RANKS AND SETS EVINCING THEM FOR THE TANGENT DEVELOPABLE

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Abstract. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. We study the complement in the $k$-secant variety of $X$ of the set of all points with $X$-rank $k$. In the case $k = 2$ this is contained in the tangent developable $\tau(X)$ of $X$ and we study its $X$-ranks and the sets evincing them when $X$ is a smooth OADP, i.e. $\dim X = n$, $r = 2n + 1$ and a general point of $\mathbb{P}^{2n+1}$ is contained in a unique secant line of $X$.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ with characteristic 0. When we will speak about homology groups or cohomology groups or the fundamental group we will always assume $\mathbb{K} = \mathbb{C}$ without further mention.

For any set $A \subset \mathbb{P}^r$ let $\langle A \rangle$ denote its linear span. Fix any $q \in \mathbb{P}^r$. The $X$-rank $r_X(q)$ of $X$ is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S \rangle$. Any set $S \subset X$ such that $\vert S \vert = r_X(q)$ and $q \in \langle S \rangle$.

Chevalley’s theorem $S(X,q)$ is a constructible set and hence (over $\mathbb{C}$) it has the homotopy type of a finite CW-complex. Set $n := \dim X$. For any integer $k > 0$ the $k$-secant variety $\sigma_k(X)$ of $X$ is the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle S \rangle$ with $S \subset X$ and $\vert S \vert = k$. The projective algebraic set $\sigma_k(X)$ is irreducible and $\dim \sigma_k(X) \leq \min\{r, (n+1)k - 1\}$. If $k \geq 2$ and $\sigma_{k-1}(X) \not\subset \mathbb{P}^r$ we have $\sigma_{k-1}(X) \subset \sigma_k(X)$ and a general element $q \in \sigma_k(X)$ has $r_X(q) = k$. If $k \geq 2$ and $\sigma_{k-1}(X) \not\subset \mathbb{P}^r$ set $\sigma^0_k(X) := \{q \in \mathbb{P}^r \mid r_X(q)\}$. By Chevalley’s theorem $\sigma^0_k(X)$ is a constructible algebraic set. By assumption $\sigma^0_k(X)$ contains a non-empty Zariski open subset. Thus $\sigma_k(X)$, $\sigma^0_k(X)$ and $\sigma_k(X) \setminus \sigma^0_k(X)$ are constructible algebraic set and hence, being homotopic to finite CW-complexes, their homology and cohomology groups with integers coefficients are finitely generated abelian groups.

There are several examples in which $\sigma_k(X) \setminus \sigma^0_k(X)$ has complex codimension 1 and there is a hypersurface $B$ of $\sigma_k(X)$ such that $B \subset \sigma_k(X) \setminus \sigma^0_k(X)$ ([3]). With the following very restrictive assumption on $k$ we ask more refined topological questions, not always true without some strong assumptions.

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Let \( \rho(X) \) be the maximal positive integer \( x \) such that every zero-dimensional scheme \( Z \subset X \) with \( \deg(Z) \leq x \) is linearly independent. If \( 2k \leq \rho(X) \) we have \( \dim \sigma_k(X) = k(n + 1) - 1 \) and for every \( q \in \sigma_k(X) \setminus \sigma_{k-1}(X) \) there is a unique zero-dimensional scheme \( Z \subset X \) such that \( \deg(Z) = k \) and \( q \in \langle Z \rangle \); moreover \( Z \) is smoothable and Gorenstein ([6, 5]). The uniqueness part shows the existence of a locally closed irreducible set \( B \subset \sigma_k(X) \setminus \sigma_k(X) \) such that \( \dim B = k(n + 1) - 2 \) and for each \( q \in B \) there is \( Z \subset B \) formed by the union \( k - 2 \) points of \( X \) and a connected degree 2 scheme whose support is a smooth point of \( X \). The closure \( \overline{B} \) of \( B \) in \( \mathbb{P}^r \) (or in \( \sigma_k(X) \)) is described in the following way. Let \( \tau(X) \subseteq \mathbb{P}^r \) denote the tangential variety, i.e. the closure in \( \mathbb{P}^r \) of the union of all tangent spaces \( T_pX \) of \( X \) at \( p \in X_{\text{reg}} \). If \( k = 2 \) we have \( \overline{B} = \tau(X) \). If \( k > 2 \) the closed set \( \overline{B} \) is the join of \( \tau(X) \) and \( \sigma_{k-2}(X) \). We proved in [2] the uniqueness of the scheme \( Z \) at a general \( q \in B \) under milder assumptions. Here we pose the following question.

