COMMON MULTIPLES OF PATHS AND STARS WITH COMPLETE GRAPHS

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**Abstract.** A graph $G$ is a common multiple of two graphs $H_1$ and $H_2$ if there exists a decomposition of $G$ into edge-disjoint copies of $H_1$ and also a decomposition of $G$ into edge-disjoint copies of $H_2$. If $G$ is a common multiple of $H_1$ and $H_2$ and $G$ has $q$ edges, then we call $G$ a $(q, H_1, H_2)$ graph. Our paper deals with the following question: ‘Given two graphs $H_1$ and $H_2$, for which values of $q$ does there exist a $(q, H_1, H_2)$ graph?’ when $H_1$ is either a path or a star with 3 or 4 edges and $H_2$ is a complete graph.

1. Introduction and preliminaries

All graphs considered here are finite and undirected, unless otherwise noted. The size of a graph $G$, denoted by $e(G)$, is its number of edges.

$K_n$ denotes the complete graph on $n$ vertices, and $K_{m,n}$ denotes the complete bipartite graph with vertex partitions of sizes $m$ and $n$.

A $k$-path, denoted by $P_k$, is a path with $k$ vertices (is a path of length $k - 1$); a $k$-star, denoted by $S_k$, is the complete bipartite graph $K_{1,k}$.

Let $G$ and $H$ be graphs. A decomposition of $G$ is a set of edge-disjoint subgraphs of $G$ whose union is $G$. An $H$-decomposition of $G$ is a decomposition of $G$ into copies of $H$. If $G$ has an $H$-decomposition, we say that $G$ is $H$-decomposable or $H$ divides $G$ and write $H | G$.

Given two graphs $H_1$ and $H_2$, one may ask for a graph $G$ that is a common multiple of $H_1$ and $H_2$ in the sense that both $H_1$ and $H_2$ divide $G$. Several authors have investigated the problem of finding least common multiples of pairs of graphs; that is, graphs of minimum size which are both $H_1$- and $H_2$-decomposable. The problem was introduced by Chartrand et al in [6] and they showed that every two nonempty graphs have a least common multiple. It is clear that least common multiple of two graphs may not be unique. The size of a least common multiple of two graphs $H_1$ and $H_2$ is denoted by lcm$(H_1,H_2)$. Also if $q_1$ and $q_2$ are two natural numbers, their number theoretic lcm is denoted by lcm$(q_1,q_2)$ as usual. Clearly, the least common multiple of two graphs $H_1$ and $H_2$, $lcm(H_1,H_2) \geq lcm(e(H_1),e(H_2))$. The problem of finding the size of least common multiples of graphs has been studied for several pairs of graphs: cycles and stars[6, 15],

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paths and complete graphs [12], pairs of cycles [10], pairs of cubes [2]. Pairs of
graphs having a unique least common multiple were investigated in [7] and least
common multiples of digraphs were considered in [8]. This is an interesting area
in decomposition of graphs like different types of domination[5, 13] in the field of
domination in graphs.

If $G$ is a common multiple of $H_1$ and $H_2$, and $G$ has $q$ edges, then we call
$G$ a ($q, H_1, H_2$) graph. An obvious necessary condition for the existence of a
($q, H_1, H_2$) graph is that $e(H_1)|q$ and $e(H_2)|q$. This obvious necessary condition
is not always sufficient. Some necessary conditions are easy to see and others
are more difficult. For example there is no $(15, K_6, K_6)$ graph as there is no $K_3$-
decomposition of $K_6$. However, non-existence of a $(36, K_3, K_4)$ graph is somewhat
less obvious. Hence a natural question is: Given two graphs $H_1$ and $H_2$, for which
values of $q$, does there exist a ($q, H_1, H_2$) graph? Adams, Bryant, and Maenhaut
[1] gave a complete solution to this problem in the case where $H_1$ is the 4-cycle
and $H_2$ is a complete graph; Bryant and Maenhaut [3] gave a complete solution to
this problem in the case where $H_1$ is the complete graph $K_3$ and $H_2$ is a complete
graph. A complete solution to this problem in the case where $H_1$ is a path and
$H_2$ is a star, is investigated in [9].

Since $S_2 = P_3$, the obvious necessary and sufficient condition for the existence
of a ($q, P_3, K_n$) graph is that $2q \equiv 0 \pmod{n(n - 1)}$ when $\binom{n}{2}$ is even
and $q \equiv 0 \pmod{n(n - 1)}$ when $\binom{n}{2}$ is odd (a nontrivial connected graph $G$ is
$P_3$-decomposable if and only if $G$ has even size). So in this paper we establish
the necessary and sufficient condition for the existence of a ($q, P_4, K_n$) graph, a
($q, P_5, K_n$) graph, a ($q, S_3, K_n$) graph and a ($q, S_4, K_n$) graph. The graph theo-
retic concepts described here are, of course, suggested by their number theoretic
counterparts.

The complete graph with vertex set $\{v_1, v_2, ..., v_m\}$ will be denoted by $[v_1, v_2, ..., v_m]$, $m$-path $P_m$ with vertex set $\{v_1, v_2, ..., v_m\}$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{m-1}, v_m\}$ will be denoted by $\langle v_1, v_2, ..., v_m \rangle$ and $m$-star $S_m$ with vertex set $\{v_0, v_1, v_2, ..., v_m\}$ and center at $v_0$ will be denoted by $[v_0; v_1, v_2, ..., v_m]$. If $G$ and
$H$ are graphs, and $H$ is a subgraph of $G$, then the graph obtained by removing
the edges of $H$ from $G$ will be denoted by $G - H$. If $G_1$ and $G_2$ are graphs,
then the union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the graph with vertex set
$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. (We
shall only be considering the union of edge-disjoint graphs.)

