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ON THE DIFFERENCE OF ZAGREB COINDICES OF GRAPH OPERATIONS

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Abstract. Recently the differences of Zagreb indices are reported in [8]. The Zagreb coindices are defined as $\overline{M_1}(G) = \sum_{uv \notin E(G)} [d(u) + d(v)]$ and $\overline{M_2}(G) = \sum_{uv \notin E(G)} d(u)d(v)$, where $d(u)$ is the degree of the vertex u of G. So far, the difference $\overline{M_1}$ and $\overline{M_2}$ has not been studied. We show that this difference is closely related to the vertex degree-based invariant $\overline{RM_2} = \sum_{uv \notin E(G)} (d(u)-1)(d(v)-1)$, and explore their basic mathematical properties and present explicit formulas for this new invariant under several graph operations.

1. Introduction and preliminaries

Let $G = (V, E)$ be a graph. The number of vertices of G is denoted by n and the number of edges is denoted by m, thus $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex v is denoted by $d_G(v)$.

The complement of a graph G, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are connected by an edge uv, if and only if they are not adjacent in G. Obviously, $E(G) \cup E(\overline{G}) = E(K_n)$, and $\overline{m} = |E(\overline{G})| = \frac{n(n-1)}{2} - m$. For undefined terminologies we refer the reader to [11].

There are two invariants called the first Zagreb index and second Zagreb index [3, 4, 5, 6, 7, 9, 10, 17, 18, 12, 13, 14, 15] defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \qquad \text{and} \qquad M_2(G) = \sum_{uv \in E(G)} d_G(u)\, d_G(v),$$

respectively.

In fact, one can rewrite the first Zagreb index as

$$M_1(G) = \sum_{uv \in E(G)} \big[d_G(u) + d_G(v)\big].$$

Notice that contribution of nonadjacent vertex pairs should be taken into account when computing the weighted Winer polynomials of certain composite

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graphs (see [1]) defined first Zagreb coindex and second Zagreb coindex as

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} \left[d_G(u) + d_G(v)\right] \qquad \text{and} \qquad \overline{M_2}(G) = \sum_{uv \notin E(G)} d_G(u)\, d_G(v),$$

respectively.

It follows directly from the definition that the second Zagreb coindex achieve its smallest possible value of zero. Therefore, throughout this paper, to avoid trivialities, we assume that the graphs considered have second Zagreb coindex value at least one.

Recently in [8] the differences of Zagreb indices are reported. In this paper, as an analogue, we study the differences of Zagreb coindicies.

2. An identity for the difference $\overline{M_2} - \overline{M_1}$

First, we define an identity for the difference of Zagreb coindices.

For the sake of brevity, we denote the difference between second and first Zagreb coindex by $\Delta\overline{M}$. That is,

$$\Delta\overline{M} \quad = \quad \Delta\overline{M}(G) = \overline{M_2}(G) - \overline{M_1}(G). \tag{1}$$

Hence,

$$\begin{aligned}
\Delta\overline{M}(G) \quad &= \quad \sum_{u,v \notin E(G)} [d_G(u)d_G(v) - d_G(u) - d_G(v)] \\
&= \quad \sum_{u,v \notin E(G)} [(d_G(u) - 1)(d_G(v) - 1) - 1] \\
&= \quad \sum_{u,v \notin E(G)} (d_G(u) - 1)(d_G(v) - 1) - \overline{m},
\end{aligned}$$

where, $\overline{m}$ is the number of edges in $\overline{G}$.

Hence we can define a new vertex-degree based graph invariant as

$$\overline{RM_2}(G) = \sum_{u,v \notin E(G)} (d_G(u) - 1)(d_G(v) - 1)$$

By means of (1), we have the identity

$$\Delta\overline{M(G)} \quad = \quad \overline{RM_2}(G) - \overline{m}. \tag{2}$$

We propose that $\overline{RM_2}(G)$ be called the reduced second Zagreb coindex of a graph G.

Theorem 1. *Let G be a graph of order n and size m. Then*

$$\overline{RM_2}(G) \quad = \quad \overline{M_2}(G) - \overline{M_1}(G) + \overline{m}.$$

Proof. By the definition of reduced second Zagreb coindex, we have

$$\begin{aligned}\overline{RM_2}(G) &= \sum_{u,v\notin E(G)} (d_G(u)-1)(d_G(v)-1)\\ &= \sum_{u,v\notin E(G)} \Big[(d_G(u)d_G(v)) - d_G(u) - d_G(v) + 1\Big]\\ &= \overline{M_2}(G) - \overline{M_1}(G) + \overline{m},\end{aligned}$$

as desired. □

3. Graph Operations

In this section, we introduce graph operations used for producing the composite graphs that are relevant for our purpose and review their basic properties. We consider five operations. Each of them is treated in a separate subsection.

