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INEQUALITIES FOR ANGULAR DERIVATIVES AT THE BOUNDARY OF THE UNIT DISC

BÜLENT NAFI ÖRNEK

Abstract. In this paper, a boundary version of Schwarz inequality is investigate. For the function $f(z) = \beta + c_p z^p + ...$ defined in the unit disk, $\Re f(z) > \alpha$, $0 \leq \alpha < \beta$ (β is any integer≥ 1), we estimate a modulus of angular derivative at the boundary point z_0, $\Re f(z_0) = \alpha$, by taking into account their first nonzero two Maclaurin coefficients. The sharpness of these estimates is also proved.

1. Introduction

In recent years, a boundary version of Schwarz lemma was investigated in D. Burns and S.G. Krantz [3], R.Osserman [14], V. N. Dubinin ([4], [5]), M. Mateljević ([10], [11], [12], [13]), G. Ren and X. Wang ([18], [20]), M. Jeong ([7], [8]) and other's studies.

As is known, one of the simplest result of the complex function theory is the classical Schwarz lemma. It is assumed that the Schwarz lemma has a very important role in geometric function theory, hyperbolic geometry, complex dynamical systems and theory of quasi-conformal mappings. We refer to ([2], [6]) for a more complete insight on the Schwarz lemma.

A general form of this lemma, which is very simple and commonly used, is as follows: Let f be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$, $f(0) = 0$, and $|f| < 1$ for $|z| < 1$. For any point z in the unit disc D, we have

$$|f(z)| \leq |z| \tag{1.1}$$

for any point $z \in D$, and moreover

$$|f'(0)| \leq 1. \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z \in D$) or in (1.2) if and only if $f(z)$ is an entire linear function of the form $f(z) = e^{i\alpha}z$, where α is a real number. ([6], p.329)

Let $f(z) = \beta + c_p z^p + c_{p+1} z^{p+1} ....$, $p \in \mathbb{N} = \{1, 2, ...\}$ be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$ and $\Re f(z) > \alpha$, $0 \leq \alpha < \beta$ (β is any integer≥ 1)

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in $|z|<1$.

$$\vartheta(z)=\frac{f(z)-\alpha}{\beta-\alpha}=1+b_p z^p+b_{p+1}z^{p+1}+...$$

is a holomorphic function and $\Re\vartheta(z)>0$ for $|z|<1$ and therefore

$$\varphi(z)=\frac{1-\vartheta(z)}{1+\vartheta(z)}=d_p z^p+d_{p+1}z^{p+1}+... \tag{1.3}$$

is a holomorphic and $|\varphi(z)|<1$ for $|z|<1$. Thus, from Schwarz lemma we obtain

$$|\varphi(z)|\leq|z|^p,$$

$$|f(z)|\leq\alpha+(\beta-\alpha)\frac{1+|z|^p}{1-|z|^p}$$

and

$$|c_p|\leq 2(\beta-\alpha)$$

If the function f can be extended by continuity to a point $z_0\in\partial D$, $|f(z_0)|=1$, and the derivative $f'(z_0)$ exists, then from (1.1) implies the inequality $|f'(z_0)|\geqslant 1$, which is know as the Schwarz lemma on the boundary. Applying inequality (5) in ([6], p. 330) to the function $\frac{f(z)}{z}$, we arrive at the following generalization of the Schwarz lemma [6]

$$|f(z)|\leq|z|\frac{|z|+|f'(0)|}{1+|z|\,|f'(0)|},\ z\in D. \tag{1.4}$$

In proving our main results, we shall need the following lemma due to Julia-Wolff [17].

