrow 0} \left(-\frac{1}{2} \int_{\Omega} [\operatorname{div}(\omega_\varepsilon)]^2 dx - \int_{\Omega} \operatorname{div}(\omega_\varepsilon) g dx - \int_{\Omega} |\omega_\varepsilon| dx \right) \\
&= -\frac{1}{2} \int_{\Omega} (\tilde{v} - g)^2 dx - \int_{\Omega} (\tilde{v} - g) g dx - \int_{\Omega} (g - \tilde{v}) \tilde{v} dx \\
&\quad - \tilde{A} \tilde{v}|_{\Gamma_N} \\
&= -\frac{1}{2} \int_{\Omega} (\tilde{v} - g)^2 dx + \int_{\Omega} (\tilde{v} - g)^2 dx - \tilde{A} \tilde{v}|_{\Gamma_N} \\
&= \frac{1}{2} \int_{\Omega} (\tilde{v} - g)^2 dx - \tilde{A} \tilde{v}|_{\Gamma_N}.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0} (-G(\omega_\varepsilon)) = J(\tilde{v}) = J(v). \tag{4.28}$$

lem 4.2 allows us to write

$$\sup_{\omega \in H_{div, \tilde{A}}(\Omega)} (-G(\omega)) \leq J(v) = \lim_{\varepsilon \rightarrow 0} (-G(\omega_\varepsilon)). \quad (4.29)$$

Since

$$J(v) = \lim_{\varepsilon \rightarrow 0} (-G(\omega_\varepsilon)) \leq \sup_{\omega \in H_{div, \tilde{A}}(\Omega)} (-G(\omega)), \quad (4.30)$$

one concludes that

$$\lim_{\varepsilon \rightarrow 0} (-G(\omega_\varepsilon)) = \sup_{\omega \in H_{div, \tilde{A}}(\Omega)} (-G(\omega)) = J(v)$$

□

Remark 4.5. From lemma 4.3 since $(w_\varepsilon)_{0 < \varepsilon}$ is uniformly bounded in $(L^1(\Omega))^s$, we can deduce that there exists $\omega \in (M_b(\Omega))^s$ weak limit of w_ε . Also thanks to Theorem 4.4 the solution v of problem (4.2) can be characterized by the following.

$$\begin{cases} v - \operatorname{div}(\omega) = g \text{ in } V^* \\ |\omega|(\Omega) = \int_{\Omega} v(g - v) dx + \tilde{A}v|_{\Gamma_N}. \end{cases} \quad (4.31)$$

where $|\omega|$ represents the total variation associated to the measure ω .

Another characterization of the $H_{div, \tilde{A}}(\Omega)$ is given by the following result which will be useful in the construction of a finite dimensional subspace of the space $H_{div, \tilde{A}}(\Omega)$.

Lemma 4.6.

$$H_{div, \tilde{A}}(\Omega) = \{\phi + \nabla Z \phi \in \mathcal{H}\},$$

where with Z is a solution of the following problem

$$\begin{cases} -\Delta Z = 0 & \text{in } \Omega \\ Z = 0 & \text{on } \Gamma_D \\ Z = \rho(\text{unknown constant}) & \text{on } \Gamma_N \\ \nabla Z \cdot \eta = \frac{\tilde{A}}{\operatorname{meas}(\Gamma_N)} & \text{on } \Gamma_N \end{cases} \quad (4.32)$$

and

$$\mathcal{H} = \left\{ \phi \in H_{div}(\Omega) : \int_{\Omega} -\operatorname{div}(\phi) \xi dx = \int_{\Omega} \phi \nabla \xi dx \quad \forall \xi \in V \right\}.$$