2. Common Multiples of $P_4$ and $K_n$

In this section we determine, for all positive integers $n$, the set of integers $q$
for which there exists a common multiple of $P_4$ and $K_n$ having precisely $q$ edges.
The following known results on the path-decomposition of complete graphs are used for the discussion.

**Theorem 2.1.** [14] Let $k$ and $n$ be positive integers. There exists a $P_{k+1}$-
decomposition of $K_n$ if and only if $n \geq k + 1$ and $n(n - 1) \equiv 0 \pmod{2k}$.

**Theorem 2.2.** [4] Let $n$ and $t$ be positive integers and let $m_1, m_2, ..., m_t$ be a
sequence of positive integers. There exist $t$ pairwise edge-disjoint paths of lengths
$m_1, m_2, ..., m_t$ in $K_n$ if and only if $m_i \leq n - 1$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + \cdots + m_t \leq {n \choose 2}$.

Characterization for the existence of a $(q, P_4, K_n)$ graph is given in the next theorem.

Theorem 2.3. There exists a graph with $q$ edges that is both $P_4$-decomposable and $K_n$-decomposable if and only if

1. $2q \equiv 0 \mod n(n - 1)$ when $n \equiv 0, 1, 3, 4 \mod 6$
2. $2q \equiv 0 \mod 3n(n - 1)$ when $n \equiv 2, 5 \mod 6$
3. $q \neq 3$ when $n = 3$.

Proof. If there exists a $(q, P_4, K_n)$ graph, then we require that $3$ divides $q$ and that $\binom{n}{2}$ divides $q$. Necessary conditions (1) and (2) follow immediately from this and will be referred to as the obvious necessary conditions. If $n = 3$, then $q \neq 3$ as $K_3$ is not $P_4$-decomposable.

Sufficient Conditions

If $n \equiv 0, 1, 3, 4 \mod 6$ and $n \geq 4$, then $P_4 | K_n$ (Theorem 2.1) and hence when $n \equiv 0, 1, 3, 4 \mod 6$, there exists a $(q, P_4, K_n)$ graph $G$ for all $q \equiv 0 \mod \binom{n}{2}$ by taking $G$ to be $\frac{q}{\binom{n}{2}}$ vertex-disjoint copies of $K_n$. If $n = 3$, it is sufficient to construct a $(6, P_4, K_3)$ graph and a $(9, P_4, K_3)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

To construct a $(6, P_4, K_3)$ graph $G$, we let $G$ be the union of the following two edge-disjoint copies of $K_3$.

\[ \begin{array}{c}
[1, 2, 3] \\
[1, 4, 5]
\end{array} \]

A $P_4$-decomposition of $G$ is given by the following two edge-disjoint copies of $P_4$.

\[ \langle 2, 3, 1, 4 \rangle \quad \langle 4, 5, 1, 2 \rangle \]

To construct a $(9, P_4, K_3)$ graph $G$, we let $G$ be the union of the following three edge-disjoint copies of $K_3$.

\[ \begin{array}{c}
[1, 2, 3] \\
[1, 4, 5] \\
[1, 6, 7]
\end{array} \]

A $P_4$-decomposition of $G$ is given by the following three edge-disjoint copies of $P_4$.

\[ \langle 7, 6, 1, 5 \rangle \quad \langle 5, 4, 1, 3 \rangle \quad \langle 3, 2, 1, 7 \rangle \]

Thus sufficient conditions (1) and (3) obtained.

If $n \equiv 2, 5 \mod 6$, then $\binom{n}{2} \equiv 1 \mod 3$ and $lcm(3, \binom{n}{2}) = 3\binom{n}{2}$. First suppose that $n = 2$. For a $(3, P_4, K_2)$ graph $G$, we let $G$ be $P_4$, which is $K_2$-decomposable.
Now suppose that \( n \equiv 2, 5 \pmod{6} \) and \( n \geq 5 \). Let \( \binom{n}{2} = 3r + 1 \), where \( r > 0 \). By Theorem 2.2, \( K_n \) can be decomposed into the paths \( P_4, P_4, \ldots, P_4, P_2 \).

Take three copies of \( K_n \). Consider the end vertices of \( P_2 \) in each copy of \( K_n \) in the decomposition. Identify these vertices to get a connected graph \( G \), which is actually three \( K_n \)'s joined by identification of end vertices of copies of \( P_2 \) serially. \( G \) is a \( (3\binom{n}{2}, P_4, K_n) \) graph, since by the construction itself \( G \) is \( K_n \)-decomposable. \( G \) is also \( P_4 \)-decomposable, since \( r \) copies of \( P_4 \) can be taken from each \( K_n \) and one copy of \( P_4 \) can be obtained by identification of the end vertices of three \( P_2 \)'s. Therefore there exists a \( (k\binom{n}{2}, P_4, K_n) \) graph for all \( k \equiv 0 \pmod{3}, n \equiv 2, 5 \pmod{6} \).

\[ \square \]

3. Common Multiples of \( P_5 \) and \( K_n \)

Following theorem gives a characterization for the existence of a \( (q, P_5, K_n) \) graph.

**Theorem 3.1.** There exists a graph with \( q \) edges that is both \( P_5 \)-decomposable and \( K_n \)-decomposable if and only if

1. \( 2q \equiv 0 \pmod{n(n-1)} \) when \( n \equiv 0, 1 \pmod{8} \)
2. \( q \equiv 0 \pmod{n(n-1)} \) when \( n \equiv 4, 5 \pmod{8} \)
3. \( q \equiv 0 \pmod{2n(n-1)} \) when \( n \equiv 2, 3, 6, 7 \pmod{8} \).

