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SOLUTION OF FRACTIONAL KINETIC EQUATION FOR HADAMARD TYPE FRACTIONAL INTEGRAL VIA MELLIN TRANSFORM

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Abstract. The aim of this paper is to introduce the generalized form of the fractional kinetic equation including Hadamard type fractional integral. We also present the solution of Kinetic equation including Hadamard type fractional integral with the help of generalized $k-$Wright function and Mellin transform. We also present the solution of Kinetic equation including Hadamard type fractional integral with the help of generalized Multindex Bessel Function and Mellin transform.

1. Introduction

The importance of fractional differential equations in the field of applied science has gained more attention not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of many physical phenomena [1–5, 7, 8, 13, 18–20]. The kinetic equations particularly describe the continuity of the substance's motion. The extension and generalization of kinetic fractional equations which involve many fractional operators was found in the value of fractional differential equations has grown more interest in applied science not only in mathematics but also in physics, dynamical systems, control systems and engineering in order to create many physical phenomena as a mathematical model [10–12, 16, 17, 21, 27].

The production and destruction of nuclei may be represented by reaction-type (kinetic) equations in thermonuclear reactions. The distribution functions of a single particle's dynamic states is determined by solutions of such reaction-type, linear and nonlinear equations. The solution of a linear kinetic equation, representing small deviations from the equilibrium solution of the nonlinear kinetic equation, is investigated more thoroughly. If we consider an arbitrary reaction characterized by a time dependent quantity $N = N(t)$. It is not impossible to equate the rate of change to a balance between the production rate p and destruction rate d which is dependent directly on the quantity N on account of

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interaction mechanisms [22, 24–26]. Hence, $d = d(N)$ and $p = p(N)$. This dependence is complicated since the destruction or production at time t depends not only on $N(t)$ but also on the past history $N(t_1),\ t_1 < t$, of the variable N .

In 2008, Saxena and Kalla [23] have considered the fractional kinetic equation involving Riemann-Liouville fractional integral of the form

$$N(t) - N_0\ \phi(t) = -\delta^\gamma\ {}_{RL}I_t^\gamma N(t), \tag{1.1}$$

where ${}_{RL}I_t^\gamma$ is the Riemann-Liouville fractional integral operator. By applying Laplace transform method, solutions of (1.1) were obtained.

In 2019, M. Chand et al. [4] have obtained the solution of the following fractional kinetic equations involving product of generalized $k-$Wright function

$$N(t) - N_0\left(\prod_{\mu=1}^{l}\ {}_{n_\mu}\psi_{m_\mu}^{k_\mu}\left[\begin{matrix}(p_{\mu i},\alpha_{\mu i})_{1,n_\mu}\\(q_{\mu j},\beta_{\mu j})_{1,m_\mu}\end{matrix}\middle| c^\gamma t^\gamma\right]\right) = -\delta^\gamma\ {}_{RL}I_t^\gamma N(t). \tag{1.2}$$

In this paper, we establish the following form of fractional kinetic equation including Hadamard type fractional integral

$$N(t) - N_0\phi(t) = -\delta^\gamma\ {}_{H}I_t^\gamma N(t), \tag{1.3}$$

where ${}_{H}I_t^\gamma$ is the Hadamard type fractional integral operator.

This paper is presented as below: Section 2 contains some preliminaries. In Section 3, we discuss the solution of equation (1.3) by using Mellin transform. In Section 4, we obtain the solution of kinetic equation including Hadamard type fractional integral with $k-$Wright function. In Section 5, we obtain the solution of kinetic equation including Hadamard type fractional integral with generalized Multindex Bessel function.

2. Preliminaries

In this section, we introduce some notations, definitions and theorems which are used throughout this paper.

Definition 2.1. [14] Let $x \in \mathbb{R}^+$ be a real variable, the mellin transform of the function $\phi(x)$ is defined as

$$\mathcal{M}\{\phi(x)\}(\tau) = \int_0^\infty x^{\tau-1}\phi(x)dx, \quad (\tau \in \mathbb{C})$$

and the inverse mellin transform

$$\mathcal{M}^{-1}\{\phi(\tau)\}(x) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} x^{-\tau}\phi(\tau)d\tau, \quad (\varepsilon = \Re(\tau)).$$

In addition, let $\theta(x)$ be a function, the mellin convolution of two functions $\theta(x)$ and $\phi(x)$ is defined as

$$\theta * \phi = (\theta * \phi)(t) := \int_0^t \theta(\frac{t}{x})\phi(x)\frac{dx}{x}.$$

Definition 2.2. [14] Let $\Re(\gamma) > 0$, the left-sided and right-sided Hadamard type fractional integrals of order $\gamma \in \mathbb{C}$ defined as

$$({}_H I^{\gamma}_{0+}\phi)(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (\log \frac{t}{s})^{\gamma-1} \quad \frac{\phi(s)}{s} ds \qquad (t > 0)$$

and

$$({}_H I^{\gamma}_{-}\phi)(t) := \frac{1}{\Gamma(\gamma)} \int_t^{\infty} (\log \frac{s}{t})^{\gamma-1} \quad \frac{\phi(s)}{s} ds \qquad (t > 0),$$

respectively.

