RATIONALIZATION OF TOEPLITZ-HANKEL OPERATORS

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ABSTRACT. In this paper, we introduce and study the rationalization of all kinds of Toeplitz and Hankel operators on the space $L^2(T)$, $T$ being the unit circle. Firstly, we define the Rationalized (Toeplitz-Hankel) operator of order $(k_1, k_2)$ on the space $L^2$, for $\varphi \in L^\infty$, as

$$R_\varphi(TH)(f) = W_{k_1}M_{\varphi}W_{k_2}^*(f), \quad \forall \ f \in L^2$$

where $k_1$ and $k_2$ are nonzero integers and $M_\varphi : L^2 \to L^2$ is the multiplication operator. For nonzero $k \in \mathbb{Z}$, $W_k$ is an operator on $L^2$ defined as

$$W_k(z^i) = \begin{cases} z^{i/k} & \text{if } i \text{ is divisible by } k \\ 0 & \text{otherwise} \end{cases}$$

We get some properties and spectral values of the operator $W_k$. It is proved that if $k_1$ and $k_2$ are relatively prime integers then a bounded operator $R$ on $L^2$ is Rationalized (Toeplitz-Hankel) operator of order $(k_1, k_2)$ on the space $L^2$ if only if $M_{z^{k_2}}R = RM_{z^{k_1}}$. Further if $k_1$ or $k_2$ are not relatively prime then also we give a characterization.

1. Introduction

The study of Toeplitz operators began almost in the beginning of 20th century by O. Toeplitz. A lot of work on Toeplitz operators has been done by different mathematicians in the world. Toeplitz operators and Hankel operators became a subject of investigations for the researchers. Motivated by these, M. C. Ho [7] in the year 1995 introduced slant Toeplitz operators and later the notion of slant Hankel operators was introduced. Further, these notions have been generalized [2] to $k$th order slant Toeplitz and slant Hankel operators and studied simultaneously. Let us recall the following:

The place $L^2(T) = L^2$ stands for the collection of all complex valued Lebesgue measurable functions on the unit circle that are square integrable. Thus

$$L^2 = \left\{ f : f \text{ complex valued measurable function on } T : \int_T |f|^2 < \infty \right\}$$

$L^2$ in a Hilbert space with $\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta$, $f, g \in L^2$, where $\overline{g}$ in complex conjugate of $g$.

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The set \( \{ z^n = e^{in\theta} : n \in \mathbb{Z} \} \) is an orthonormal basis of \( L^2 \). If \( f \in L^2 \), the complex numbers

\[
a_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta
\]

are called Fourier Coefficients of \( f \) and \( \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \) is called Fourier expansion of \( f \). We define the space \( H^2(T) \) as

\[
H^2 = \left\{ f \in L^2 : \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta = 0 \text{ for } n < 0 \right\}
\]

Elements of \( H^2 \) are called analytic functions.

The space \( L^\infty = L^\infty(T) \) is the collection of all essentially bounded complex valued measurable functions on the unit circle.

The multiplication operator \( M_\varphi \), \( \varphi \in L^\infty \) on \( L^2 \) is defined as \( M_\varphi f = \varphi f \) \( \forall f \in L^2 \). In [7] M. C. Ho defined the slant Toeplitz operator on the space \( L^2 \) as \( A_\phi : L^2 \to L^2 \) as

\[
A_\varphi f = WM_\varphi f \quad \forall f \in L^2
\]

where

\[
W : L^2 \to L^2 \text{ is the operator defined as } W(z^{2n}) = z^n \forall n \in \mathbb{Z}
\]

\[
W(z^{2n+1}) = 0
\]

In [4] the slant Hankel operator on \( L^2 \) is defined as \( K_\varphi : L^2 \to L^2 \) as

\[
K_\varphi f = JA_\varphi f
\]

where \( J \) is the reflection operator on \( L^2 \).

The notion of Generalized slant Toeplitz operator on \( L^2 \) [2] was introduced as

\[
U_\varphi : L^2 \to L^2
\]

\[
U_\varphi f = W_k M_\varphi f \quad \forall f \in L^2
\]

where for \( k \geq 2 \), \( W_k(z^n) = \begin{cases} z^{n/k} & \text{if } n \text{ is divisible by } k, \\ 0 & \text{otherwise.} \end{cases} \)

In this paper, the rationalization of all such operators is given and a new notion of Rationalized (Toeplitz-Hankel) operator of order \((k_1, k_2)\) on the space \( L^2 \) is introduced.