If there exists a \( (q, P_5, K_n) \) graph, then we require that \( 4 \) divides \( q \) and that \( \binom{n}{2} \) divides \( q \). Necessary conditions follow immediately from this and will be referred to as the obvious necessary conditions.

**Sufficient Conditions**

To show that the stated necessary conditions are sufficient we consider each in turn and construct the \( (q, P_5, K_n) \) graphs required to prove Theorem 3.1. First note that if \( n \equiv 0, 1 \pmod{8} \), then \( P_5 | K_n \) (Theorem 2.1) and hence when \( n \equiv 0, 1 \pmod{8} \), there exists a \( (q, P_5, K_n) \) graph \( G \) for all \( q \equiv 0 \pmod{\binom{n}{2}} \). (Take \( G \) be \( \frac{q}{\binom{n}{2}} \) vertex-disjoint copies of \( K_n \)). Thus sufficient condition (1) obtained. We require a few lemmas to construct the graphs for the remaining congruence classes of \( n \pmod{8} \).

**Lemma 3.2.** There exists a \( (6k, P_5, K_4) \) graph for all even \( k \).

*Proof.* \( \text{lcm}(4, 6) = 12 \). So it is sufficient to construct a \( (12, P_5, K_4) \) graph \( G \), as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of this.

To construct a \( (12, P_5, K_4) \) graph \( G \), we let \( G \) be the union of the following two edge-disjoint copies of \( K_4 \).

\[ [1, 2, 3, 4] \quad [1, 5, 6, 7] \]
A \( P_5 \)-decomposition of \( G \) is given by the following three edge-disjoint copies of \( P_5 \).

\[
\langle 2, 4, 3, 1, 6 \rangle \quad \langle 3, 2, 1, 5, 7 \rangle \quad \langle 4, 1, 7, 6, 5 \rangle
\]

\[\square\]

**Lemma 3.3.** For all \( k \equiv 0 \pmod{4} \), there exists a \( (k(n), P_5, K_n) \) graph, when \( n = 2, 3 \).

**Proof.** If \( n = 2, 3 \), then \( \gcd(4, \binom{n}{2}) = 1 \) and hence \( \text{lcm}(4, \binom{n}{2}) = 4\binom{n}{2} \).

First suppose that \( n = 2 \). For a \( (4, P_5, K_2) \) graph \( G \), we let \( G \) be \( P_5 \), which is \( K_2 \)-decomposable.

Now suppose that \( n = 3 \). To construct a \( (12, P_5, K_3) \) graph \( G \), we let \( G \) be the union of the following four edge-disjoint copies of \( K_3 \).

\[
[1, 2, 3] \quad [3, 4, 5] \quad [5, 6, 7] \quad [7, 8, 9]
\]

A \( P_5 \)-decomposition of \( G \) is given by the following three edge-disjoint copies of \( P_5 \).

\[
\langle 1, 2, 3, 4, 5 \rangle \quad \langle 5, 6, 7, 8, 9 \rangle \quad \langle 9, 7, 5, 3, 1 \rangle
\]

Hence, if \( n = 2, 3 \), we can construct a \( (k(n), P_5, K_n) \) graph \( G \), for all \( k \equiv 0 \pmod{4} \).

\[\square\]

**Lemma 3.4.** For all \( n \geq 5 \), there exists a \( (q, P_4, K_n) \) graph if

1. \( q \equiv 0 \pmod{n(n-1)} \) and \( n \equiv 4, 5 \pmod{8} \); or
2. \( q \equiv 0 \pmod{2n(n-1)} \) when \( n \equiv 2, 3, 6, 7 \pmod{8} \).

**Proof.** If \( n \equiv 4, 5 \pmod{8} \), then \( \binom{n}{2} \equiv 2 \pmod{4} \) and \( \text{lcm}(4, \binom{n}{2}) = 2\binom{n}{2} \).

If \( n \equiv 2, 7 \pmod{8} \), then \( \binom{n}{2} \equiv 1 \pmod{4} \) and \( \text{lcm}(4, \binom{n}{2}) = 4\binom{n}{2} \).

If \( n \equiv 3, 6 \pmod{8} \), then \( \binom{n}{2} \equiv 3 \pmod{4} \) and \( \text{lcm}(4, \binom{n}{2}) = 4\binom{n}{2} \).

Consider the following cases.

**Case 1:** \( n \equiv 4, 5 \pmod{8}, n \geq 5 \)

Let \( \binom{n}{2} = 4r + 2 \), where \( r > 0 \).

Since \( n \geq 5 \), by Theorem 2.2, \( K_n \) can be decomposed into the edge-disjoint paths

\[
P_5, P_5, \ldots, P_5, P_3\underbrace{\text{r copies}}_{r-copies}
\]

Take two copies of \( K_n \) and identify the end vertices of two \( P_3 \) in each copy of \( K_n \) to get the required graph \( G = (2\binom{n}{2}, P_5, K_n) \). By the construction itself \( G \) is \( K_n \)-decomposable. \( G \) is \( P_5 \)-decomposable, since \( G \) can be decomposed into \( 2r + 1 \) edge-disjoint copies of \( P_5 \). The identified vertex joined two copies of \( P_3 \) to make a \( P_5 \). Then there exists a \( (k(n), P_5, K_n) \) graph for all even \( k \).