Lemma 2.3. *[14] Let $\Re(\gamma) > 0$, $\tau \in \mathbb{C}$ the mellin transform $(\mathcal{M}\phi)(\tau)$ exists for a function $\phi(x)$.*

(a) If $\Re(\tau) < 0$, then

$$(\mathcal{M}\ {}_H I^{\gamma}_{0+}\phi)(\tau) = (-\tau)^{-\gamma}(\mathcal{M}\phi)(\tau).$$

(b) If $\Re(\tau) > 0$, then

$$(\mathcal{M}\ {}_H I^{\gamma}_{-}\phi)(\tau) = (\tau)^{-\gamma}(\mathcal{M}\phi)(\tau).$$

Theorem 2.4. *[15] Let $x \in [0, \xi]$, the Mellin integral transform of a function $\phi(x)$ in $[0, \xi]$ is*

$$\mathcal{M}\big[\phi(x), \tau, 0, \xi\big] = \Phi(\tau) := \int_0^{\xi} \xi^{-\tau} x^{\tau-1} \phi(x) dx.$$

Then,

$$\phi(x) = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{x^{-\tau}}{\tau} \mathcal{M}\big[\phi(x), \tau, 0, c\big] d\tau.$$

Definition 2.5. [9] For $k \in \mathbb{R}^+$; $z \in \mathbb{C}$; $p_i, q_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$ $(\alpha_i,\ \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ and $(p_i + \alpha_i r)$, $(q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, the generalized $k-$Wright function ${}_n\psi^k_m$ is defined by

$${}_n\psi^k_m(z) = \ {}_n\psi^k_m \left[\begin{matrix} (p_i, \alpha_i)_{1,n} \\ (q_j, \beta_j)_{1,m} \end{matrix} \middle| z \right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n}\ \Gamma_k(p_i + \alpha_i r)}{\prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r)} \frac{z^r}{r!},$$

where $k-$Gamma function defined as

$$\Gamma_k(y) = \int_0^{\infty} t^{y-1} e^{-\frac{t^k}{k}} dt := k^{\frac{y}{k}-1} \Gamma(\frac{y}{k}) \quad (\Re(y) > 0).$$

Definition 2.6. [6] The generalized Multindex Bessel function $J^{(\xi_j)_m,\gamma}_{(\zeta_j)_m,q}[z]$ is defined as the form

$$J^{\ (\xi_j)_m,\gamma}_{\ (\zeta_j)_m,q}[z] = \sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^{m} \Gamma(\xi_j r + \zeta_j + 1)} \frac{(-z)^r}{(r!)} \quad (m \in \mathbb{N}) \qquad (2.1)$$

for $\xi_j,\ \zeta_j,\ q,\ z \in \mathbb{C}$, $(j = 1, 2, ..., m)$ such that

$$\sum_{j=1}^{m} \Re(\xi_j) > \max\{0,\ \Re(q)-1\},\ q > 0,\ \Re(\zeta_j) > -1,\ \Re(\gamma) > 0\ and\ q \in (0, 1) \cup \mathbb{N}.$$

Next, we introduce the generalized form of the fractional kinetic equation including Hadamard type fractional integral.

3. Kinetic Equation Including Hadamard type Fractional Integral

Given the following equation

$$N(t) - N_0\phi(t) = -\delta^\gamma \ {}_HI_t^\gamma N(t). \tag{3.1}$$

So, from definition 2.1 and Lemma 2.3, it follows that

$$\mathcal{M}\{N(t)\} - N_0\Phi(\tau) = -\delta^\gamma(\tau)^{-\gamma}\mathcal{M}\{N(t)\},$$

that is

$$\mathcal{M}\{N(t)\} = \frac{N_0}{1+\delta^\gamma(\tau)^{-\gamma}}\Phi(\tau), \tag{3.2}$$

provided that $\tau > \delta$, we can write previous equation as the following:

$$\mathcal{M}\{N(t)\} = N_0\sum_{\mu=0}^{\infty}(-1)^\mu\big(\frac{\delta}{\tau}\big)^{\mu\gamma}\Phi(\tau), \quad where \quad |\frac{\delta}{\tau}| < 1. \tag{3.3}$$