Lemma 1.1 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in D, $f(0)=0$ and $f(D)\subset D$. If, in addition, the function f has an angular limit $f(z_0)$ at $z_0\in\partial D$, $|f(z_0)|=1$, then the angular derivative $f'(z_0)$ exists and $1\leq|f'(z_0)|\leq\infty$.*

Then, passing to the angular limit in (1.4), we arrive at the boundary Schwarz lemma [14]

$$|f'(z_0)|\geq\frac{2}{1+|f'(0)|}. \tag{1.5}$$

In (1.4) for real z and in the left-hand-side inequality in (1.5) for $z_0=1$, an equality occurs for the function $f(z)=z\frac{z+a}{1+az}$, $0\leq a\leq 1$.

It follows that

$$|f'(z_0)|\geqslant 1, \tag{1.6}$$

with equality only if f is of the form $f(z)=ze^{i\theta}$, θ real.

In the last 15 years, there have been tremendous studies on Schwarz lemma at the boundary (see,[1], [2], [3], [4], [5], [7], [8], [14], [15], [16], [18], [19], [20] and references therein). Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies $|f(z_0)|=1$ condition of the boundary of the unit circle.

Also, M. Jeong [8] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [7] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

Furthermore, X. Tang, T. Liu and J. Lu [19] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions. T. Liu, J. Wang, X. Tang [9] established a new type of the classical boundary Schwarz lemma for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$. They then applied their new Schwarz lemma to study problems from the geometric function theory in several complex variables.

2. Main Results

In the following theorems, we estimate a modulus of angular derivative at the boundary point z_0, $\Re f(z_0) = \alpha$, by taking into account their first nonzero two Maclaurin coefficients. The sharpness of these estimates is also proved.

Theorem 2.1. *Let $f(z) = \beta + c_p z^p + c_{p+1} z^{p+1} \ldots$, $c_p \neq 0$, $p \geq 2$ be a holomorphic function in the unit disc D and $\Re f(z) > \alpha$, $0 \leq \alpha < \beta$ (β is any integer≥ 1) for $|z| < 1$. Suppose that, for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $\Re f(z_0) = \alpha$. Then*

$$|f'(z_0)| \geqslant \frac{\beta - \alpha}{2} \left(p + \frac{2\left(2\left(\beta - \alpha\right) - |c_p|\right)^2}{4\left(\beta - \alpha\right)^2 - |c_p|^2 + 2\left(\beta - \alpha\right)|c_{p+1}|} \right). \tag{2.1}$$

Moreover, the equality in (2.1) *occurs for the function*

$$f(z) = \alpha + (\beta - \alpha)\frac{1 - z^p}{1 + z^p}.$$

Proof. Let $\varphi(z)$ be defined as in (1.3).

$$B(z) = z^p$$

is a holomorphic functions in D, $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have

$$|\varphi(z)| \leq |B(z)|.$$

Therefore,

$$h(z) = \frac{\varphi(z)}{B(z)}$$

is a holomorphic function in D and $|h(z)| < 1$ for $|z| < 1$. In particular, we have

$$|h(0)| = \frac{|c_p|}{2\left(\beta - \alpha\right)} \leq 1 \tag{2.2}$$

and

$$|h'(0)| = \frac{|c_{p+1}|}{2\left(\beta - \alpha\right)}.$$

Moreover, it can be seen that

$$\frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = |\varphi'(z_0)| \geq |B'(z_0)| = \frac{z_0 B'(z_0)}{B(z_0)}.$$

The function

$$\Phi(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}$$

is a holomorphic function in the unit disc D, $|\Phi(z)|<1$, $\Phi(0)=0$ and $|\Phi(z_0)|=1$ for $z_0\in\partial D$.