**Case 2:** \( n \equiv 2, 7 \pmod{4} \)
Let \( \binom{n}{2} = 4r+1 \), where \( r > 0 \). By Theorem 2.2, \( K_n \) can be decomposed into the edge-disjoint paths
\[
P_5, P_5, ..., P_5, P_2
\]
Take four copies of \( K_n \). Consider the edges of \( P_2 \) in each copy of \( K_n \) in the above decomposition. Concatenate these four edges at the end vertices to make a path of length 4. This is a \((4 \binom{n}{2}, P_5, K_n)\) graph \( G \). By the construction itself \( G \) is \( K_n \)-decomposable. \( G \) is \( P_5 \)-decomposable, since \( G \) can be decomposed into \( 4r + 1 \) edge-disjoint copies of \( P_5 \). Among these copies of \( P_5 \) in \( G \), one copy of \( P_5 \) is obtained by the concatenation of four \( P_2 \)’s. Then there exists a \((k \binom{n}{2}, P_5, K_n)\) graph for all \( k \equiv 0 \) (mod 4).

**Case 3:** \( n \equiv 3, 6 \) (mod 4).
Let \( \binom{n}{2} = 4r+3 \), where \( r > 0 \). By Theorem 2.2, \( K_n \) can be decomposed into the edge-disjoint paths
\[
P_5, P_5, ..., P_5, P_4
\]
Take four copies of \( K_n \). Consider the end vertices of \( P_4 \) in each copy of \( K_n \) in the decomposition. Concatenate the four paths at these end vertices to make a path of length 12. This is the required graph \( G = (4 \binom{n}{2}, P_5, K_n) \). By the construction itself \( G \) is \( K_n \)-decomposable. \( G \) is \( P_5 \)-decomposable, since \( G \) can be decomposed into \( 4r + 3 \) edge-disjoint copies of \( P_5 \). Three copies of \( P_5 \) is obtained by decomposing the above path of length 12. Thus there exists a \((k \binom{n}{2}, P_5, K_n)\) graph for all \( k \equiv 0 \) (mod 4).

Lemmas 3.2, 3.3 and 3.4 allow us to construct all of the \((q, P_5, K_n)\) graphs that we require for the sufficient conditions (2) and (3) of Theorem 3.1. Thus proof of Theorem 3.1 is completed.

### 4. Common Multiples of \( S_3 \) and \( K_n \)

In this section we determine, for all positive integers \( n \), the set of integers \( q \) for which there exists a common multiple of \( S_3 \) (3-star) and complete graph \( K_n \) having precisely \( q \) edges. The following known results on the star-decomposition of complete graphs are used for the discussion.

**Theorem 4.1.** [16] A complete graph, \( K_n \) can be decomposed into \( \frac{\binom{n}{2}}{k} \) edge-disjoint stars, \( S_k \) if and only if \( \binom{n}{2} \equiv 0 \) (mod \( k \)) and \( n \geq 2k \).

**Theorem 4.2.** [11] Let \( m_1, m_2, ..., m_l \) be non-negative integers such that \( \sum_{i=1}^{l} m_i = \binom{n}{2} \) and \( 2m_i \leq n \) for \( 1 \leq i \leq l \). Then \( K_n \) can be decomposed into \( S_{m_1}, S_{m_2}, ..., S_{m_l} \).

**Theorem 4.3.** [6] For every integer \( l \geq 1 \), \( \text{lcm}(C_3, K_{1,l}) = \frac{3kl}{d} \), where \( d = \gcd(3, l) \) and \( k = \left\lceil \frac{(d^2+3)}{9} \right\rceil \).

The following theorem gives the necessary and sufficient condition for the existence of a \((q, S_3, K_n)\) graph.
Theorem 4.4. There exists a graph with \( q \) edges that is both \( S_3 \)-decomposable and \( K_n \)-decomposable if and only if

1. \( 2q \equiv 0 \pmod{n(n-1)} \) when \( n \equiv 0, 1, 3, 4 \pmod{6} \)
2. \( 2q \equiv 0 \pmod{3n(n-1)} \) when \( n \equiv 2, 5 \pmod{6} \)
3. \( 2q \geq 3n(n-1) \) when \( n = 3, 4 \).

Proof. The given conditions are necessary for the following reasons.

If there exists a \((q, S_3, K_n)\) graph, then we require that 3 divides \( q \) and that \( \binom{n}{2} \) divides \( q \). Conditions (1) & (2) follow immediately from this and will be referred to as the obvious necessary conditions.

If \( n = 3 \) and there exists a \((q, S_3, K_3)\) graph, then \( q \neq 3, 6 \) since \( \text{lcm}(S_3, K_3) = 9 \) by Theorem 4.3.

If \( n = 4 \) and there exists a \((q, S_3, K_4)\) graph, then \( q \neq 6 \), since \( K_4 \) is not \( S_3 \)-decomposable, and \( q \neq 12 \), since two copies of \( K_4 \) intersecting in at most one vertex is not \( S_3 \)-decomposable. Hence \( q \geq 3\binom{n}{2} \) when \( n = 3, 4 \).

Sufficient Conditions

To show that the stated necessary conditions are sufficient we consider each in turn and construct the \((q, S_3, K_n)\) graphs required to prove Theorem 4.4.

Case 1: \( n = 2, 3, 4, 5 \)

When \( n = 2, K_2 | S_3 \) and hence there exists a \((q, S_3, K_2)\) graph \( G \) for all \( q \equiv 0 \pmod{3} \) (Take \( G \) to be \( q \) vertex disjoint copies of \( S_3 \)).