Now, applying the inverse Mellin transform, we obtain

$$N(t) = N_0\sum_{\mu=0}^{\infty}(-1)^\mu\delta^{\mu\gamma}\Gamma(1-\mu\gamma)\,\big(\log t\big)^{\mu\gamma-1} * \phi(t), \tag{3.4}$$

where

$$\big(\log t\big)^{\mu\gamma-1} * \phi(t) = \int_1^t \big(\log\frac{t}{x}\big)^{\mu\gamma-1}\phi(x)\frac{dx}{x}. \tag{3.5}$$

Special case : If $\phi(t) \equiv 1$, then the convolution integral becomes

$$\big(\log t\big)^{\mu\gamma-1} * 1 = \int_1^t \big(\log\frac{t}{x}\big)^{\mu\gamma-1}\frac{dx}{x}. \tag{3.6}$$

Then, equation (3.4) becomes

$$N(t) = N_0\sum_{\mu=0}^{\infty}(-1)^\mu\frac{[(\delta\log t)^\gamma]^\mu}{\Gamma(\gamma\mu+1)}\ \frac{\pi}{\sin(\pi\mu\gamma)}. \tag{3.7}$$

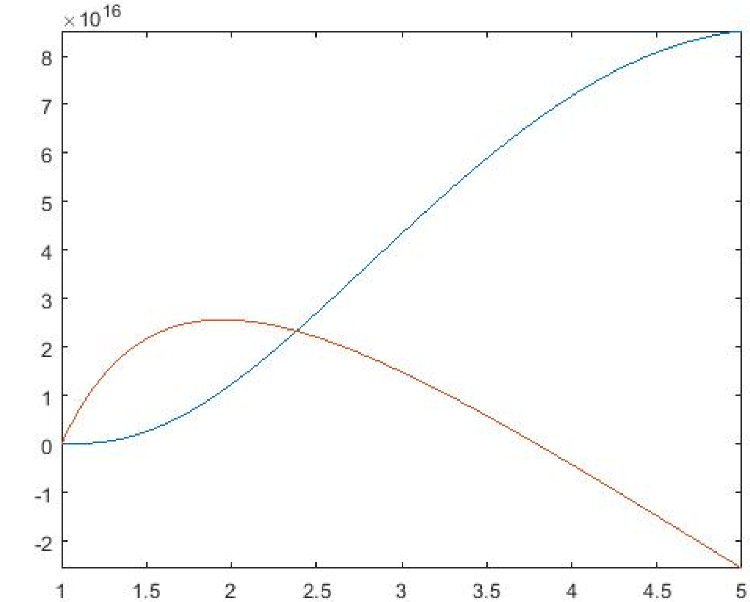

FIGURE 1. Solution (3.7) for $N_0 = 1, \delta = 3, \gamma = \frac{1}{2}, \frac{3}{2}$

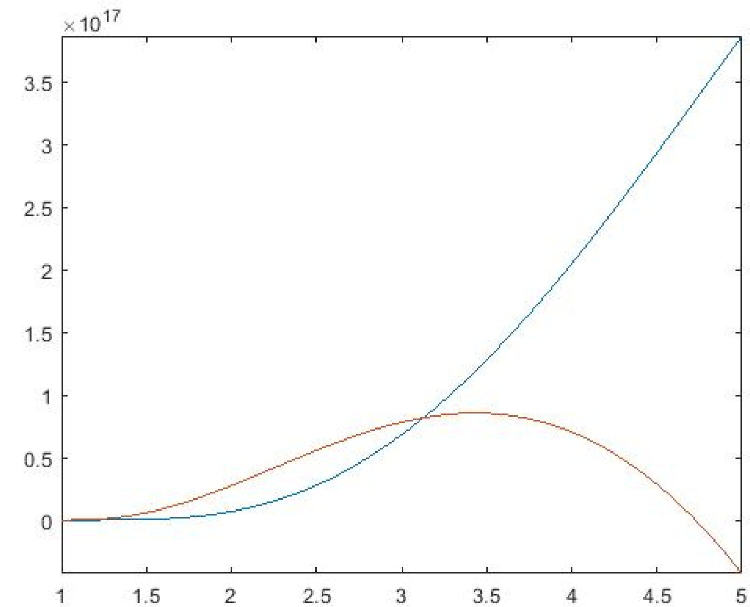

FIGURE 2. Solution (3.7) for $N_0 = 1, \delta = 4, \gamma = \frac{3}{2}, \frac{5}{2}$

Next, we investigate solutions of a generalized Hadamard type fractional Kinetic equations involving a generalized $k-$Wright function.