**Question 1.1.** Under which assumption on \( X \) and \( k \) the natural map of abelian groups \( H_2(\sigma_k(X),\mathbb{Z}) \to H_2(\sigma_k(X),\overline{B},\mathbb{Z}) \) has infinite cokernel (or the corresponding statement with \( \sigma_k^0(X) \cup B \) instead of \( \sigma_k(X) \) and \( \overline{B} \))? Do \( H_1(\overline{B},\mathbb{Z}) \) contains at least one copy of \( \mathbb{Z} \)?

The reader may also consider the dual question for cohomology groups and/or for other coefficient groups instead of integers and the query about \( \pi_1(\overline{B}) \). See Remark 2.3 for an explanation of why \( \overline{B} \) should matter.

The question is prompted by [9, 13] in which the authors showed how important for the applications is the path-connectedness of \( \sigma_k^0(X) \). We give a case in which Question 1.1 has a negative answer (Proposition 2.4).

**Question 1.2.** Is \( S(X,q) \) infinite for a general \( q \in \tau(X) \)?

Question 1.2 has an easy affirmative answer in several cases (see [3] when \( X \) is a curve with sufficiently large \( \rho(X) \)). In many cases there are some \( q \in \tau(X) \) such that \( |S(X,q)| = 1 \) (see Example 2.1). Thus it is really important to assume that we are requiring that \( q \) is general in \( \tau(X) \).

We prove the following case in which Question 1.2 has a positive answer.

**Theorem 1.3.** Let \( X \subset \mathbb{P}^{2n+1} \) be a smooth \( n \)-dimensional OADP. We have \( r_X(q) = 3 \) and \( \dim S(X,q) > 0 \) for a general \( q \in \tau(X) \).

2. The proofs

**Example 2.1.** Fix an integer \( r \geq 3 \). Let \( \nu_{r+1} : \mathbb{P}^1 \to \mathbb{P}^{r+1} \) be the order \( r + 1 \) Veronese embedding of \( \mathbb{P}^1 \). Set \( Y := \nu_{r+1}(\mathbb{P}^1) \). Fix \( a, b \in Y \) such that \( a \neq b \) and set \( Z := 2a + b \) \( Z \) is a degree 3 zero-dimensional scheme spanning a plane \( \langle Z \rangle \). Take a general \( o \in \langle Z \rangle \) and call \( \ell_o : \mathbb{P}^{r+1} \setminus \{o\} \to \mathbb{P}^r \) the linear projection for \( q \). Set \( X := \ell_o(Y) \) and \( L := \ell_o(\langle Z \rangle \setminus \{o\}) \). Take a general \( q \in L \). \( L \) is the tangent line of \( X \) at \( \ell_o(a) \) and hence \( q \in \tau(X) \setminus X \). Since \( L \) is spanned by \( \{\ell_o(a), \ell_o(b)\} \), we have \( r_X(q) = 2 \). Since \( \rho(Y) = r + 2 \), it is easy to check that \( \{a, b\} \) is the only element of \( S(X,q) \).

We recall the following definition ([8]). Let \( X \subset \mathbb{P}^{2n+1} \) be an integral and non-degenerate \( n \)-dimensional variety. \( X \) is said to be a variety with **only one**
**apparent double point** or an OADP for short if for a general \( q \in \mathbb{P}^r \) there is a unique secant line of \( X \). \( X \) is said to be a smooth OADP if it is smooth and it is an OADP variety. The existence of at least one secant line of \( X \) passing through a general \( q \in \mathbb{P}^{2n+1} \) is equivalent to \( \sigma_2(X) = \mathbb{P}^{2n+1} \), i.e. to \( r_{X,\text{gen}} = 2 \), and it implies that \( S(X,q) \) is finite. The condition \( |S(X,q)| = 1 \) for a general \( q \in \mathbb{P}^r \) is very strong: it obviously implies \( r + 1 \equiv 0 \pmod{n+1} \), \( n := \dim X \), and that \( X \) is not defective. When \( \dim X = 1 \) it implies that \( X \) is a rational normal curve ([7, Theorem 3.1]). For \( \dim X > 1 \) there is hope of a complete classification (at least for low \( \dim X \) and \( X \) smooth) only if \( r = 2n + 1 \).