When \( n = 3 \), it is sufficient to construct a \((9, S_3, K_3)\) graph, a \((12, S_3, K_3)\) graph and a \((15, S_3, K_3)\) graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

For a \((9, S_3, K_3)\) graph \( G \), we let \( G \) be the union of the following three edge-disjoint copies of \( K_3 \).

\[
[1, 2, 3] \quad [2, 4, 5] \quad [3, 5, 6]
\]

An \( S_3 \)-decomposition of \( G \) is given by the following three edge-disjoint copies of \( S_3 \).

\[
[2; 1, 3, 4] \quad [3; 1, 5, 6] \quad [5; 2, 4, 6]
\]

For a \((12, S_3, K_3)\) graph \( G \), we let \( G \) be \( K_2, 2, 2, 2 \), the graph of the octahedron. It is easy to see that \( K_2, 2, 2, 2 \) is \( S_3 \)-decomposable and \( K_3 \)-decomposable.

To construct a \((15, S_3, K_3)\) graph \( G \), we let \( G \) be the union of the following five edge-disjoint copies of \( K_3 \).

\[
[1, 2, A] \quad [1, 3, B] \quad [2, 4, B] \quad [3, 5, A] \quad [4, 5, C]
\]

An \( S_3 \)-decomposition of \( G \) is given by the following five edge-disjoint copies of \( S_3 \).

\[
[1; 2, A, B] \quad [2; 4, A, B] \quad [3; 1, A, B] \quad [4; 5, C, B] \quad [5; 3, A, C]
\]

Hence there exists a \((3k, S_3, K_3)\) graph \( G \) for all \( k \geq 3 \).

When \( n = 4 \), \( \text{lcm}(3, \binom{n}{2}) = 6 \). It is sufficient to construct a \((18, S_3, K_4)\) graph,
a \((24, S_3, K_4)\) graph and a \((30, S_3, K_4)\) graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these. To construct a \((18, S_3, K_4)\) graph \(G\), we let \(G\) be the union of the following three edge-disjoint copies of \(K_4\).

\[
\begin{align*}
[1, 2, 3, A] & \quad [1, 4, 5, B] & [2, 4, 6, C] \\
[1; 2, A, B] & \quad [2; 4, C, A] & [3; 2, 1, A] \\
[4; 1, C, B] & \quad [5; 4, 1, B] & [6; 4, 2, C]
\end{align*}
\]

An \(S_3\)-decomposition of \(G\) is given by the following six edge-disjoint copies of \(S_3\).

\[
\begin{align*}
[1; 2, A, B] & \quad [2; 4, C, A] & [3; 2, 1, A] \\
[4; 1, C, B] & \quad [5; 4, 1, B] & [6; 4, 2, C]
\end{align*}
\]

To construct a \((24, S_3, K_4)\) graph \(G\), we let \(G\) be the union of the following four edge-disjoint copies of \(K_4\).

\[
\begin{align*}
[0, 2, 3, A] & \quad [2, 4, 5, B] & [4, 6, 7, A] & [1, 3, 7, B] \\
[0; 2, 4, A] & \quad [1; 3, 5, A] & [2; 4, A, B] & [3; 2, A, B] \\
\end{align*}
\]

To construct a \((30, S_3, K_4)\) graph \(G\), we let \(G\) be the union of the following five edge-disjoint copies of \(K_4\).

\[
\begin{align*}
[0, 2, 4, A] & \quad [1, 3, 5, A] & [4, 6, 8, B] & [5, 7, 9, B] & [2, 3, 6, 7] \\
[0; 2, 4, A] & \quad [1; 3, 5, A] & [2; 4, 7, A] & [3; 6, 2, A] \\
[8; 6, 4, B] & \quad [9; 7, 5, B]
\end{align*}
\]

An \(S_3\)-decomposition of \(G\) is given by the following ten copies of \(S_3\).

\[
\begin{align*}
[0; 2, 4, A] & \quad [1; 3, 5, A] & [2; 4, 7, A] & [3; 6, 2, A] \\
[8; 6, 4, B] & \quad [9; 7, 5, B]
\end{align*}
\]

Therefore there exists a \((6k, S_3, K_4)\) graph \(G\) for all \(k \geq 3\).

When \(n = 5\), \(lcm(3, \binom{n}{2}) = 30\). It is easy to see that \(K_5\) can be decomposed into three edge-disjoint copies of \(S_3\) and an \(S_1\). Take three copies of \(K_5\) and identify them at one vertex of \(S_1\) in the decomposition of each \(K_5\), to get the required graph \(G = (30, S_3, K_5)\) graph. Then \(G\) is \(K_5\)-decomposable, by its construction and \(G\) can be decomposed into 10 edge-disjoint copies of \(S_3\) (one \(S_3\) centered at
the identified vertex and 9 from three copies of $K_5$). Therefore there exists a $(10k, S_3, K_5)$ graph $G$ for all $k \equiv 0 \pmod{3}$.

**Case 2:** $n \geq 6$ and $n \equiv 0, 1, 3, 4 \pmod{6}$.

Then $S_3/K_n$, by Theorem 4.1. Hence in this case there exists a $(q, S_3, K_n)$ graph $G$ for all $q \equiv 0 \pmod{\binom{n}{2}}$ (Take $G$ to be $q/3$ vertex-disjoint copies of $K_n$).