4. Solution Of Kinetic Equation Including Hadamard type Fractional Integral with $k-$Wright Function

Theorem 4.1. *If $\delta > 0, c > 0, \Re(\gamma) > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ and $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, then the solution of the equation*

$$N(t) - N_0\left(\prod_{\mu=1}^{l} {}_{n_\mu}\psi_{m_\mu}^{k_\mu}\left[\begin{matrix}(p_{\mu i}, \alpha_{\mu i})_{1,n_\mu}\\(q_{\mu j}, \beta_{\mu j})_{1,m_\mu}\end{matrix}\middle| c^\gamma t^\gamma\right]\right) = -\delta^\gamma\ {}_H I_t^\gamma N(t) \tag{4.1}$$

given by the following formula

$$N(t) = N_0\left(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\Gamma_{k_\mu}(p_{\mu i} + \alpha_{\mu i} r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j} + \beta_{\mu j} r)}\right)\left(\frac{c^{r\gamma}\xi^{r\gamma}}{r!}\right)^\mu log(t)$$

$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\left[-\left[\log(t)^\delta\right]^\gamma\right]^\nu\left[\log(t)^{-\gamma r\mu}\right]^l\Gamma[1 - (\gamma\nu + l + 2)]. \tag{4.2}$$

Proof. The Mellin transform of Hadamard fractional integral operator $\left(\mathcal{M}\ {}_H I_-^\gamma \phi\right)$ is given as

$$(\mathcal{M}\ {}_H I_-^\gamma \phi)(\tau) = (\tau)^{-\gamma}\Phi(\tau),$$

where $\Phi(\tau) = \int_0^\xi \xi^{-\tau} t^{\tau-1}\phi(t)dt$.
Now, applying the Mellin transform on (4.1), gives

$$\mathcal{M}\Big(N(t);\tau\Big) = N_0\mathcal{M}\left(\prod_{\mu=1}^{l} {}_{n_\mu}\psi_{m_\mu}^{k_\mu}\left[\begin{matrix}(p_{\mu i}, \alpha_{\mu i})_{1,n_\mu}\\(q_{\mu j}, \beta_{\mu j})_{1,m_\mu}\end{matrix}\middle| c^\gamma t^\gamma;\tau\right]\right) - \delta^\gamma\mathcal{M}\Big({}_H I_t^\gamma N(t);\tau\Big),$$

that is

$$N(\tau) = N_0\int_0^\xi \xi^{-\tau} t^{\tau-1}\left(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\Gamma_{k_\mu}(p_{\mu i} + \alpha_{\mu i} r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j} + \beta_{\mu j} r)}\right)\left(\frac{c^{r\gamma}t^{r\gamma}}{r!}\right)^\mu dt - \delta^\gamma(\tau)^{-\gamma}N(\tau). \tag{4.3}$$

By interchanging the order of integration and summation in the equation (4.3), we obtain

$$\begin{aligned}N(\tau)\left[1 + \delta^\gamma(\tau)^{-\gamma}\right] &= N_0\left(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\Gamma_{k_\mu}(p_{\mu i} + \alpha_{\mu i} r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j} + \beta_{\mu j} r)}\right)\int_0^\xi \xi^{-\tau} t^{\tau-1}\left(\frac{c^{r\gamma}t^{r\gamma}}{r!}\right)^\mu dt\\ &= N_0\left(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\mu}(p_{\mu i} + \alpha_{\mu i} r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j} + \beta_{\mu j} r)}\right)\frac{(c^{\gamma r}\xi^{\gamma r})^\mu}{(\tau + \gamma r\mu)(r!)^\mu}.\end{aligned} \tag{4.4}$$

Equation (4.4) leads to

$$N(\tau) = N_0\left(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\Gamma_{k_\mu}(p_{\mu i} + \alpha_{\mu i} r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j} + \beta_{\mu j} r)}\right)\frac{(c^{\gamma r}\xi^{\gamma r})^\mu}{(\tau + \gamma r\mu)(r!)^\mu}\sum_{\nu=0}^{\infty}(-1)^\nu\left(\frac{\delta}{\tau}\right)^{\gamma\nu}. \tag{4.5}$$

Now, taking Mellin inverse of the equation (4.5), we have

$$N(t)=N_0\Bigg(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\ \Gamma_{k_\mu}(p_{\mu i}+\alpha_{\mu i}r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j}+\beta_{\mu j}r)}\Bigg)\Bigg(\frac{c^{r\gamma}\xi^{r\gamma}}{r!}\Bigg)^{\mu}\sum_{\nu=0}^{\infty}(-1)^{\nu}\delta^{\gamma\nu}\mathcal{M}^{-1}\Bigg(\frac{\tau^{-\gamma\nu}}{\tau+\gamma r\mu}\Bigg). \tag{4.6}$$

Since

$$\begin{aligned}\mathcal{M}^{-1}\Bigg(\frac{\tau^{-\gamma\nu}}{\tau+\gamma r\mu}\Bigg)&=\int_0^\infty\frac{t^{-\tau}}{\tau}\frac{\tau^{-\gamma\nu}}{\tau+\gamma r\mu}d\tau\\&=\sum_{l=0}^{\infty}\big(-\gamma r\mu\big)^{l}\int_0^\infty t^{-\tau}\tau^{-(\gamma\nu+l+2)}d\tau\\&=\sum_{l=0}^{\infty}\big(-\gamma r\mu\big)^{l}\big(\log t\big)^{\gamma\nu+l+1}\Gamma\big[1-(\gamma\nu+l+2)\big].\end{aligned}$$

