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ON ALMOST $*$-RICCI SOLITON

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ABSTRACT. In the present paper, we prove three fundamental results concerning almost $*$-Ricci soliton in the framework of para-Sasakian manifold. The paper is organised as follows:

- If a para-Sasakian metric g represents an almost $*$-Ricci soliton with potential vector field V is Jacobi along Reeb vector field ξ, then g becomes a $*$-Ricci soliton.
- If a para-Sasakian metric g represents an almost $*$-Ricci soliton with potential vector field V as infinitesimal paracontact transformation, then V is killing and g is η-Einstein.
- If a para-Sasakian metric g represents an almost $*$-Ricci soliton with potential vector field V being collinear with the Reeb vector field ξ, then $\lambda = 0$, V is strict and g is η-Einstein.

1. INTRODUCTION

The study of Ricci solitons on pseudo-Riemannian (Riemannian) manifold is holding the attention of modern differential geometers because it is considered as a natural generalization of Einstein metric (i.e. the Ricci tensor is a constant multiple of the Riemannian metric g). It is a special self similar solution of the Hamilton's Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

with initial condition $g(0) = g$. We say that the Ricci soliton is shrinking when $\lambda < 0$, steady when $\lambda = 0$, and expanding when $\lambda > 0$. It is viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. A pseudo-Riemannian metric g on a smooth manifold (M^n, g) of dimension n is said to be a Ricci soliton if there exists a vector field V and a soliton constant λ satisfying

$$\frac{1}{2}\pounds_V g + S = \lambda g,$$

where $\pounds_V g$ denotes the Lie-derivative of g in the direction of V and S is the Ricci tensor of g. Ricci solitons were introduced in Riemannian geometry [10]

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as the self-similar solutions of the Ricci flow, and play an important role in understanding its singularities.

In parallel with contact and complex structures in the Riemannian case, paracontact metric structures were introduced by Kaneyuki and Williams [12] in pseudo-Riemannian manifold, as a natural odd-dimensional counterpart to para Hermitian structures. Zamkovoy [21] started the study of paracontact metric manifolds, which become essential for further investigations of paracontact metric geometry. Calvaruso and Perrone [3] had taken the initiative of studying Ricci solitons in the context of paracontact metric geometry. The case of Ricci solitons in the framework paracontact geometry was treated in the paper [1], [4] and [15].

In [17], Pigola et al. introduced and studied almost Ricci soliton, where they modified the definition of Ricci solitons by allowing the constant λ to be a smooth function on M.

In [18], Tachibana introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds and Hamada [9] on real hypersurfaces in non-flat complex space forms (for details see [5],[8]). The $*$-Ricci tensor S^* can be viewed as analogy of the usual Ricci tensor S in contact geometry. In view of this, Kaimakamis et al. [11] introduced the $*$-Ricci soliton, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor S in soliton equation with the $*$-Ricci tensor S^* and defined by

$$\frac{1}{2}\pounds_V g + S^* = \lambda g, \tag{1.1}$$

where

$$S^*(X,Y) = \frac{1}{2}\text{trace}\{Z \mapsto R(X,\phi Y)\phi Z\}. \tag{1.2}$$

Many authors [13], [16], [7] have made a rigorous study of $*$-Ricci soliton in the context of both contact and paracontact geometry.

In the present paper, generalizing the notion of $*$-Ricci soliton to almost $*$-Ricci soliton, the author treated almost $*$-Ricci soliton in the framework of para-Sasakian manifold.

2. PRELIMNARIES

A $(2n+1)$-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (ϕ,ξ,η) if there exists a $(1,1)$-type tensor field ϕ, a characteristic (Reeb) vector field ξ and a 1-form η satisfying the following conditions:

$$\phi(\xi) = 0, \quad \eta\circ\phi = 0 \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi^2 = I - \eta\otimes\xi. \tag{2.2}$$

Let $D = \ker\eta$ be the horizontal distribution generated by η. Then the tensor field ϕ induces an almost paracomplex structure(refer to [6], [21]) on every fibre on D. An almost complex paracomplex structure on an $2n$-dimensional manifold $(1,1)$-tensor J such that $J^2 = +1$ and the eigen subbundles T^+ and T^- corresponding to the eigen values $1,-1$ of J respectively have equal dimension n. The Nijenhius tensor N of J given by

$$N_J(\cdot,\cdot) = [J\cdot, J\cdot] - J[J\cdot,\cdot] - J[\cdot, J\cdot] + [\cdot,\cdot],$$

is an obstruction for the integrability of the eigen subbundles T^+ and T^-. If $N = 0$, then the almost paracomplex structure is called paracomplex or integrable.