From (1.5), we obtain

$$\begin{aligned}
\frac{2}{1+|\Phi'(0)|} &\leq |\Phi'(z_0)|=\frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(z_0)\right|^2}|h'(z_0)| \\
&= \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(z_0)\right|^2}\left|\frac{\varphi'(z_0)}{B(z_0)}-\frac{B'(z_0)\varphi(z_0)}{B^2(z_0)}\right| \\
&= \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(z_0)\right|^2}\left|\frac{\varphi(z_0)}{z_0B(z_0)}\right|\left|\frac{z_0\varphi'(z_0)}{\varphi(z_0)}-\frac{z_0B'(z_0)}{B(z_0)}\right| \\
&\leq \frac{1+|h(0)|}{1-|h(0)|}\left\{|\varphi'(z_0)|-|B'(z_0)|\right\}
\end{aligned}$$

and

$$\frac{2}{1+|\Phi'(0)|}\leq\frac{1+|h(0)|}{1-|h(0)|}\left\{|\varphi'(z_0)|-|B'(z_0)|\right\}. \tag{2.3}$$

With the simple calculations, we take

$$|\varphi'(z_0)|=\frac{2\,|\vartheta'(z_0)|}{|\vartheta(z_0)+1|^2},$$

$$|\Phi'(0)|=\frac{2\,(\beta-\alpha)\,|c_{p+1}|}{4\,(\beta-\alpha)^2-|c_p|^2},$$

$$|\vartheta(z_0)+1|^2\geq\left[\Re\left(\vartheta(z_0)+1\right)\right]^2=\left[1+\Re\vartheta(z_0)\right]^2=\left[1+\Re\left(\frac{f(z_0)-\alpha}{\beta-\alpha}\right)\right]^2=1$$

and for $z_0\in\partial D$

$$|B'(z_0)|=p.$$

Let's substitute the values of $|\Phi'(0)|$, $|\varphi'(z_0)|$, $|B'(z_0)|$ and $|h(0)|$ into (2.3). We obtain

$$\begin{aligned}
\frac{2}{1+\frac{2(\beta-\alpha)|c_{p+1}|}{4(\beta-\alpha)^2-|c_p|^2}} &\leq \frac{2\,(\beta-\alpha)+|c_p|}{2\,(\beta-\alpha)-|c_p|}\left\{\frac{2\,|\vartheta'(z_0)|}{|\vartheta(z_0)+1|^2}-p\right\} \\
&\leq \frac{2\,(\beta-\alpha)+|c_p|}{2\,(\beta-\alpha)-|c_p|}\left\{\frac{2\,|f'(z_0)|}{(\beta-\alpha)}-p\right\}.
\end{aligned}$$

So, we get the inequality (2.1).

To show that the inequality (2.1) is sharp, take the holomorphic function

$$f(z)=\alpha+(\beta-\alpha)\,\frac{1-z^p}{1+z^p}.$$

Then

$$f'(z) = (\beta - \alpha)\frac{-pz^{p-1}(1+z^p) - pz^{p-1}(1-z^p)}{(1+z^p)^2},$$

$$f'(1) = -\frac{p}{2}(\beta - \alpha)$$

and

$$|f'(1)| = \frac{p}{2}(\beta - \alpha).$$

Since $|c_p| = 2(\beta - \alpha)$, (2.1) is satisfied with equality. $\square$

If $f(z) - \beta$ has no zeros different from z $= 0$ in Theorem 2.1, the inequality (2.1) can be further strengthened. This is given by the following theorem.

Theorem 2.2. *Let $f(z) = \beta + c_p z^p + c_{p+1}z^{p+1}....,\ c_p > 0,\ p \geq 2$ be a holomorphic function in the unit disc D, $f(z) - \beta$ has no zeros in D except $z = 0$ and $\Re f(z) > \alpha$, $0 \leq \alpha < \beta$ (β is any integer≥ 1) for $|z| < 1$. Suppose that, for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $\Re f(z_0) = \alpha$. Then we have the inequality*

$$|f'(z_0)| \geqslant \frac{\beta - \alpha}{2}\left(p - \frac{2|c_p|\left(\ln\frac{|c_p|}{2(\beta-\alpha)}\right)^2}{2|c_p|\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right) - |c_{p+1}|}\right) \tag{2.4}$$

and

$$|c_{p+1}| \leq 2\left|c_p \ln\left(\frac{|c_p|}{2(\beta - \alpha)}\right)\right|. \tag{2.5}$$