Proof. For all $\sigma \in H_{div, \tilde{A}}(\Omega)$, we have $\sigma = (\sigma - \nabla Z) + \nabla Z$, where Z is a solution of the problem (4.32). The term $(\sigma - \nabla Z)$ belongs to $H_{div, \tilde{A}}(\Omega)$ and

$$\begin{aligned} \int_{\Omega} -\operatorname{div}(\sigma - \nabla Z) \xi dx &= \int_{\Omega} -\operatorname{div}(\sigma) \xi dx + \int_{\Omega} \operatorname{div}(\nabla Z) \xi dx \\ &= \int_{\Omega} \sigma \cdot \nabla \xi dx - \tilde{A} \xi|_{\Gamma_N} - \int_{\Omega} \nabla Z \cdot \nabla \xi dx + \tilde{A} \xi|_{\Gamma_N} \\ &= \int_{\Omega} (\sigma - \nabla Z) \cdot \nabla \xi dx, \quad \forall \xi \in V. \end{aligned}$$

Hence, $\sigma \in \{\phi + \nabla Z : \phi \in \mathcal{H}\}$. Therefore, $H_{div, \tilde{A}}(\Omega) \subset \{\phi + \nabla Z : \phi \in \mathcal{H}\}$. Note also that, for any $\phi \in \mathcal{H}$, we have $\phi + \nabla Z \in H_{div}(\Omega)$ and

$$\begin{aligned} \int_{\Omega} -div(\phi + \nabla Z)\xi dx &= \int_{\Omega} -div(\phi)\xi dx + \int_{\Omega} -div(\nabla Z)\xi dx \\ &= \int_{\Omega} \phi \cdot \nabla \xi dx + \int_{\Omega} \nabla Z \nabla \xi dx - \tilde{A}\xi|_{\Gamma_N} \\ &= \int_{\Omega} (\phi + \nabla Z) \cdot \nabla \xi dx - \tilde{A}\xi|_{\Gamma_N}, \quad \forall \xi \in V, \end{aligned}$$

which implies that $\{\phi + \nabla Z : \phi \in \mathcal{H}\} \subset H_{div, \tilde{A}}(\Omega)$. $\square$

As consequence of the previous result, we have

$$\sup\{-G(\sigma); \sigma \in H_{div, \tilde{A}}\} = \sup\{\mathcal{G}(\phi) = G(\phi + \nabla Z) : \phi \in \mathcal{H}\} \quad (4.33)$$

5. SPACE DISCRETIZATION

The solution of the dual problem (4.9) is computed using Raviart-Thomas finite element of the lowest order [23]. Let T_h be a regular partitioning (quadrangulation) of Ω consisting of disjoint open simplexes (or quadrilateral) τ of diameter no greater than a given real h , with $\Omega = \cup_{\tau \in T_h} \bar{\tau}$. Let $V_h \subset \mathcal{H}$ be the space of lowest-order Raviart-Thomas finite elements [23]:

$$V_h = \{q_h \in (L^2(\Omega))^2 : q_{\tau}^h = a_{\tau} + b_{\tau}x, \quad a_{\tau} \in \mathbb{R}^2, b_{\tau} \in \mathbb{R}, \quad \forall \tau \in T_h, \text{ and } q_h \cdot \nu \text{ is continuous across element boundaries}\}$$

where ν represents the outward unit normal to τ . The space V_h is a finite dimensional subspace of $\mathcal{H}$ with dimension equal to $N = N(h)$.

We denote by r_h the interpolation operator onto V_h given in Theorem 6.1 in [23]. Then, thanks to [23], we have for all $w \in H_{div}(\Omega)$,

$$r_h(w) \rightarrow w \text{ in } (L^2(\Omega))^2 \text{ and } div(r_h(w)) \rightarrow div(w), \text{ in } L^2(\Omega) \text{ as } h \rightarrow 0. \quad (5.1)$$

Let now by w_h the of the optimization problem

$$G(w_h) = \inf\{\mathcal{G}(q_h); q_h \in V_h\}. \quad (5.2)$$

By the same method used in the proof of [8, Theorem 3.8], we obtain the following convergence result.