The definition of the almost paracontact structure implies that the endomorphism ϕ has rank $2n$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. If a manifold M^{2n+1} with (ϕ, ξ, η)-structure admits a semi-Riemannian metric g satisfying $g(\phi\cdot, \phi\cdot) = -g(\cdot, \cdot) + \eta(\cdot)\eta(\cdot)$, then M^{2n+1} has an almost paracontact metric structure and g is called compatible metric. Any compatible metric g with a given almost paracontact structure is of signature $(n+1, n)$. Any almost paracontact structure always admits a compatible metric. The fundamental 2-form F is given by $F(\cdot, \cdot) = g(\cdot, \phi\cdot)$ is non-degenerate on the horizontal distribution D and $\eta \wedge F^n \neq 0$.

If $g(\cdot, \phi\cdot) = d\eta(\cdot, \cdot) = \frac{1}{2}(\cdot\eta(\cdot) - \cdot\eta(\cdot) - \eta[\cdot, \cdot])$, then η is paracontact form and the almost paracontact metric manifold (M, ϕ, ξ, η) is said to be paracontact metric manifold.

A paracontact structure for which ξ is Killing vector field is called a K-paracontact structure. A paracontact structure on M^{2n+1} cannonically gives rise to an almost paracomplex structure on the product $M^{2n+1} \times \mathbb{R}$. If this almost paracomplex structure is integrable, the given paracontact metric manifold is said to be a para-Sasakian. Equivalently, (see [20], [21]) a paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \ \forall X, Y \in \mathfrak{X}(M). \tag{2.3}$$

Any para-Sasakian manifold is K-paracontact, and the converse also holds for three-dimensional space. From (2.3), it follows that

$$\nabla_X \xi = -\phi X, \tag{2.4}$$

for any $X \in \mathfrak{X}(M)$. For details we refer the reader to ([21]). Also in a $(2n+1)$-dimensional para-Sasakian manifold, the following relations hold:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.5}$$

$$R(X, \xi)\xi = -X + \eta(X)\xi, \tag{2.6}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.7}$$

$$Q\xi = -2n\xi, \tag{2.8}$$

for any $X, Y \in \mathfrak{X}(M)$. Here R is the curvature tensor of g and S denotes the Ricci tensor defined by $S(\cdot, \cdot) = g(Q\cdot, \cdot)$, Q being the Ricci operator.

For any $X \in \mathfrak{X}(M)$, $\pounds_X f = Xf = g(X, grad\ f)$ for some smooth function f, where $grad\ f$ is the gradient of f. The Hessian of a smooth function f on M is defined by

$$H^f(X, Y) = (\nabla_X grad\ f)Y = XY(f) - (\nabla_X Y)(f),$$

for every $X, Y \in \mathfrak{X}(M)$. It is well known that H^f is symmetric and that

$$g(\nabla_X grad\ f, Y) = g(\nabla_Y grad\ f, X),$$

which shows that it can be regarded as a solution of $T^*(M) \otimes T^*(M)$ [refer to [14]].

A vector field W on a semi-Riemannian manifold (M^{2n+1}, g) is said to be paracontact or infinitesimal paracontact transformation if it preserves the paracontact form η, i.e., there exists a smooth function $f : M^{2n+1} \to \mathbb{R}$ satisfying

$$\pounds_W \eta = f\eta. \tag{2.9}$$

If the vector field W is strict, then $f = 0$.

A para-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions on M.

3. MAIN RESULTS

Before we proceed for the main results of the paper, let us state and prove few important lemmas:

Lemma 3.1. *Let $M(\phi, \xi, \eta, g)$ be a para-Sasakian manifold. Then the following relations are true* [16]*:*
(i) $(\nabla_X Q)\xi = Q\phi X + 2n\phi X$ *(ii)* $\nabla_\xi Q = 0$.

Proof. Differentiating covariantly (2.8) and thereby using (2.4), we obtain (i). Since ξ is killing, therefore by virtue of (2.4) we have $\nabla_\xi Q = Q\phi - \phi Q$. On a para-Sasakian manifold, we have $Q\phi - \phi Q = 0$ [for details refer to Lemma 3.15, Page 48, [21]]. Hence using this, we have the desired result (ii).