**Proof of Theorem 1.3:** Fix a general \( q \in \tau(X) \).

(a) In this step we prove that \( r_X(q) = 3 \). Assume \( r_X(q) = 2 \). Since \( q \in \langle Z \rangle \) for some connected zero-dimensional scheme, we get the existence of at least 2 degree 2 smoothable zero-dimensional subscheme, \( Z \), of \( X \) whose linear span contains \( q \). The abstract join contains all such pairs \((Z,q)\). Since \( X \) is an OADP, we get that the set of all such \( Z \)'s has positive dimension and a general element of one of its component is formed by 2 points. Thus \( S(X,q) \) is infinite. We recall that the focus \( F(X) \) of \( X \) is the closure in \( \mathbb{P}^{2n+1} \) of the union of all focal secant lines, i.e. the union of all secant lines \( L \subset \mathbb{P}^{2n+1} \) such that a general \( o \in L \) is contained in infinitely many secant lines of \( X \) ([8, page 480]). A general secant line is not a focal line. Thus the set of all focal lines have dimension at most \( 2n - 1 \). Any two secant lines meeting outside \( X \) are focal lines. Since any \( p \in F(X) \) is contained in infinitely many focal lines, we get \( \dim F(X) \leq 2n - 1 \). Since \( \dim \tau(X) = 2n \), we get \( r_X(q) > 2 \). Since \( r_{X,\text{gen}} = 2 \), the inequality \( r_X(q) \leq 4 \) is true for all \( q \in \mathbb{P}^{2n+1} \) by a general result due the Blekherman and Teitler ([4]). As in many other papers the proofs in [4] help to get a stronger statement in specific situations. Remember that \( r_X(a) = 2 \) for all \( a \in \mathbb{P}^{2n+1} \setminus \tau(X) \). Thus to prove that \( r_X(q) \leq 3 \) (and hence \( r_X(q) = 3 \)) it is sufficient to show that for a general \( p \in X \) the line \( L_p := \langle \{p,q\} \rangle \) is not contained in \( \tau(X) \). This is true, because \( q \) is general in \( \tau(X) \) and the join of \( X \) and \( \tau(X) \) contains \( \sigma_2(X) = \mathbb{P}^{2n+1} \).

(b) Now we observe that the proof of step (a) gave a one-dimensional family of elements of \( S(X,q) \). Call again \( p \) the only point of \( X \) such that \( q \in T_pX \). Take \( o \in T_pX \setminus F(X) \). Call \( S_o \) the only subset of \( X \) such that \( |S_o| = 2 \) and \( o \in \langle S_o \rangle \). We proved that for a general \( q \), i.e. for a general \( p \in X \) and a general \( q \in T_pX \) we have \( r_X(q) = 3 \) and \( p \not\in S_o \), i.e. \( p \cup S_o \in S(X,q) \). \( \square \)

**Remark 2.2.** The inequality \( r_X(q) \leq 3 \) for a general \( q \in \tau(X) \) is true (with the same proof) for all integral \( X \subset \mathbb{P}^r \) such that \( \sigma_2(X) = \mathbb{P}^r \).

**Remark 2.3.** Assume \( 2 \leq k \leq \rho(X) \). For any positive integer \( x \) let \( \Delta_x \subset \mathbb{C}^x \) denote the open unit ball. Fix \( q \in B \). There is a fundamental family of open neighborhoods \( U \) of \( q \) in \( \sigma_k(X) \) for the euclidean topology such that for each \( U \in U \) the pair \( (U, U \cap \sigma_k^0) \) is biholomorphic to \( (\Delta_{2k-1}, \Delta_{2k-2} \times (\Delta_1 \setminus \{0\})) \). Hence the local fundamental group of the pair \((\sigma_k(X), \sigma_k^0(X))\) at \( q \) is isomorphic to \( \mathbb{Z} \). We think that in general these local contributions will not cancel globally in \((\sigma_k(X), \sigma_k^0(X))\).
Proposition 2.4. Let $X \subset \mathbb{P}^{2n+1}$, $n \geq 3$, be a smooth $n$-dimensional OADP. Then the natural map $H_2(\sigma_k(X), \mathbb{Z}) \to H_2(\sigma_k(X), \mathcal{B}, \mathbb{Z})$ is surjective and $\pi_1(\mathcal{B}) = 0$.

Proof. Recall that $\mathcal{B} = \tau(X)$. By the definition of OADP we have $\sigma_2(X) = \mathbb{P}^{2n+1}$ and $\sigma_2(X) \setminus \sigma_0^2(X) \subseteq \tau(X)$. $\tau(X)$ is an integral hypersurface and hence it is simply connected ([10, Corollary 5.3]). Use the homology sequence of a pair ([12, Ch. VII, §5]). □

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References

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