As the commutant of the bilateral shift is the set of all multiplications on \( L^2 \). M. C. Ho [7] proved that \( A \) is slant Toeplitz operator if and only if \( M_z A = AM_z \).

Further, in [1] it has been proved that \( A \) is a slant Hankel operator if and only if \( M_z A = AM_z \). That is \( A = W_2 M_\varphi \).

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Also in [3] it has been proved that \( A \) is a \( k \)th order slant Toeplitz operator if and only if \( M_z A = AM_z^k \).

Further it has been proved that \( A \) is a \( k \)th order slant Hankel operator if and only if \( M_z A = AM_z^k \).
Motivated by all these, we give rationalization of these operators and generalized the characteristic equations of these operators $M_{z^{k_2}}A = AM_{z^{k_1}}$ where $k_1$ and $k_2$ are any non-zero integers. In this paper, we have proved that for any non-zero integers $k_1$ and $k_2$, $M_{z^{k_2}}R = RM_{z^{k_1}}$ if and only if

$$A/\bar{N}_i = W_m \bar{M}_{\bar{\varphi}_i} W_n^*/\bar{N}_i$$

where $L^2 = \bar{N}_0 \oplus \bar{N}_1 \oplus \ldots \bar{N}_{d-1}$, $k_1 = dm$, $k_2 = dn$, $d =$ greatest common divisor of $k_1$ and $k_2$.

If $k_1$ and $k_2$ are relatively prime then a bounded operator $R$ an $L^2$ is Rationalized (Toeplitz-Hankel) operator of order $(k_1, k_2)$ if and only if $R = W_{k_1} M_{\varphi} W_{k_2}^*$. That is, if and only if

$$M_{z^{k_2}}R = RM_{z^{k_1}}.$$

We also discuss some properties of the operator $W_k$ and find spectral values of $W_k$. Precisely we prove

$$\sigma(W_k) = \sigma_a(W_k) = \sigma_e(W_k) = \mathbb{D},$$

$$\sigma_p(W_k) = \mathbb{D} \cup \{1\}$$

$$\sigma_c(W_k) = \{1\}$$

where $\mathbb{D}$, $\mathbb{D}$, $\sigma$, $\sigma_a$, $\sigma_e$, $\sigma_p$, $\sigma_c$ denote the unit disc, the closed unit disc, spectrum, the approximate point spectrum, the essential spectrum, the point spectrum, the compression spectrum, respectively. Let us start with the following:

**Definition 1.1.** For any non-zero integer $k$. Define

$$W_k : L^2 \to L^2$$

as

$$W_k(z^i) = \begin{cases} \frac{z^i}{k}, & i \text{ is divisible by } k \\ 0, & \text{otherwise} \end{cases}$$


2. Properties of $W_k$

**Theorem 2.1.** (i) For any non-zero integer $k$, we have

$$\|W_k\| = 1$$

**Proof.** As

$$\|W_k f\|^2 = \left\| \sum_{i=\infty}^{\infty} a_{ki} z^i \right\|^2$$

$$= \sum_{i=\infty}^{\infty} |a_{ki} z^i|^2$$

$$\leq \sum_{i=\infty}^{\infty} |a_i|^2 = \|f\|^2$$

and $\|W_k z^0\| = 1$, where $f(z) = \sum_{i=\infty}^{\infty} a_i z^i$ is the Fourier expansion of $f$. Hence $\|W_k\| = 1$

(ii) $W_k^*(z^i) = z^{ki}$ for any $i \in \mathbb{Z}$.
Proof. For \( f, g \in L^2 \), we have
\[
(W_k^* f, g) = \langle f, W_k g \rangle = \langle f, g_0 \rangle
\]
where \( g(z) = g_0(z^k) + zg_1(z^k) + \cdots + z^{k-1}g_{k-1}(z^k) \) and \( g_0, g_1, \ldots, g_{k-1} \) are in \( L^2 \). So
\[
(W_k^* f, g) = \langle f(z^k), g_0(z^k) \rangle = \langle f(z^k), g_0(z^k) + zg_1(z^k) + \cdots + z^{k-1}g_{k-1}(z^k) \rangle = \langle f(z^k), g \rangle.
\]
Thus \( W_k^*(f) = f(z^k) \) and therefore \( W_k^*(z^j) = z^{kj} \), for any \( j \in \mathbb{Z} \).