□

Though the proof of the following Lemma has been in [16], but here the proof has been done in the different manner.

Lemma 3.2. *The $*$-Ricci tensor on a $(2n+1)$-dimensional para-Sasakian manifold $M(\phi, \xi, \eta, g)$ is given by*

$$S^*(X, Y) = -S(X, Y) - (2n-1)g(X, Y) - \eta(X)\eta(Y), \tag{3.1}$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. Let us begin by taking covariant differentiation of (2.5) along an arbitrary vector field $Z \in \mathfrak{X}(M)$ and using (2.4), we have

$$(\nabla_Z R)(X, Y)\xi - R(X, Y)\phi Z = g(X, \phi Z)Y - g(Y, \phi Z)X, \tag{3.2}$$

for every $X, Y \in \mathfrak{X}(M)$. Contracting (3.2) with respect to an orthonormal frame $\{e_i\}$, we acquire

$$(\operatorname{div} R)(X, Y)\xi - g(R(X, Y)\phi e_i, e_i) = 2g(\phi X, Y),$$

for every $X, Y \in \mathfrak{X}(M)$. At this point, by virtue of the contracted second Bianchi identity, the foregoing equation reduces to

$$g((\nabla_X Q)Y - (\nabla_Y Q)X, \xi) - g(R(X, Y)\phi e_i, e_i) = 2g(\phi X, Y), \tag{3.3}$$

for every $X, Y \in \mathfrak{X}(M)$. Now in view of Lemma 3.1(i) and (3.3), it follows

$$g(R(X, Y)\phi e_i, e_i) = g(Q\phi X, Y) + g(\phi QX, Y) + 2(2n-1)g(\phi X, Y), \tag{3.4}$$

for every $X, Y \in \mathfrak{X}(M)$. Now replacing Y by ϕY in (3.4) and recalling the definition of $*$-Ricci tensor (1.2), we obtain the $*$-Ricci tensor on a $(2n+1)$-dimensional para-Sasakian manifold as

$$S^*(X,Y) = -S(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y), \ \forall\, X, Y \in \mathfrak{X}(M)$$

where we have used the property $Q\phi - \phi Q = 0$ for para-Sasakian manifold [for details refer to Lemma 3.15, Page 48, [21]]. □

By virtue of Lemma 3.2, the $*$-Ricci soliton can be expressed as

$$(\pounds_V g)(X,Y) = 2S(X,Y) + 2(\lambda + 2n - 1)g(X,Y) + 2\eta(X)\eta(Y), \tag{3.5}$$

for any $X, Y \in \mathfrak{X}(M)$.

Proposition 3.1. *If a para-Sasakian metric g represents an almost $*$-Ricci soliton, then we have the following relation*

$$(\pounds_V \nabla)(X,\xi) - 2Q\phi X = -2(2n+1)\phi X + (grad\ \lambda)\eta(X) - (X\lambda)\xi - (\xi\lambda)X, \tag{3.6}$$

for all $X \in \mathfrak{X}(M)$

Proof. Let us begin with the covariant derivative of (3.5) along an arbitrary $Z \in \mathfrak{X}(M)$ and using (2.3), we get

$$(\nabla_Z \pounds_V g)(X,Y) = 2(\nabla_Z S)(X,Y) + 2[(Z\lambda)g(X,Y) - \eta(X)g(Y,\phi Z) - \eta(Y)g(X,\phi Z)], \tag{3.7}$$

for every $X, Y \in \mathfrak{X}(M)$. According to Yano (see [19], p. 23), we write

$$(\pounds_V \nabla_Z g - \nabla_Z \pounds_V g - \nabla_{[V,Z]} g)(X,Y) = -g((\pounds_V \nabla)(Z,X),Y) - g((\pounds_V \nabla)(Z,Y),X),$$

for all $X, Y, Z \in \mathfrak{X}(M)$. From (3.7) and the last equation, on applying parallel Riemannian metric we obtain

$$\begin{aligned} g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X) - 2(\nabla_Z S)(X,Y) \\ = -2(Z\lambda)g(X,Y) + 2\{\eta(X)g(Y,\phi Z) + \eta(Y)g(X,\phi Z)\}, \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In view of the symmetric property of the Ricci operator Q, interchanging cyclically the roles of X, Y, Z in the preceding equation and by a direct calculation, we obtain