3.1. **Union.** The simplest operation we consider here is a union of two graphs. A union $G_1 \cup G_2$ of the graphs G_1 and G_2 is the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. Here we assume that V_1 and V_2 are disjoint.

Proposition 2. *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be two simple graphs. Then*

$$\overline{RM_2}(G_1 \cup G_2) = \overline{RM_2}(G_1) + \overline{RM_2}(G_2) + 4m_1m_2 - 2(m_1n_2 + m_2n_1) + n_1n_2.$$

Proof. The degree $d_{G_1\cup G_2}(u)$ of a vertex u is equal to the degree of u in the component G_i that contains it. The reduced second Zagreb coindex of $G_1 \cup G_2$ is then equal to the sum of the reduced second Zagreb coindex of the components plus the contribution from the missing edges between the components. But the missing edges make the edge set of K_{n_1,n_2}. Hence the missing edges between G_1 and G_2 is given by

$$\begin{aligned}\sum_{u\in V(G_1)} \Big[\sum_{v\in V(G_2)} (d(u)-1)(d(v)-1)\Big] &= \sum_{u\in V(G_1)} (d_{G_1}(u)-1) \cdot \sum_{v\in V(G_2)} (d_{G_2}(v)-1)\\ &= (2m_1 - n_1)(2m_2 - n_2)\\ &= 4m_1m_2 - 2(m_1n_2 + m_2n_1) + n_1n_2.\end{aligned}$$

□

3.2. **Join.** The join $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 is the graph on the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2 \cup \{u_1u_2 \colon u_1 \in V_1, u_2 \in V_2\}$.

Proposition 3. *Let G be graph of order n and size m. Then*

$$\overline{RM_2}(G + K_1) = \overline{M_2}(G)$$

Proof. Note that $d_{G+K_1}(u) = d_G(u)+1$. Hence by the definition of reduced second Zagreb index, we have

$$\begin{aligned}
\overline{RM_2}(G+K_1) &= \sum_{u,v\notin E(G+K_1)} (d_G(u)+1-1)(d_G(v)+1-1) \\
&= \sum_{u,v\notin E(G)} (d_G(u))(d_G(v)) \\
&= \overline{M_2}(G),
\end{aligned}$$

as desired. □

Proposition 4. *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be two simple graphs. Then*

$$\begin{aligned}
\overline{RM_2}(G+G_2) &= \overline{M_2}(G_1) + \overline{M_2}(G_2) + n_2\overline{M_1}(G_1) + n_1\overline{M_1}(G_2) + \overline{m_1}n_2^2 \\
&+ \overline{m_2}n_1^2 - 4(m_1+m_2) - 4n_1n_2 + n_1 + n_2.
\end{aligned}$$

Proof. Notice that $d_{G_1+G_2}(u) = d_{G_1}(u) + n_2$, and $d_{G_1+G_2}(v) = d_{G_2}(v) + n_1$ for $u \in V(G_1)$ and $v \in V(G_2)$. Since all possible edges between G_1 and G_2 are present in $G_1 + G_2$, there are no missing edges, and hence their contribution is zero. The remaining two contributions, one from the edges missing in G_1 and the other from the edges missing in G_2, are given by

$$\begin{aligned}
\sum_{e\in E(G_1)} (d_{G_1}(u)+n_2-1)(d_{G_1}(v)+n_2-1) &= \overline{M_2}(G_1) + n_2\overline{M_1}(G_1) + \overline{m_1}n_2^2 \\
&- 4m_1 - 2n_1n_2 + n_1,
\end{aligned}$$

and similarly for the sum taken over the missing edges in G_2 are given by

$$\begin{aligned}
\sum_{e\in E(G_2)} (d_{G_2}(u)+n_1-1)(d_{G_2}(v)+n_1-1) &= \overline{M_2}(G_2) + n_1\overline{M_1}(G_2) + \overline{m_2}n_1^2 \\
&- 4m_2 - 2n_1n_2 + n_2.
\end{aligned}$$