Then, equation (4.6) can be written as

$$\begin{aligned}N(t)=N_0\Bigg(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\ \Gamma_{k_\mu}(p_{\mu i}+\alpha_{\mu i}r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j}+\beta_{\mu j}r)}\Bigg)\Bigg(\frac{c^{r\gamma}\xi^{r\gamma}}{r!}\Bigg)^{\mu}log(t)\\\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\big[-\big[\log(t)^{\delta}\big]^{\gamma}\big]^{\nu}\big[\log(t)^{-\gamma r\mu}\big]^{l}\Gamma[1-(\gamma\nu+l+2)].\end{aligned}$$

□

Theorem 4.2. *If $c>0, k\in\mathbb{R}^+;\Re(\gamma)>0,\ c,z\in\mathbb{C};\ p_i,q_j\in\mathbb{C},\ \alpha_i,\beta_j\in\mathbb{R}\ (\alpha_i,\beta_j\neq 0;\ i=1,2,...,n;\ j=1,2,...,m)$ and $(p_i+\alpha_i r),\ (q_j+\beta_j r)\in\mathbb{C}\setminus k\mathbb{Z}^-$, then the solution of the equation*

$$N(t)-N_0\Bigg(\prod_{\mu=1}^{l}\ {}_{n_\mu}\psi_{m_\mu}^{k_\mu}\Bigg[\begin{matrix}(p_{\mu i},\alpha_{\mu i})_{1,n_\mu}\\(q_{\mu j},\beta_{\mu j})_{1,m_\mu}\end{matrix}\Bigg|c^{\gamma}t^{\gamma}\Bigg]\Bigg)=-c^{\gamma}\ {}_{H}I_t^{\gamma}N(t) \tag{4.7}$$

given by the following formula

$$N(t)=N_0\Bigg(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\ \Gamma_{k_\mu}(p_{\mu i}+\alpha_{\mu i}r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j}+\beta_{\mu j}r)}\Bigg)\Bigg(\frac{c^{r\gamma}\xi^{r\gamma}}{r!}\Bigg)^{\mu}log(t)$$

$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\big[-\big[\log(t)^{c}\big]^{\gamma}\big]^{\nu}\big[\log(t)^{-\gamma r\mu}\big]^{l}\Gamma[1-(\gamma\nu+l+2)]. \tag{4.8}$$

Theorem 4.3. *If $c>0, k\in\mathbb{R}^+;\Re(\gamma)>0,\ c,z\in\mathbb{C};\ p_i,q_j\in\mathbb{C},\ \alpha_i,\beta_j\in\mathbb{R}\ (\alpha_i,\beta_j\neq 0;\ i=1,2,...,n;\ j=1,2,...,m)$ and $(p_i+\alpha_i r),\ (q_j+\beta_j r)\in\mathbb{C}\setminus k\mathbb{Z}^-$, then the solution of the equation*

$$N(t)-N_0\Bigg(\prod_{\mu=1}^{l}\ {}_{n_\mu}\psi_{m_\mu}^{k_\mu}\Bigg[\begin{matrix}(p_{\mu i},\alpha_{\mu i})_{1,n_\mu}\\(q_{\mu j},\beta_{\mu j})_{1,m_\mu}\end{matrix}\Bigg|t\Bigg]\Bigg)=-c^{\gamma}\ {}_{H}I_t^{\gamma}N(t) \tag{4.9}$$

given by the following formula

$$N(t) = N_0\left(\prod_{\mu=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\mu}\Gamma_{k_\mu}(p_{\mu i}+\alpha_{\mu i}r)}{\prod_{j=1}^{m_\mu}\Gamma_{k_\mu}(q_{\mu j}+\beta_{\mu j}r)}\right)\left(\frac{\xi^r}{r!}\right)^{\mu} log(t)$$
$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\left[-\left[\log(t)^c\right]^{\gamma}\right]^{\nu}\left[\log(t)^{-r\mu}\right]^{l}\Gamma[1-(\gamma\nu+l+2)]. \tag{4.10}$$

Proof. Proof of Theorem 4.2 and 4.3 are similar to the proof of Theorem 4.1. So, it is omitted here. □

From previous Theorems 4.1, 4.2 and 4.3 if we choose $\mu = 1$, then we have the following corollaries:

Corollary 4.4. *If $\delta > 0, c > 0, \Re(\gamma) > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ and $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C}\setminus k\mathbb{Z}^-$, then the solution of the equation*

$$N(t) - N_0\left({}_n\psi_m^k\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle| c^{\gamma}t^{\gamma}\right]\right) = -\delta^{\gamma}\ {}_HI_t^{\gamma}N(t) \tag{4.11}$$

given by the following formula

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\right)\left(\frac{c^{r\gamma}\xi^{r\gamma}}{r!}\right) log(t)$$
$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\left[-\left[\log(t)^{\delta}\right]^{\gamma}\right]^{\nu}\left[\log(t)^{-\gamma r}\right]^{l}\Gamma[1-(\gamma\nu+l+2)]. \tag{4.12}$$

Corollary 4.5. *If $\delta > 0, c > 0, \Re(\gamma) > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ and $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C}\setminus k\mathbb{Z}^-$, then the solution of the equation*

$$N(t) - N_0\left({}_n\psi_m^k\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle| c^{\gamma}t^{\gamma}\right]\right) = -c^{\gamma}\ {}_HI_t^{\gamma}N(t) \tag{4.13}$$

given by the following formula

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\right)\left(\frac{c^{r\gamma}\xi^{r\gamma}}{r!}\right) log(t)$$
$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\left[-\left[\log(t)^{c}\right]^{\gamma}\right]^{\nu}\left[\log(t)^{-\gamma r}\right]^{l}\Gamma[1-(\gamma\nu+l+2)]. \tag{4.14}$$

Corollary 4.6. *If $\delta > 0, c > 0, k \in \mathbb{R}^+; \Re(\gamma) > 0,\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ and $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C}\setminus k\mathbb{Z}^-$, then the solution of the equation*

$$N(t) - N_0\left({}_n\psi_m^k\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle| t\right]\right) = -c^{\gamma}\ {}_HI_t^{\gamma}N(t) \tag{4.15}$$

given by the following formula:

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\ \Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\right)\left(\frac{\xi^r}{r!}\right)log(t)$$

$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\big[-\big[\log(t)^c\big]^{\gamma}\big]^{\nu}\big[\log(t)^{-r}\big]^{l}\Gamma[1-(\gamma\nu+l+2)]. \tag{4.16}$$

Next, we investigate solutions of a generalized Hadamard type fractional Kinetic equations involving a generalized Multindex Bessel function.

5. Solution Of Kinetic Equation Including Hadamard type Fractional Integral with Generalized Multindex Bessel Function

Theorem 5.1. *If* $\delta > 0\ , c > 0, \gamma > 0; \delta \neq c\ , m \in \mathbb{N}, \xi_j, \zeta_j, \gamma, q \in \mathbb{C}, (j = 1, 2, ..., m)$ *such that* $\Re(\gamma) > 0$ *and* $\sum_{j=1}^{m}\Re(\xi_j) > \max\ \{0,\ \Re(q)-1\},\ q > 0, \Re(\zeta_j) > -1$ *and* $q \in (0,1)\cup\mathbb{N}$ *then the solution of the equation*

$$N(t) - N_0 \mathrm{J}\ _{(\zeta_j)_m, q}^{(\xi_j)_m,\gamma}(c^{\gamma}t^{\gamma}) = -\delta^{\gamma}\ {}_H I_t^{\gamma} N(t) \tag{5.1}$$

given by the following formula

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\left(\frac{(-c^{\gamma}\xi^{\gamma})^r}{(r!)}\right)log(t)$$

$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\big[-\big[\log(t)^{\delta}\big]^{\gamma}\big]^{\nu}\big[\log(t)^{-\gamma r}\big]^{l}\Gamma[1-(\gamma\nu+l+2)]. \tag{5.2}$$

Proof. The Mellin transform of Hadamard fractional integral operator $\big(\mathcal{M}\ {}_H I_{-}^{\gamma}\phi\big)$ is given as

$$(\mathcal{M}\ {}_H I_{-}^{\gamma}\phi)(\tau) = (\tau)^{-\gamma}\Phi(\tau),$$

where $\Phi(\tau) = \int_0^{\xi}\xi^{-\tau}t^{\tau-1}\phi(t)dt$.
Now, applying the Mellin transform on (5.1), gives

$$\mathcal{M}\bigg(N(t);\tau\bigg) = N_0\mathcal{M}\bigg(J\ _{(\zeta_j)_m,q}^{(\xi_j)_m,\gamma}(c^{\gamma}t^{\gamma});\tau\bigg) - \delta^{\gamma}\mathcal{M}\bigg({}_H I_t^{\gamma}N(t);\tau\bigg),$$

that is

$$N(\tau) = N_0\int_0^{\xi}\xi^{-\tau}t^{\tau-1}\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\left(\frac{(-c^{\gamma}t^{\gamma})^r}{r!}\right)dt - \delta^{\gamma}(\tau)^{-\gamma}N(\tau). \tag{5.3}$$