$$\begin{aligned} g((\pounds_V \nabla)(X,Y),Z) = &-(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) \\ &+ (Z\lambda)g(X,Y) - (X\lambda)g(Y,Z) - (Y\lambda)g(Z,X) \\ &- 2\{\eta(X)g(Y,\phi Z) + \eta(Y)g(X,\phi Z)\}, \end{aligned} \tag{3.8}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where we have used the symmetric property of $(1,2)$-type tensor filed $\pounds_V \nabla$, i.e., $(\pounds_V \nabla)(X,Y) = (\pounds_V \nabla)(Y,X)\ \forall\, X, Y \in \mathfrak{X}(M)$. On the other hand, recalling Lemma 3.1 and taking ξ instead of Y in (3.8), one can deduce the required formula.

□

Theorem 3.1. *If a para-Sasakian metric g represents an almost $*$-Ricci soliton with potential vector field V is Jacobi along Reeb vector field ξ, then g becomes a $*$-Ricci soliton.*

Proof. Let us begin, by feeding the Proposition 3.1 in the well known formula (see [19], p.23):

$$\nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V - R(V, X)Y = (\pounds_V \nabla)(X, Y),$$

we obtain

$$\nabla_X \nabla_\xi V - \nabla_{\nabla_X \xi} V - R(V, X)\xi = 2Q\phi X - 2(2n+1)\phi X + (grad\ \lambda)\eta(X) - (X\lambda)\xi - (\xi\lambda)X. \tag{3.9}$$

By hypothesis: the potential vector V is a Jacobi field along ξ, i.e.
$\nabla_\xi \nabla_\xi V - R(V, \xi)\xi = 0$, replacing X by ξ in the (3.9) and recalling $\nabla_\xi \xi = 0$, we acquire

$$grad\ \lambda = 2(\xi\lambda)\xi. \tag{3.10}$$

Taking its covariant derivative along $X \in \mathfrak{X}(M)$ and then inner product with $Y \in \mathfrak{X}(M)$, we deduce

$$g(\nabla_X grad\ \lambda, Y) = 2X(\xi(\lambda))\eta(Y) + 2\xi(\lambda)d\eta(X, Y), \tag{3.11}$$

where we have used (2.4) and the formula $d\eta(X, Y) = g(X, \phi Y)$. By the symmetric property of H^λ, i.e. $g(\nabla_X grad\ \lambda, Y) = g(\nabla_Y grad\ \lambda, X)$, (3.11) gives us $\xi(\lambda)d\eta(X, Y) = 0$ all X, Y orthogonal to ξ. This implies that $\xi(\lambda) = 0$ on M, as $d\eta$ is non-vanishing on M. It follows from (3.10) that $grad\ \lambda = 0$ on M, and therefore λ is constant on M. Hence g is a $*$-Ricci soliton. □

Theorem 3.2. *If a para-Sasakian metric g represents an almost $*$-Ricci soliton with potential vector field V as infinitesimal paracontact transformation, then V is killing and g is η-Einstein with constant scalar curvature.*

Proof. Since the exterior derivative d commutes with the Lie-derivative, i.e.
$\pounds_V \circ d = d \circ \pounds_V$, operating (2.9) by d, we acquire

$$(\pounds_V d\eta)(X, Y) = \frac{1}{2}\{X(\nu)\eta(Y) - Y(\nu)\eta(X)\} + \nu d\eta(X, Y),$$

for any $X, Y \in \mathfrak{X}(M)$. Taking Lie derivative of $d\eta(X, Y) = g(X, \phi Y)$ along V and combining the foregoing equation with (3.5) and (2.9), we obtain

$$2(\pounds_V \phi)(X) + 2(2\lambda - \nu + 2(2n-1))\phi X = -4Q\phi X + \eta(X) grad\ \nu - X(\nu)\xi. \tag{3.12}$$