**Case 3:** $n \geq 6$ and $n \equiv 2, 5 \pmod{6}$.

In this case $\binom{n}{2} \equiv 1 \pmod{3}$ and $lcm(3, \binom{n}{2}) = 3\binom{n}{2}$. Let $\binom{n}{2} = 3r + 1$, where $r > 0$. By Theorem 4.2, $K_n$ can be decomposed into the edge-disjoint stars $S_3, S_3, \ldots, S_3, S_1$. Take 3 copies of $K_n$. Consider the centers of $S_1$ in each copy of $K_n$ in the decomposition of $K_n$. Identify these three vertices from 3 copies of $K_n$ to get the required graph $G = (3\binom{n}{2}, S_3, K_n)$. By construction itself $G$ is $K_n$-decomposable. Also, $G$ can be decomposed into $3r + 1$ copies of $S_3$ (r copies of $S_3$ from each $K_n$ and one copy of $S_3$ obtained at the identified vertex). Therefore there exists a $(k\binom{n}{2}, S_3, K_n)$ for all $k \equiv 0 \pmod{3}$.

5. **Common Multiples of $S_4$ and $K_n$**

In this section we determine, for all positive integers $n$, the set of integers $q$ for which there exists a common multiple of $S_4(4 - \text{star})$ and $K_n$ having precisely $q$ edges. The following theorem gives necessary and sufficient condition for the existence of a $(q, S_4, K_n)$ graph.

**Theorem 5.1.** There exists a graph with $q$ edges that is both $S_4$-decomposable and $K_n$-decomposable if and only if

1. $2q \equiv 0 \pmod{n(n-1)}$ when $n \equiv 0, 1 \pmod{8}$
2. $q \equiv 0 \pmod{n(n-1)}$ when $n \equiv 4, 5 \pmod{8}$
3. $q \equiv 0 \pmod{2n(n-1)}$ when $n \equiv 2, 3, 6, 7 \pmod{8}$
4. $q \neq 12$ when $n = 3$
5. $q > n(n-1)$ when $n = 4, 5$.

**Proof.** If there exists a $(q, S_4, K_n)$ graph, then we require that 4 divides $q$ and that $\binom{n}{2}$ divides $q$. Conditions (1) - (3) follow immediately from this and will be referred to as the obvious necessary conditions. The following two lemmas establish the remaining necessary conditions (4) and (5).

**Lemma 5.2.** If there exists a $(q, S_4, K_3)$ graph, then $q \neq 12$.

**Proof.** If $n = 3$, then the obvious necessary condition for the existence of a $(q, S_4, K_n)$ graph is that $q \equiv 0 \pmod{4\binom{n}{2}}$. Suppose that $n = 3$, and there exists a $(12, S_4, K_3)$ graph. This is impossible since $lcm(S_4, K_3) = 24$ by Theorem 4.3. So $q \neq 12$. □

**Lemma 5.3.** If $n = 4, 5$ and there exists a $(q, S_4, K_n)$ graph, then $q > 2\binom{n}{2}$.
Proof. If \( n = 4, 5 \), then the obvious necessary condition for the existence of a \((q, S_4, K_n)\) graph is that \( q \equiv 0 \pmod{2^{\binom{n}{2}}} \). First suppose that \( n = 4 \), and there exists a \((12, S_4, K_4)\) graph \( G \). Such a graph consists of two copies of \( K_4 \) intersecting in at most one vertex (in order for \( G \) to be \( K_n \)-decomposable), then there is only one vertex with degree greater than 4. So, if we remove a copy of \( S_4 \) from \( G \), we can not obtain two more edge-disjoint copies of \( S_4 \)'s from \( G \).

Now suppose that \( n = 5 \), and there exists a \((20, S_4, K_5)\) graph \( G \). Then \( G \) consists of two copies of \( K_5 \) intersecting in at most one vertex, and hence \( G \) has eight vertices of degree 4 and one vertex of degree 8. If we remove copies of \( S_4 \)'s one by one from \( G \) we could obtain only three \( S_4 \)'s. But \( G \) must contain five \( S_4 \)'s. Thus \( q > 2^{\binom{n}{2}} \) when \( n = 4, 5 \).

\(\square\)

Sufficient Conditions
To show that the stated necessary conditions are sufficient we consider each in turn and construct the \((q, S_4, K_n)\) graphs required to prove Theorem 5.1. First note that if \( n \equiv 0, 1 \pmod{8} \), then \( S_4 | K_n \) (Theorem 4.1) and hence when \( n \equiv 0, 1 \pmod{8} \), there exists a \((q, S_4, K_n)\) graph \( G \) for all \( q \equiv 0 \pmod{2^{\binom{n}{2}}} \)(Take \( G \) to be \( q \left(\binom{n}{2}\right) \) vertex-disjoint copies of \( K_n \)). Thus sufficient condition (1) obtained. We require a few lemmas to construct the graphs for the remaining congruence classes of \( n \pmod{8} \).

Lemma 5.4. For all \( k > 1 \), there exists a \((12k, S_4, K_4)\) graph.

Proof. It is sufficient to construct a \((24, S_4, K_4)\) graph and a \((36, S_4, K_4)\) graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

To construct a \((24, S_4, K_4)\) graph \( G \), we let \( G \) be the union of the following four edge-disjoint copies of \( K_4 \).

\[
[1, 2, 3, A] \quad [1, 4, 5, B] \quad [2, 4, 6, C] \quad [3, 5, 6, D]
\]

An \( S_4 \)-decomposition of \( G \) is given by the following six edge-disjoint copies of \( S_4 \).

\[
[1; 2, 4, A, B] \quad [2; 3, 6, A, C] \quad [3; 1, 5, A, D] \quad [4; 2, 5, B, C] \\
[5; 1, 6, B, D] \quad [6; 3, 4, C, D]
\]

To construct a \((36, S_4, K_4)\) graph, we let \( G \) be the union of the following six edge-disjoint copies of \( K_4 \).

\[
[1, 2, 3, A] \quad [4, 5, 6, A] \quad [7, 8, 9, A] \quad [1, 4, 7, B] \\
[2, 5, 8, B] \quad [3, 6, 9, B]
\]
An $S_4$-decomposition of $G$ is given by the following nine edge-disjoint copies of $S_4$.