By interchanging the order of integration and summation in the equation (5.3), we obtain

$$N(\tau)\big[1+\delta^{\gamma}(\tau)^{-\gamma}\big] = N_0\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\int_0^{\xi}\xi^{-\tau}t^{\tau-1}\left(\frac{-(c^{\gamma}t^{\gamma})^r}{r!}\right)dt$$
$$= N_0\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\frac{(-c^{\gamma}\xi^{\gamma})^r}{(\tau+\gamma r)(r!)}. \quad (5.4)$$

Equation (5.4) leads to

$$N(\tau) = N_0\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\frac{(-c^{\gamma}\xi^{\gamma})^r}{(\tau+\gamma r)(r!)}\sum_{\nu=0}^{\infty}(-1)^{\nu}\left(\frac{\delta}{\tau}\right)^{\gamma\nu}. \quad (5.5)$$

Now, taking Mellin inverse of the equation (5.5), we have

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\frac{(-c^{\gamma}\xi^{\gamma})^r}{(r!)}\sum_{\nu=0}^{\infty}(-1)^{\nu}\delta^{\gamma\nu}\mathcal{M}^{-1}\left(\frac{\tau^{-\gamma\nu}}{\tau+\gamma r}\right). \quad (5.6)$$

Since

$$\mathcal{M}^{-1}\left(\frac{\tau^{-\gamma\nu}}{\tau+\gamma r}\right) = \int_0^{\infty}\frac{t^{-\tau}}{\tau}\frac{\tau^{-\gamma\nu}}{\tau+\gamma r}d\tau$$
$$= \sum_{l=0}^{\infty}\big(-\gamma r\big)^l\int_0^{\infty}t^{-\tau}\tau^{-(\gamma\nu+l+2)}d\tau$$
$$= \sum_{l=0}^{\infty}\big(-\gamma r\big)^l\big(\log t\big)^{\gamma\nu+l+1}\Gamma\big[1-(\gamma\nu+l+2)\big].$$

Then, equation (5.6) can be written as

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\left(\frac{(-c^{\gamma}\xi^{\gamma})^r}{(r!)}\right)log(t)$$
$$\times\sum_{\nu=0}^{\infty}\sum_{l=0}^{\infty}\big[-\big[\log(t)^{\delta}\big]^{\gamma}\big]^{\nu}\big[\log(t)^{-\gamma r}\big]^{l}\Gamma[1-(\gamma\nu+l+2)].$$

□

Theorem 5.2. *If* $c>0, \gamma>0$, $m\in\mathbb{N}, \xi_j, \zeta_j, \gamma, q\in\mathbb{C}, (j=1,2,...,m)$ *such that* $\Re(\gamma)>0$ *and* $\sum_{j=1}^{m}\Re(\xi_j)>\max\{0,\ \Re(q)-1\}$, $q>0, \Re(\zeta_j)>-1$ *and* $q\in(0,1)\cup\mathbb{N}$ *then the solution of the equation*

$$N(t)-N_0\mathrm{J}\,{}^{(\xi_j)_m,\gamma}_{(\zeta_j)_m,q}(c^{\gamma}t^{\gamma}) = -c^{\gamma}\ {}_{H}I_t^{\gamma}N(t) \quad (5.7)$$

given by the following formula

$$N(t) = N_0\left(\sum_{r=0}^{\infty}\frac{(\gamma)_{qr}}{\prod_{j=1}^{m}\Gamma(\xi_j r+\zeta_j+1)}\right)\left(\frac{(-c^{\gamma}\xi^{\gamma})^r}{(r!)}\right)log(t)$$

$$\times \sum_{\nu=0}^{\infty} \sum_{l=0}^{\infty} \left[- \left[\log(t)^{c} \right]^{\gamma} \right]^{\nu} \left[\log(t)^{-\gamma r} \right]^{l} \Gamma[1 - (\gamma\nu + l + 2)]. \tag{5.8}$$

Theorem 5.3. *If* $c > 0, \gamma > 0$ $, m \in \mathbb{N}, \xi_j, \zeta_j, \gamma, q \in \mathbb{C}, (j = 1, 2, ..., m)$ *such that* $\Re(\gamma) > 0$ *and* $\sum_{j=1}^{m} \Re(\xi_j) > \max\{0,\ \Re(q) - 1\},\ q > 0, \Re(\zeta_j) > -1$ *and* $q \in (0, 1) \cup \mathbb{N}$ *then the solution of the equation*

$$N(t) - N_0 \mathrm{J}\ {}^{(\xi_j)_m,\gamma}_{(\zeta_j)_m,q}(c^{\gamma} t^{\gamma}) = -c^{\gamma}\ {}_{H}I_{t}^{\gamma} N(t) \tag{5.9}$$

given by the following formula

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\prod_{j=1}^{m} \Gamma(\xi_j r + \zeta_j + 1)} \right) \left(\frac{(-\xi)^r}{(r!)} \right) log(t)$$

$$\times \sum_{\nu=0}^{\infty} \sum_{l=0}^{\infty} \left[- \left[\log(t)^{c} \right]^{\gamma} \right]^{\nu} \left[\log(t)^{-\gamma r} \right]^{l} \Gamma[1 - (\gamma\nu + l + 2)]. \tag{5.10}$$