On the other hand from (2.2), we get

$$(\pounds_V \phi)(\phi X) + \phi\big((\pounds_V \phi)(X)\big) = -(\pounds_V \eta)(X)\xi - \eta(X)\pounds_V \xi, \tag{3.13}$$

for any $X \in \mathfrak{X}(M)$. Now by virtue of (3.5) and (2.8), we obtain

$$(\pounds_V g)(X, \xi) = 2\lambda\eta(X).$$

In consideration with this, and (2.9), we have

$$g(X, \pounds_V \xi) = (\nu - 2\lambda)\eta(X). \tag{3.14}$$

In view of this and $\phi\xi = 0$, by a direct calculation, we get $(\pounds_V \phi)(\xi) = 0$, and therefore (3.12) yields that $grad\ \nu = \xi(\nu)\xi$. Thus, it follows from (2.4) that

$$H^\nu(X, Y) = X\xi(\nu)\eta(Y) - \xi(\nu)g(\phi X, Y). \tag{3.15}$$

Since H^ν is symmetric and ϕ is skew-symmetric, we achieve $\xi(\nu)d\eta(X,Y) = 0$ for all X, Y orthogonal to ξ, and therefore, $\xi(\nu) = 0$, as $d\eta$ is non-vanishing everywhere on a para-contact metric manifold M. Consequently, $grad\ \nu = 0$. Hence ν is constant on M. Now, it suffices to combine (1.1), (3.14) and $\eta(\xi) = 1$ to arrive at $\nu = \lambda$. Using this and (2.8) in (3.12), we compute

$$(\pounds_V \phi)(\phi X) + 2QX = -(\lambda + 2(2n-1))X + (\lambda - 2)\eta(X)\xi. \tag{3.16}$$

Also operating by ϕ, (3.12) yields

$$\phi(\pounds_V \phi)(X) + 2QX = -(\lambda + 2(2n-1))X + (\lambda - 2)\eta(X)\xi, \tag{3.17}$$

where we have used the property $Q\phi = \phi Q$ for para-Sasakian manifold [for details refer to [21]]. In addition, it follows directly from (3.5) and (3.14) that

$$(\pounds_V \eta)(X) = (\pounds_V g)(X, \xi) + g(X, \pounds_V \xi) = \lambda\eta(X), \tag{3.18}$$

where we have used $\nu = \lambda$. Now, it suffices to feed (3.14), (3.16), (3.17) and (3.18) in (3.13) to arrive at the relation

$$2S(X,Y) = -(\lambda + 2(2n-1))X + (\lambda - 2)\eta(X)\xi, \tag{3.19}$$

which shows M is an η-Einstein manifold with constant scalar curvature $r = -n(\lambda + 4n)$. Moreover, inserting (3.19) into (3.12), we achieve $\pounds_V \phi = 0$, and therefore, V is killing. Hence the proof.

□

Theorem 3.3. *If a para-Sasakian metric g represents an almost $*$-Ricci soliton with potential vector field V is collinear with the Reeb vector field ξ, then $\lambda = 0$, V is strict and g is η-Einstein with constant scalar curvature $-4n^2$.*

Proof. Assume the potential vector field V is collinear with ξ, i.e., $V = \sigma\xi$ for a non-zero smooth-function σ on M. In view of its covariant derivative and (2.4), we can compute

$$(\pounds_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_V Y, X) = X(\sigma)\eta(Y) + Y(\sigma)\eta(X), \tag{3.20}$$

where we have used the anti-symmetric property of ϕ. In view of this, (3.5) becomes

$$X(\sigma)\eta(Y) + Y(\sigma)\eta(X) - 2S(X,Y) = 2(\lambda + 2n - 1)g(X,Y) + 2\eta(X)\eta(Y). \tag{3.21}$$

At this point, replacing Y by ξ in the preceding equation and recalling (2.8) yields that

$$X(\sigma) = (2\lambda - \xi(\sigma))\eta(X), \tag{3.22}$$

for any $X \in \mathfrak{X}(M)$. Again, substituting ξ for X we get $\xi(\sigma) = \lambda$, and hence (3.22) follows $X(\sigma) = \lambda\eta(X)$, i.e., $d\sigma = \xi(\sigma)\eta$, where d is the exterior derivative operator. Taking its exterior derivative, using Poincare lemma: $d^2 = 0$ and then taking the wedge product with η, we acquire $\xi(\sigma)\eta \wedge d\eta = 0$. Consequently, $\xi(\sigma) = 0$ as $\eta \wedge d\eta$ never vanishes anywhere on a paracontact metric manifold M. This implies that $d\sigma = 0$, and hence, σ is a constant on M. It follows from (3.22) that $\lambda = 0$, which reduces the (3.21) to the relation

$$S(X,Y) = -(2n-1)g(X,Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, (M^{2n+1}, g) is η-Einstein with constant scalar curvature $r = -4n^2$, and complete the proof. $\square$

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