(iii) \( M_zW_k = W_kM_z \)

Proof. If \( m \) is divisible by \( k \) then \( m = ki \) for some \( i \) in \( \mathbb{Z} \). Then
\[
M_zW_k(z^m) = M_zz^i = z^{i+1},
W_kM_z(z^m) = W_kz^{m+k} = z^{i+1}.
\]
If \( m \) is not divisible by \( k \), then
\[
M_zW_k(z^m) = 0,
W_kM_z(z^m) = W_k(z^{m+k}) = 0.
\]
Hence \( M_zW_k = W_kM_z \).

(iv) \( W_kM_\varphi W_k^* = M_{W_k\varphi} \) and hence \( W_k\varphi \in L^\infty \).

Proof. For \( i, j \in \mathbb{Z} \)
\[
\langle W_kM_\varphi W_k^* z^j, z^i \rangle = \langle \varphi z^{kj}, z^{ki} \rangle = \langle \varphi, z^{k(i-j)} \rangle
\]
\[
\langle M_{W_k\varphi} z^j, z^i \rangle = \langle z^j W_k\varphi, z^i \rangle = \langle W_k\varphi, z^{i-j} \rangle = \langle \varphi, z^{k(i-j)} \rangle
\]
Therefore \( W_kM_\varphi W_k^* = M_{W_k\varphi} \) and hence \( [6] \) \( W_k\varphi \in L^\infty \).

Remark 2.2. From above, we note that \( W_k^*(z^i) = z^{ki} \) for any \( i \in \mathbb{Z} \). In fact some other properties are true for all \( k \in \mathbb{Z}, k \neq 0 \).

(v) \( W_kW_k^* = I \)
\( W_k^*W_k = P_k \) where \( P_k \) is the projection of \( L^2 \) onto the closed linear span of \( \{ z^{ki} : i \in \mathbb{Z} \} \)

(vi) \( W_k^*(\varphi g) = (W_k^*\varphi)(W_k^*g) \)

(vii) \( W_k\varphi g = (W_k\varphi)(W_kg) + W_k((I - P_k)\varphi)(I - P_k)g) \).

In particular,
\[
W_k(\varphi(z^k)g) = \varphi W_k g,
W_k(\varphi(z)g(z^k)) = gW_k\varphi.
\]
Theorem 2.3. For any integer $k$ different from $0, 1, -1$ we have the $\sigma_p(W_k) = D \cup \{1\}$.

Proof. Since $W_k(z^i) = \begin{cases} z^{i/k} & \text{if } i \text{ is divisible by } k \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \quad W_k(z^{ki+1}) = 0 \quad \forall \ i \in \mathbb{Z}.$$ 

This follows that $0$ is an eigen value of $W_k$ and it is of infinite multiplicity.

For $0 < |\lambda| < 1$ and for each $n$ in $\mathbb{Z}$, define a function

$$f_n(z) = \sum_{i=0}^{\infty} \lambda^i z^{k(i+1)}$$

As $|f_n(z)| \leq \frac{1}{1-|\lambda|} < \infty$, so for each $n$, $f_n$ is in $L^2$ and it is nonzero. Now

$$W_k f_n = \sum_{i=1}^{\infty} \lambda^i z^{k(i+1)} = \lambda f_n.$$ 

This implies $\lambda \in \sigma_p(W_k)$. Since $W_k z^0 = z^0$, we have $1 \in \sigma_p(W_k)$. Therefore $D \cup \{1\} \subseteq \sigma_p(W_k)$.

We claim that if $|\lambda| = 1$ and $\lambda \neq 1$ then $\lambda \notin \sigma_p(W_k)$. Because if $\lambda \in \sigma_p(W_k)$, then $W_k f = \lambda f$, for some nonzero $f$ in $L^2$. If $\sum_{i=-\infty}^{\infty} a_i z^i$ is the Fourier expansion of $f$, then we have

$$W_k \left( \sum_{i=-\infty}^{\infty} a_i z^i \right) = \lambda \sum_{i=-\infty}^{\infty} a_i z^i.$$ 

This implies that

$$a_0 = \lambda a_0 \text{ and } a_{k+n} = \lambda a_n \text{ for } n = \pm 1, \pm 2, \pm 3.$$ 

This implies $a_n \neq 0$ for some $n$, otherwise $f$ would be zero.