In addition, the equality in (2.4) *occurs for the function*

$$f(z) = \alpha + (\beta - \alpha)\frac{1 - z^p}{1 + z^p}$$

and the equality in (2.5) *occurs for the function*

$$f(z) = \alpha + (\beta - \alpha)\frac{1 - z^p e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)}}{1 + z^p e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)}},$$

where $0 < c_p < 1,\ \ln\left(\frac{c_p}{2(\beta-\alpha)}\right) < 0.$

Proof. Let $c_p > 0$. Let $\varphi(z),\ h(z)$ and $B(z)$ be as in the proof of Theorem2.1. Having the mind the inequality (2.2), we denote by $\ln h(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln h(0) = \ln\left(\frac{c_p}{2(\beta - \alpha)}\right) < 0.$$

The auxiliary function

$$\mu(z) = \frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)}$$

is a holomorphic function in the unit disc D, $|\mu(z)| < 1$, $\mu(0) = 0$ and $|\mu(z_0)| = 1$ for $z_0 \in \partial D$.

From (1.5), we obtain

$$\begin{aligned}\frac{2}{1+|\mu'(0)|} &\leq |\mu'(z_0)| = \frac{|2\ln h(0)|}{|\ln h(z_0)+\ln h(0)|^2}\left|\frac{h'(z_0)}{h(z_0)}\right| \\ &= \frac{|2\ln h(0)|}{|\ln h(z_0)+\ln h(0)|^2}|h'(z_0)| \\ &= \frac{|2\ln h(0)|}{|\ln h(z_0)+\ln h(0)|^2}\left\{|\varphi'(z_0)|-|B'(z_0)|\right\}\end{aligned}$$

and

$$\frac{2}{1+|\mu'(0)|} \leq \frac{|2\ln h(0)|}{|\ln h(z_0)+\ln h(0)|^2}\left\{|\varphi'(z_0)|-|B'(z_0)|\right\}. \tag{2.6}$$

It can be seen that

$$|\mu'(0)| = -\frac{|c_{p+1}|}{2\,|c_p|\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}.$$

Let's substitute the values of $|\mu'(0)|$, $|\varphi'(z_0)|$, $|B'(z_0)|$ and $\ln h(0)$ into (2.6). We obtain

$$\frac{2}{1-\frac{|c_{p+1}|}{2|c_p|\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}} \leq \frac{-2\ln h(0)}{\ln^2 h(0)+\arg^2 h(z_0)}\left\{\frac{2\,|\vartheta'(z_0)|}{|\vartheta(z_0)+1|^2}-p\right\}.$$

Replacing $\arg^2 h(z_0)$ by zero, we take

$$\begin{aligned}\frac{2}{1-\frac{|c_{p+1}|}{2|c_p|\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}} &\leq \frac{-2}{\ln^2 h(0)}\left\{\frac{2\,|\vartheta'(z_0)|}{|\vartheta(z_0)+1|^2}-p\right\} \\ &= \frac{-2}{\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}\left\{\frac{2\,|\vartheta'(z_0)|}{|\vartheta(z_0)+1|^2}-p\right\}.\end{aligned}$$

and

$$\frac{2}{1-\frac{|c_{p+1}|}{2|c_p|\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}} \leq \frac{-2}{\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}\left(2\frac{f'(z_0)}{\beta-\alpha}-p\right).$$

Therefore, we obtain (2.4) with an obvious equality case.

Similary, $\mu(z)$ function satisfies the assumptions of the Schwarz lemma, we obtain

$$1 \geq |\mu'(0)| = \frac{|2\ln h(0)|}{|\ln h(0)+\ln h(0)|^2}\left|\frac{h'(0)}{h(0)}\right|$$

and

$$1 \geq \frac{-1}{2\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}\frac{|c_{p+1}|}{|c_p|}.$$

Therefore, we have the inequality (2.5).