Proof. Proof of Theorems 5.2 and 5.3 are similar to the proof of Theorem 5.1. So, it is omitted here. □

From previous Theorems 5.1,5.2and 5.3 if we choose $m = 1$, then we have the following corollaries:

Corollary 5.4. *If* $\delta > 0$ $, c > 0, \gamma > 0; \delta \neq c$ $, m \in \mathbb{N}, \xi, \zeta, \gamma, q \in \mathbb{C}$ *such that* $\Re(\gamma) > 0$ *and* $\Re(\xi) > \max\{0,\ \Re(q) - 1\},\ q > 0, \Re(\zeta) > -1$ *and* $q \in (0, 1) \cup \mathbb{N}$ *then the solution of the equation*

$$N(t) - N_0 \mathrm{J}\ {}^{(\xi),\gamma}_{(\zeta),q}(c^{\gamma} t^{\gamma}) = -\delta^{\gamma}\ {}_{H}I_{t}^{\gamma} N(t) \tag{5.11}$$

given by the following formula

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\Gamma(\xi r + \zeta + 1)} \right) \left(\frac{(-c^{\gamma}\xi^{\gamma})^r}{(r!)} \right) log(t)$$

$$\times \sum_{\nu=0}^{\infty} \sum_{l=0}^{\infty} \left[- \left[\log(t)^{\delta} \right]^{\gamma} \right]^{\nu} \left[\log(t)^{-\gamma r} \right]^{l} \Gamma[1 - (\gamma\nu + l + 2)]. \tag{5.12}$$

Corollary 5.5. *If* $c > 0, \gamma > 0; \delta \neq c$ $, m \in \mathbb{N}, \xi, \zeta, \gamma, q \in \mathbb{C}$ *such that* $\Re(\gamma) > 0$ *and* $\Re(\xi) > \max\{0,\ \Re(q) - 1\},\ q > 0, \Re(\zeta) > -1$ *and* $q \in (0, 1) \cup \mathbb{N}$ *then the solution of the equation*

$$N(t) - N_0 \mathrm{J}\ {}^{(\xi),\gamma}_{(\zeta),q}(c^{\gamma} t^{\gamma}) = -c^{\gamma}\ {}_{H}I_{t}^{\gamma} N(t) \tag{5.13}$$

given by the following formula

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\Gamma(\xi r + \zeta + 1)} \right) \left(\frac{(-c^{\gamma}\xi^{\gamma})^r}{(r!)} \right) log(t)$$

$$\times \sum_{\nu=0}^{\infty} \sum_{l=0}^{\infty} \left[- \left[\log(t)^{c} \right]^{\gamma} \right]^{\nu} \left[\log(t)^{-\gamma r} \right]^{l} \Gamma[1 - (\gamma\nu + l + 2)]. \tag{5.14}$$

Corollary 5.6. *If $c > 0, \gamma > 0; \delta \neq c$, $m \in \mathbb{N}, \xi, \zeta, \gamma, q \in \mathbb{C}$ such that $\Re(\gamma) > 0$ and $\Re(\xi) > \max\{0,\ \Re(q) - 1\}$, $q > 0, \Re(\zeta) > -1$ and $q \in (0,1) \cup \mathbb{N}$ then the solution of the equation*

$$N(t) - N_0 \mathrm{J}\ ^{(\xi),\gamma}_{(\zeta),q}(t) = -c^{\gamma}\ _{H}I_t^{\gamma} N(t) \tag{5.15}$$

given by the following formula

$$N(t) = N_0 \left(\sum_{r=0}^{\infty} \frac{(\gamma)_{qr}}{\Gamma(\xi r + \zeta + 1)} \right) \left(\frac{(-\xi)^r}{(r!)} \right) log(t)$$
$$\times \sum_{\nu=0}^{\infty} \sum_{l=0}^{\infty} \left[- \left[\log(t)^c \right]^{\gamma} \right]^{\nu} \left[\log(t)^{-\gamma r} \right]^{l} \Gamma[1 - (\gamma\nu + l + 2)]. \tag{5.16}$$

6. Conclusion

In this paper, we introduced the generalized form of fractional Kinetic equation including fractional integral of Hadamard type. Also, we presented the Kinetic equation solution including the Hadamard type fractional integral by using generalized $k-$Wright function and Mellin transform as well as the Kinetic equation solution including the Hadamard type fractional integral by using generalized Multindex Bessel function and Mellin transform. In fact, the findings obtained here are very capable of generating a quite large number of established and presumably new results.

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