Hence since $a_i \to 0$ as $i \to \infty$, it follows that $|\lambda| < 1$. Consequently

$$\sigma_p(W_k) = D \cup \{1\}.$$ 

This completes the proof.

Remark 2.4. Since by [6], we have

$$\sigma_c(W_k) = \overline{\sigma_p(W_k^*)}$$ 

it follows that $\lambda \in \sigma_c(W_k)$ if and only if $|\lambda| = 1$ (as $W_k^*$ is an isometry). Since $W_k^* z^0 = z^0$, we have $1 \in \sigma_p(W_k^*)$. Here we can also see that if $|\lambda| = 1$ and $\lambda \neq 1$ then $\lambda \notin \sigma_p(W_k^*)$.

If $\lambda \in \sigma_p(W_k^*)$. Then $W_k^* f = \lambda f$ for some nonzero $f$ in $L^2$. If $\sum a_i z^i$ is the Fourier expansion of $f$ then $W_k^* f = \lambda f$ gives as $a_i = \lambda a_{ki}$ and $a_{ki-n} = 0 \forall n = 1, 2, 3, \ldots, k - 1$. So proceeding as above as we get $\lambda \notin \sigma_p(W_k^*)$. This yields to the following
Theorem 2.5. For $k \neq 0, 1, -1$, $\sigma_e(W_k) = \{1\}$.

Remark 2.6. Since the boundary of a spectrum is contained in the approximate point spectrum $[6]$, it follows that $\sigma_a(W_k) = \overline{D} = \sigma(W_k)$. Since for $|\lambda| < 1$, $\langle f_k, f_l \rangle = 0$ for $k \neq l$ we get $\lambda$ is of infinite multiplicity and hence $\lambda \in \sigma_e(W_k)$.

Also as the essential spectrum is closed we have $\sigma_e(W_k) = \overline{D}$.

Remark 2.7. For $k = 1$, $W_k = I$. All the spectral parts of the spectrum of $I = \{1\}$. If $k = -1$ then $W_{-1}^2 = I$ and $W_{-1}^* = W_{-1}$

$W_{-1}(z^i + z^{-i}) = z^i + z^{-i}$

$W_{-1}^*(z^i - z^{-i}) = -(z^i - z^{-i}) \quad \forall \ i \in \mathbb{Z}$

Thus the spectral parts of $W_{-1} = \{1, -1\}$.

3. Rationalized Toeplitz-Hankel operators

We define a Rationalized (Toeplitz-Hankel) operator of order $(k_1, k_2)$ on the space $L^2$ as follows. For $\varphi \in L^\infty$, $R_\varphi : L^2 \to L^2$ as $R_\varphi(f) = W_{k_1}M_\varphi W_{k_2}^*(f)$ for $f \in L^2$, where $k_1$ and $k_2$ are nonzero integers and $M_\varphi$ is the multiplication operator on $L^2$. We denote the set of all Rationalized Toeplitz Hankel operators on $L^2$ as $RTHO(L^2)$.

Thus $R_\varphi$ is the Rationalization of all Toeplitz Hankel operators, due to the following.

(i) If $k_1 = k_2 = 1$ then

$R_\varphi f = W_1M_\varphi W_1^*$

$= M_\varphi f \quad \forall \ f \in L^2$

$\Rightarrow \ R_\varphi = M_\varphi$, multiplication operator on $L^2$

(ii) If $k_1 = -1, k_2 = 1$ then

$R_\varphi f = W_{-1}M_\varphi W_1^*f$

$= W_{-1}M_\varphi f$ which is a Hankel operator on $L^2$

$R_\varphi = W_{-1}M_\varphi$

(iii) If $k_1 = 2, k_2 = 1$ then $R_\varphi = W_2M_\varphi$ which is a slant Toeplitz operator

(iv) If $k_1 = -2$ and $k_2 = 1$ then $R_\varphi = W_2M_\varphi$ which is a slant Hankel operator

(v) If $k_1 = k$ then $R_\varphi$ is $k$th order slant Toeplitz operator

(vi) If $k_1 = -k$ then $R_\varphi$ is $k$th order slant Hankel operator
So, for non zero integers \(k_1, k_2\), we give the characterisation of Rationalized (Toeplitz-Hankel) operators as follows:

**Theorem 3.1.** Let \(R\) be bounded operator on \(L^2\). Then for a given relatively prime integers \(k_1\) and \(k_2\), \(M_{z^{k_2}}R = RM_{z^{k_1}}\) if and only if \(R = W_{k_1}M_\varphi W_{k_2}^*\). That is \(R\) is a Rationalized (Toeplitz-Hankel) operator if and only if \(M_{z^{k_2}}R = RM_{z^{k_1}}\).