Now, we shall show that the inequality (2.5) is sharp. Let

$$f(z)=\alpha+(\beta-\alpha)\,\frac{1-z^{p}e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)}}{1+z^{p}e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)}}.$$

From here, we take

$$f(z)=\beta+z^{p}w(z), \tag{2.7}$$

where

$$w(z)=-2\,(\beta-\alpha)\,\frac{e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)}}{1+z^{p}e^{\frac{1+z}{1-z}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)}}.$$

Therefore, from (2.7), we obtain

$$w'(0)=c_{p+1}.$$

Under the simple calculations, we take

$$c_{p+1}=-2c_p\ln\left(\frac{c_p}{2\,(\beta-\alpha)}\right).$$

Thus, we obtain

$$|c_{p+1}|=2\left|c_p\ln\left(\frac{c_p}{2\,(\beta-\alpha)}\right)\right|.$$

□

The following inequality (2.8) is weaker, but is simpler than (2.4) and does not contain the coeffient c_{p+1}.

Theorem 2.3. *Under hypotheses of Theorem 2.2, we have*

$$|f'(z_0)|\geqslant\frac{\beta-\alpha}{2}\left(p-\frac{1}{2}\ln\frac{|c_p|}{2\,(\beta-\alpha)}\right). \tag{2.8}$$

The equality in (2.8) *holds if and only if*

$$f(z)=\alpha+(\beta-\alpha)\,\frac{1-z^{p}e^{Q}}{1+z^{p}e^{Q}},$$

where $0<c_p<1,\ \ln\left(\frac{c_p}{2(\beta-\alpha)}\right)<0$, $Q=\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)$ *and* θ *is a real number.*

Proof. From proof of the Theorem 2.2, using the inequality (1.6) for the function $\mu(z)$, we obtain

$$1\leq|\mu'(z_0)|=\frac{-2\ln h(0)}{\ln^2 h(0)+\arg^2 h(z_0)}\left\{|\varphi'(z_0)|-|B'(z_0)|\right\}.$$

So, replacing $\arg^2 h(z_0)$ by zero, we take

$$1\leq\frac{-2}{\ln\left(\frac{|c_p|}{2(\beta-\alpha)}\right)}\left(\frac{2}{(\beta-\alpha)}\,|f'(z_0)|-p\right). \tag{2.9}$$

Therefore, we have the inequality (2.8).

If $|f'(z_0)| = \frac{\beta-\alpha}{2}\left(p - \frac{1}{2}\ln\frac{|c_p|}{2(\beta-\alpha)}\right)$ from (2.9) and $|\mu'(z_0)| = 1$, we obtain

$$\frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)} = ze^{i\theta},$$

$$\ln h(z) = \frac{1+ze^{i\theta}}{1+ze^{i\theta}} \ln\left(\frac{c_p}{2(\beta-\alpha)}\right),$$

$$h(z) = e^{\frac{1+ze^{i\theta}}{1+ze^{i\theta}}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)},$$

$$\frac{\varphi(z)}{B(z)} = e^{\frac{1+ze^{i\theta}}{1+ze^{i\theta}}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)},$$

$$\frac{1-\vartheta(z)}{1+\vartheta(z)} = ze^{\frac{1+ze^{i\theta}}{1+ze^{i\theta}}\ln\left(\frac{c_p}{2(\beta-\alpha)}\right)}$$

and

$$f(z) = \alpha + (\beta-\alpha)\frac{1-z^pe^Q}{1+z^pe^Q}.$$

□

Theorem 2.4. *Let $f(z) = \beta + c_pz^p + c_{p+1}z^{p+1}....,$ $c_p \neq 0$, $p \geq 2$ be a holomorphic function in the unit disc D, and $\Re f(z) > \alpha$, $0 \leq \alpha < \beta$ (β is any integer≥ 1) for $|z| < 1$. Suppose that for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $\Re f(z_0) = \alpha$. Let $a_1, a_2, ..., a_n$ be zeros of the function $f(z) - \beta$ in D that are different from zero. Then we have the inequality*