Hence by adding these two contributions, we get the required result. □

3.3. **Cartesian product.** For given graphs G_1 and G_2, we define their Cartesian product $G_1\Box G_2$ as the graph on the vertex set $V(G_1) \times V(G_2)$ with vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ connected by an edge if and only if $[u_1 = v_1$ and $\{u_2v_2 \in E(G_2)\}]$ or $[u_2 = v_2$ and $\{u_1v_1 \in E(G_1)\}]$

Proposition 5.

$$\overline{RM_2}(G_1\Box G_2) = \overline{M_2}(G_1\Box G_2) - \overline{M_1}(G_1) - \overline{M_1}(G_2) + n_1m_2 + n_2m_1.$$

Proof. The result follows from Theorem 1 and expression of $\overline{M_2}(G_1\Box G_2)$ from Proposition 13 of [1]. Bearing in mind that $d_{G_1\Box G_2}(u_1, u_2) = d_{G_1}(u_1)+d_{G_2}(u_2)$. □

3.4. **Disjunction.** The disjunction $G_1 \vee G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent with (v_1, v_2), where u_1 is adjacent with v_1 in G_1 or u_2 is adjacent with v_2 in G_2.

Proposition 6. *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be two simple graphs. Then*

$$\begin{aligned}\overline{RM_2}(G_1 \vee G_2) &= \overline{M_2}(G_1 \vee G_2) + \overline{M_1}(G_1) + \overline{M_1}(G_2) + n_1^2 m_2 + n_2^2 m_1 \\ &- (n_1 + n_2)\overline{m_1} - (n_1 + n_2)\overline{m_2} - 2m_1 m_2.\end{aligned}$$

Proof. It is easy to see that $d_{G_1 \vee G_2}(u_1, u_2) = n_2 d_{G_1}(u_1) + n_1 d_{G_2}(u_2) - d_{G_1}(u_1) d_{G_2}(u_2)$. Therefore, by the definition of reduced second Zagreb index, we have

$$\begin{aligned}\overline{RM_2}(G_1 \vee G_2) &= \sum_{(u_1,u_2),(v_1,v_2) \notin E(G_1 \vee G_2)} \Big[\big(n_2 d_{G_1}(u_1) + n_1 d_{G_2}(u_2) - d_{G_1}(u_1) d_{G_2} u_2 - 1\big)\big(n_2 d_{G_1}(v_1) \\ &+ n_1 d_{G_2}(v_2) - d_{G_1}(v_1) d_{G_2}(v_2) - 1\big)\Big] \\ &= \sum_{(u_1,u_2),(v_1,v_2) \notin E(G_1 \vee G_2)} \Big[\big(n_2 d_{G_1}(u_1) + n_1 d_{G_2}(u_2) - d_{G_1}(u_1) d_{G_2} u_2\big)\big(n_2 d_{G_1}(v_1) \\ &+ n_1 d_{G_2}(v_2) - d_{G_1}(v_1) d_{G_2}(v_2)\big)\Big] \\ &- \sum_{(u_1,u_2),(v_1,v_2) \notin E(G_1 \vee G_2)} \Big[\big(n_2 d_{G_1}(u_1) + n_1 d_{G_2}(u_2) - d_{G_1}(u_1) d_{G_2}(u_2)\big) \\ &- \sum_{(u_1,u_2),(v_1,v_2) \notin E(G_1 \vee G_2)} \Big[\big(n_2 d_{G_1}(v_1) + n_1 d_{G_2}(v_2) - d_{G_1}(v_1) d_{G_2}(v_2)\big) \\ &+ n_1^2 m_2 + n_2^2 m_1 - 2m_1 m_2.\end{aligned}$$

□

3.5. **Symmetric difference.** The symmetric difference $G_1 \oplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent with (v_1, v_2), where u_1 is adjacent with v_1 in G_1 or u_2 is adjacent with v_2 in G_2 but not both.

Proposition 7.

$$\begin{aligned}\overline{RM_2}(G_1 \oplus G_2) &= \overline{M_2}(G_1 \oplus G_2) + 2(\overline{M_1}(G_1) + \overline{M_1}(G_2)) + n_1^2 m_2 + n_2^2 m_1 \\ &- (n_1 + n_2)\overline{m_1} - (n_1 + n_2)\overline{m_2} - 4m_1 m_2\end{aligned}$$

Proof. Notice that $d_{G_1 \oplus G_2}(u_1, u_2) = n_2 d_{G_1}(u_1) + n_1 d_{G_2}(u_2) - 2d_{G_1}(u_1) d_{G_2}(u_2)$. Hence the result follows in the same lines of the proof of Proposition 6. □

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