$[1; 2, 4, A, B] \quad [2; 3, 5, A, B] \quad [3; 1, 6, A, B] \quad [4; 5, 7, A, B]$

$[5; 6, 8, A, B] \quad [6; 4, 9, A, B] \quad [7; 1, 8, A, B] \quad [8; 2, 9, A, B]$

$[9; 3, 7, A, B]$

\[\Box\]

**Lemma 5.5.** For all $k > 1$, there exists a $(20k, S_4, K_5)$ graph.

**Proof.** It is sufficient to construct a $(40, S_4, K_5)$ graph and a $(60, S_4, K_5)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

To construct a $(40, S_4, K_5)$ graph $G$, we let $G$ be the union of the following four edge-disjoint copies of $K_5$.

$[0, 1, 2, 3, A] \quad [0, 4, 5, 6, B] \quad [1, 4, 7, 8, C] \quad [2, 5, 7, 9, D]$

An $S_4$-decomposition of $G$ is given by the following ten edge-disjoint copies of $S_4$.

$[0; 1, 2, B, A] \quad [1; 2, 4, C, A] \quad [2; 3, 5, D, A] \quad [3; 0, 1, 2, A]$

$[4; 0, 7, C, B] \quad [5; 0, 4, D, B] \quad [6; 4, 5, 0, B] \quad [7; 1, 2, D, C]$

$[8; 1, 7, 4, C] \quad [9; 2, 5, 7, D]$

To construct a $(60, S_4, K_5)$ graph $G$, we let $G$ be the union of the following six edge-disjoint copies of $K_5$.

$[1, 2, 3, 4, 5] \quad [1, 6, 7, 8, 9] \quad [2, 6, 10, 11, 12] \quad [3, 7, 10, 13, 14]$

$[4, 8, 11, 13, 15] \quad [5, 9, 12, 14, 15]$

An $S_4$-decomposition of $G$ is given by the following fifteen edge-disjoint copies of $S_4$.

$[1; 2, 3, 4, 5] \quad [2; 3, 4, 5, 6] \quad [3; 4, 5, 7, 10] \quad [4; 5, 8, 11, 13]$

$[5; 9, 12, 14, 15] \quad [6; 1, 10, 11, 12] \quad [7; 1, 10, 13, 14] \quad [8; 1, 6, 7, 9]$

$[9; 1, 6, 7, 12] \quad [10; 2, 11, 12, 13] \quad [11; 2, 8, 12, 13] \quad [12; 2, 9, 14, 15]$

$[13; 3, 8, 14, 15] \quad [14; 3, 9, 10, 15] \quad [15; 4, 8, 9, 11]$

\[\Box\]
Lemma 5.6. There exists a \((q, S_4, K_n)\) graph if

1. \(q \equiv 0 \pmod{n(n-1)}\), \(q > n(n-1)\) and \(n = 4, 5\); or
2. \(q \equiv 0 \pmod{2n(n-1)}\) and \(n = 2, 6, 7\); or
3. \(q \equiv 0 \pmod{2n(n-1)}\), \(q \neq 12\) and \(n = 3\).

Proof. The result is true for \(n = 4\) and 5 by Lemmas 5.4 and 5.5 respectively.

If \(n = 2, 3, 6, 7\), then \(\gcd(4, n^2) = 1\) and hence \(\text{lcm}(4, n^2) = 4n^2\).

Case 1: \(n = 2\)

For every \(k \equiv 0 \pmod{4}\) let \(G_k\) be \(k/4\) vertex-disjoint copies of \(S_4\), which is both \(S_4\)-decomposable and \(K_2\)-decomposable. Hence a \((k, S_4, K_2)\) graph exists for all \(k \equiv 0 \pmod{4}\).

Case 2: \(n = 3\)

By Lemma 5.2, \(q \neq 12\). It is sufficient to construct a \((24, S_4, K_3)\) graph and a \((36, S_4, K_3)\) graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

A \((24, S_4, K_3)\) graph \(G\) is obtained from \(K_2, 4\) by adding a new vertex for each of its eight edges and joining the vertex to the two vertices incident with the corresponding edge. For a \((36, S_4, K_3)\) graph \(G\), we take \(G = K_9\), which is \(S_4\)-decomposable (by Theorem 4.1) and \(K_3\)-decomposable (\(K_3|K_n\) if and only if \(n \equiv 1, 3 \pmod{6}\)).

Case 3: \(n = 6\)

Then \(\text{lcm}(4, n^2) = 60\).

To construct a \((60, S_4, K_6)\) graph \(G\), we let \(G\) be the union of the following four edge-disjoint copies of \(K_6\).

\[
\begin{align*}
&[1, 2, 3, 4, 5, 0] & [1, 6, 7, 8, 9, A] \\
&[1, 10, 11, 12, 13, B] & [5, 7, 13, 14, 15, C] \\
\end{align*}
\]

An \(S_4\)-decomposition of \(G\) is given by the following fifteen edge-disjoint copies of \(S_4\).

\[
\begin{align*}
&[1; 0, 5, A, B] & [2; 0, 5, 3, 1] & [3; 0, 5, 4, 1] & [4; 0, 5, 2, 1] \\
&[5; 0, 7, 15, C] & [6; 7, 8, 1, A] & [7; 1, 13, A, C] & [8; 9, 1, 7, A] \\
&[9; 7, 6, 1, A] & [10; 11, 13, 1, B] & [11; 12, 13, 1, B] & [12; 13, 10, 1, B] \\
&[13; 1, 5, B, C] & [14; 5, 7, 13, C] & [15; 7, 13, 14, C] \\
\end{align*}
\]

Hence there exists a \((15k, S_4, K_6)\) graph exists for all \(k \equiv 0 \pmod{4}\).