Theorem 5.1. *Let $g \in L^2(\Omega)$, v a solution of the minimization problem (4.2) and ω_h a solution of the following optimization problem*

$$\sup\{\mathcal{G}(q_h); q_h \in V_h\}. \quad (5.3)$$

Then, as $h \rightarrow 0$,

$$div(\omega_h) \rightarrow v - g \text{ in } L^2(\Omega) \quad (5.4)$$

and

$$-\mathcal{G}(\omega_h) \rightarrow \min_{z \in K} J(z) = \frac{1}{2} \int_{\Omega} |v - g|^2 dx - \tilde{A}z|_{\Gamma_N}. \quad (5.5)$$

We denote by $\mathcal{G}_h$, the approximation of $\mathcal{G}$ in the space V_h . The term $\int_\tau |q_h + \nabla Z| dx$ on each element τ of the partition T_h ,

$$\int_\tau |q_h + \nabla Z| \sim |\tau| |q_h + \nabla Z|(P_\tau),$$

where $|\tau|$ represents the area of simplex τ and P_τ is one of the vertices of τ . Using this approximation, at each time $n \times dt$, $N \in \mathbb{N}$ and dt the time step, the solution of (5.2) is a minimizer of the non-differentiable functional:

$$\begin{aligned} \mathcal{G}_h &: \mathbb{R}^2 \rightarrow \mathbb{R} \\ w_h &\mapsto \mathcal{G}_h(w_h) := \frac{1}{2}(Aw_h, w_h) + (dt f_h^n + u_{n-1}, \operatorname{div}(w_h)) + h^2 \sum_{i,j} |w_h + \nabla Z|, \end{aligned}$$

where A is an $N \times N$ positive semi-definite matrix and $P_{\tau_{ij}}$ is one of the vertices of τ_{ij} .

The minimization of the functional $\mathcal{G}$ is done according to the Gauss Seidel type algorithm which is described in the following way.

- We start the algorithm by the initial vector $q_0 \in \mathbb{R}^n$, for any $k \geq 0$ until convergence,
- we choose a canonical direction $e_j \in \mathbb{R}^n$ and we find ρ_{jk} minimizing the functional defined by

$$\begin{aligned} \varphi_{jk} &: \mathbb{R} \rightarrow \mathbb{R} \\ \rho &\mapsto G_h(q_k + \rho e_j), \end{aligned}$$

- we take $q_{k+1} = q_k + \omega \rho_{jk} e_j$, with ω over relaxation parameter,
- algorithm is performed until $\|q_{k+1} - q_k\|_{L^2(\Omega)} \leq \varepsilon$, ε is the convergence criterion. Afterward, take $w_h = q_{k+1}$.

Then, knowing a minimizer w_h of (5.3), the solution u_n of Euler implicit time discretization of (1.1) is computed using extremality relation (4.31) in a weak sense with piecewise finite elements P_0 .

6. NUMERICAL RESULTS

We start by giving the main lines of our approach for numerical computation of problem (1.1). For the discretization of the domain $\Omega =]-1, 1[\times]-1, 1[$, we take $N \in \mathbb{N}$ and the step of discretization is $h = \frac{2}{N+1}$. Let $\partial\Omega^x = \{x_i = x_0 + ih : 0 \leq i \leq N+1\}$ be a grid on x -axis and let $\partial\Omega^y = \{y_j = y_0 + jh : 0 \leq j \leq N+1\}$ be a grid on y -axis. We work with simplexes $\tau_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $1 \leq i, j \leq N+1$ having uniform size h^2 .

For numerical tests, we consider $\omega = 0.5$, $N = 60$, $dt = 10^{-3}$, $k(t) = 0$, $k(t) = \frac{t}{T}$ for all t and $\varepsilon = 10^{-3}$ the convergence criterion.