$$|f'(z_0)| \geqslant \frac{\beta-\alpha}{2}\left\{p + \sum_{k=1}^{n}\frac{1-|a_k|^2}{|z_0-a_k|^2} + \frac{2\left(2(\beta-\alpha)\prod_{k=1}^{n}|a_k| - |c_p|\right)^2}{4(\beta-\alpha)^2\left(\prod_{k=1}^{n}|a_k|\right)^2 - |c_p|^2 + 2(\beta-\alpha)|c_{p+1}|\prod_{k=1}^{n}|a_k|}\right\}. \tag{2.10}$$

In addition, the equality in (2.10) *occurs for the function*

$$f(z) = \alpha + (\beta-\alpha)\frac{1 - z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1 + z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}},$$

where $a_1, a_2, ..., a_n$ are positive real numbers.

Proof. Let $\varphi(z)$ be defined as in (1.3).

$$B_0(z) = z^p\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}.$$

is a holomorphic functions in D, $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have

$$|\varphi(z)| \leq |B_0(z)|.$$

The function

$$k(z) = \frac{\varphi(z)}{B_0(z)}$$

is a holomorphic function in D, and $|k(z)| \leq 1$ for $|z| < 1$. In particular, we have

$$|k(0)| = \frac{|c_p|}{2\left(\beta - \alpha\right) \prod\limits_{k=1}^{n} |a_k|} \leq 1$$

and

$$|k'(0)| = \frac{|c_{p+1}|}{2\left(\beta - \alpha\right) \prod\limits_{k=1}^{n} |a_k|}.$$

Moreover, it can be seen that

$$\frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = |\varphi'(z_0)| \geq |B_0'(z_0)| = \frac{z_0 B_0'(z_0)}{B_0(z_0)}.$$

It is obviously that

$$|B_0'(z_0)| = \frac{z_0 B_0'(z_0)}{B_0(z_0)} = p + \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|z_0 - a_k|^2}.$$

The composite function

$$\Omega(z) = \frac{k(z) - k(0)}{1 - \overline{k(0)} k(z)}$$

is a holomorphic function in the unit disc D, $|\Omega(z)| < 1$, $\Omega(0) = 0$ and $|\Omega(z_0)| = 1$ for $\mathrm{z}_0 \in \partial D$.

From (1.5), we obtain

$$\frac{2}{1 + |\Omega'(0)|} \leq |\Omega'(z_0)| = \frac{1 - |k(0)|^2}{\left|1 - \overline{k(0)} k(z_0)\right|^2} |k'(z_0)| \leq \frac{1 + |k(0)|}{1 - |k(0)|} \left\{ |\varphi'(z_0)| - |B_0'(z_0)| \right\}$$

and

$$\frac{2}{1 + \frac{2(\beta - \alpha)\left|c_{p+1}\right| \prod\limits_{k=1}^{n} |a_k|}{4(\beta - \alpha)^2 \left(\prod\limits_{k=1}^{n} |a_k| \right)^2 - |c_p|^2}} \leq \frac{2(\beta - \alpha) \prod\limits_{k=1}^{n} |a_k| + |c_p|}{2(\beta - \alpha) \prod\limits_{k=1}^{n} |a_k| - |c_p|} \left\{ \frac{2|\vartheta'(z_0)|}{|\vartheta(z_0) + 1|^2} - p - \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|z_0 - a_k|^2} \right\}.$$

Therefore, we take the inequality (2.10) with an obvious equality case. □

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DEPARTMENT OF COMPUTER ENGINEERING, AMASYA UNIVERSITY, MERKEZ-AMASYA 05100, TURKEY

E-mail address: nafi.ornek@amasya.edu.tr