Case 4: \(n = 7\)

In this case \(\text{lcm}(4, n^2) = 84\). We can easily verify that \(K_7\) can be decomposed into five edge-disjoint copies of \(S_4\) and one copy of \(S_1\). To construct a \((84, S_4, K_7)\) graph \(G\), take four copies of \(K_7\) and identify them at the one vertex of \(S_1\) in the
decomposition of \( K_7 \). By the construction itself \( G \) is \( K_7 \)-decomposable. \( G \) can be decomposed into 21 edge-disjoint copies of \( S_4 \) (five copies of \( S_4 \) from each \( K_7 \) and one copy of \( S_4 \) at the identified vertex). Therefore there exists a \((21k, S_4, K_7)\) graph for all \( k \equiv 0 \pmod{4} \).

\[ \square \]

**Lemma 5.7.** For all \( n \geq 8 \), there exists a \((q, S_4, K_n)\) graph if

1. \( q \equiv 0 \pmod{n(n-1)} \) and \( n \equiv 4, 5 \pmod{8} \); or
2. \( q \equiv 0 \pmod{2n(n-1)} \) when \( n \equiv 2, 3, 6, 7 \pmod{8} \);

**Proof.** If \( n \equiv 4, 5 \pmod{8} \), then \( \binom{n}{2} \equiv 2 \pmod{4} \) and \( \operatorname{lcm}(4, \binom{n}{2}) = 2 \binom{n}{2} \).

If \( n \equiv 2, 7 \pmod{8} \), then \( \binom{n}{2} \equiv 1 \pmod{4} \) and \( \operatorname{lcm}(4, \binom{n}{2}) = 4 \binom{n}{2} \).

If \( n \equiv 3, 6 \pmod{8} \), then \( \binom{n}{2} \equiv 3 \pmod{4} \) and \( \operatorname{lcm}(4, \binom{n}{2}) = 4 \binom{n}{2} \).

Consider the following cases.

**Case 1:** \( n \equiv 4, 5 \pmod{8}, n \geq 8 \).

Let \( \binom{n}{2} = 4r + 2 \), where \( r > 0 \).

Since \( n \geq 8 \), by Theorem 4.2, \( K_n \) can be decomposed into the edge-disjoint stars

\[
\underbrace{S_4, S_4, \ldots, S_4}_{r\text{-copies}}, S_2.
\]

Take two copies of \( K_n \) and identify the centers of two \( S_2 \) in each copy of \( K_n \) to get the required graph \( G = \left(2\binom{n}{2}, S_4, K_n\right)\). By the construction itself \( G \) is \( K_n \)-decomposable. \( G \) is \( S_4 \)-decomposable, since \( G \) can be decomposed into \( 2r + 1 \) edge-disjoint copies of \( S_4 \). The identified vertex can be considered as the center of one copy of \( S_4 \). Then there exists a \((k\binom{n}{2}, S_4, K_n)\) graph for all even \( k \).

**Case 2:** \( n \equiv 2, 7 \pmod{4} \).

Let \( \binom{n}{2} = 4r + 1 \), where \( r > 0 \). By Theorem 4.2, \( K_n \) can be decomposed into the edge-disjoint stars

\[
\underbrace{S_4, S_4, \ldots, S_4}_{r\text{-copies}}, S_1
\]

Take four copies of \( K_n \). Consider the centers of \( S_1 \) in each copy of \( K_n \) in the decomposition. Identify these four vertices from each copy of \( K_n \) to get the required graph \( G = \left(4\binom{n}{2}, S_4, K_n\right)\). By the construction itself \( G \) is \( K_n \)-decomposable. \( G \) is \( S_4 \)-decomposable, since \( G \) can be decomposed into \( 4r + 1 \) edge-disjoint copies of \( S_4 \). The identified vertex can be considered as the center of one copy of \( S_4 \). Then there exists a \((k\binom{n}{2}, S_4, K_n)\) graph for all \( k \equiv 0 \pmod{4} \).

**Case 3:** \( n \equiv 3, 6 \pmod{4} \).

Let \( \binom{n}{2} = 4r + 3 \), where \( r > 0 \). By Theorem 4.2, \( K_n \) can be decomposed into the edge-disjoint stars

\[
\underbrace{S_4, S_4, \ldots, S_4}_{r\text{-copies}}, S_3
\]

Take four copies of \( K_n \). Consider the centers of \( S_3 \) in each copy of \( K_n \) in the decomposition. Identify these four vertices from each copy of \( K_n \) to get the required graph \( G = \left(4\binom{n}{2}, S_4, K_n\right)\). By the construction itself \( G \) is \( K_n \)-decomposable. \( G \) is \( S_4 \)-decomposable, since \( G \) can be decomposed into \( 4r + 3 \) edge-disjoint copies
of $S_4$. The identified vertex can be considered as the center of three copies of $S_4$. Then there exists a \((k, S_4, K_n)\) for all \(k \equiv 0 \pmod{4}\).

Lemmas 5.6 and 5.7 allow us to construct all of the \((q, S_4, K_n)\) graphs that we require for the sufficient conditions (2), (3), (4) and (5) of Theorem 5.1.

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