In first time, we consider $\Gamma_N = \{y \in]-1, 1[, x = -1\}$ with $A_0(t) = 0$

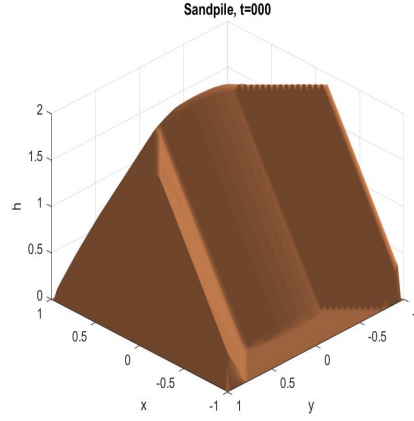


FIGURE 1. Initial unstable sandpile

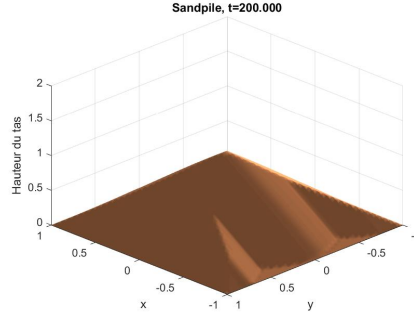
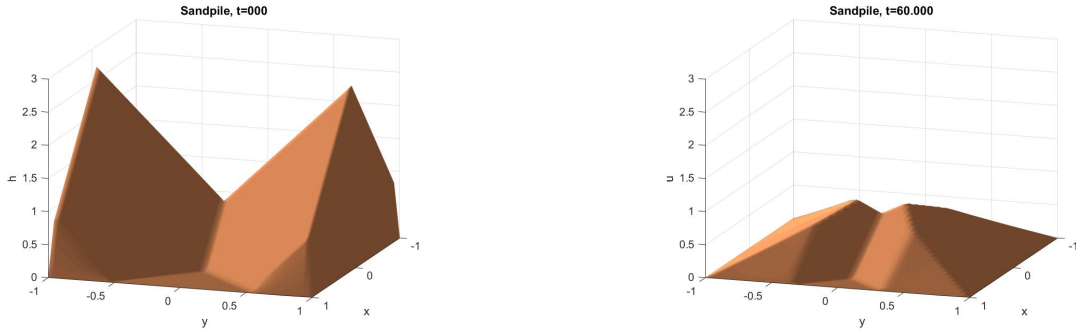


FIGURE 2. Final stable sandpile

In the following tests we take that $\Gamma_N = \{y \in]-1, 1[, x = -1 \text{ and } x = 1\}$.

FIGURE 3. Sandpile surface at time $t = 0$ and $t = 60$ for $A_0 \equiv 0$

Figures 3 and 4 show the unstable and stable configurations of the sandpile. There is a difference between the two final sand pile configurations. We observe that the convergence towards the state of equilibrium is much slower in the

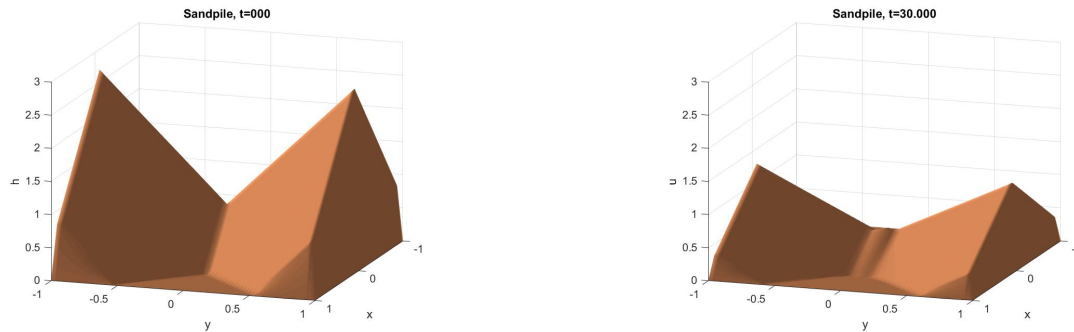


FIGURE 4. Sandpile surface at time $t = 0$ and $t = 30$ for $A_0(t) = t^2 + 2t$

second simulation (see Figure 3). This is certainly due to the condition that we have imposed on the border Γ_N : the total outgoing flux must be zero.

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