**Proof.** Suppose that \(R = W_{k_1}M_\varphi W_{k_2}^*\). Then from the properties of \(W_k\) we get

\[
\begin{align*}
M_{z^{k_2}}(W_{k_1}M_\varphi W_{k_2}^*) &= W_{k_1}M_{z^{k_1}k_2}M_\varphi W_{k_2}^* \\
&= W_{k_1}M_\varphi M_{z^{k_1}k_2}W_{k_2}^* \\
&= (W_{k_1}M_\varphi W_{k_2}^*)M_{z^{k_1}}.
\end{align*}
\]

So, \(M_{z^{k_2}}R = RM_{z^{k_1}}\).

Conversely, let \(k_1\) and \(k_2\) be positive integers and let \(f \in L^2\) and \(\sum_{i=-\infty}^{\infty} a_i z^i\) be its Fourier expansion, where \(\{z^i : i \in \mathbb{Z}\}\) is an orthonormal basis of \(L^2\). Then as we have \(M_{z^{k_2}}R = RM_{z^{k_1}}\), so we get

\[
R(f(z^{k_1})) = R\left(\sum_{i=-\infty}^{\infty} a_i z^{k_1i}\right)
= \sum_{i=-\infty}^{\infty} a_i R z^{k_1i}
= \sum_{z=-\infty}^{\infty} a_i z^{k_2i} R1
= f(z^{k_2}) R1, \quad R1 = R z^0
\]

This implies that

\[
\|f(z^{k_2}) R1\| = \|R(f(z^{k_1}))\| \leq \|R\| \|f(z^{k_1})\| = \|R\| \|f(z^{k_2})\|
\]

**Claim.** \(R1\) is bounded.

Let \(K_2 = \{f(z^{k_2}) : f \in L^2\}\) and \(B : K_2 \to L^2\) be such that

\[
B(f(z^{k_2})) = f(z^{k_2}) \varphi_0
\]

where \(\varphi_0 = R1\). Then \(B\) is bounded. If possible, let \(N\) be a set of positive measure on which \(|\varphi_0| > \|B\|\) and \(f(z^{k_2})\) is the characteristic function of \(N\). Then

\[
\|B(f(z^{k_2}))\|^2 = \int_T |\varphi_0(z)f(z^{k_2})|^2 d\mu
= \int_N |\varphi_0|^2 d\mu
> \|B\|^2 \mu(N)
= \|B\|^2 \|f(z^{k_2})\|^2
\]
This contradicts that $B$ is bounded. Hence $\mu(N) = 0$. This implies that $|\varphi_0| \leq \|B\|$ almost everywhere.

Proceeding with the same arguments, we get

$$R(zf(z^{k_1})) = f(z^{k_2})Rz$$

$$R(z^2f(z^{k_1})) = f(z^{k_2})Rz^2$$

$$\vdots$$

$$R(z^{k_1-1}f(z^{k_1})) = f(z^{k_2})Rz^{k_1-1}$$

$\Rightarrow Rz, Rz^2, \ldots, Rz^{k_1-1}$ are all bounded.

We denote $R1 = \varphi_0, Rz = \varphi_1, Rz^{k_1-1} = \varphi_{k_1-1}$, and

$$\varphi(z) = \varphi_0(z^{k_1}) + z^{-2k_2}\varphi_1(z^{k_1}) + z^{-2k_2}\varphi_2(z^{k_1}) + \cdots + z^{-(k_1-1)k_2}\varphi_{k_1-1}(z^{k_1})$$

Then $\varphi$ is bounded. Also $f$ in $L^2$ can be written as

$$f(z) = f_0(z^{k_1}) + zf_1(z^{k_1}) + z^2f_2(z^{k_2}) + \cdots + z^{k_1-1}f_{k_1-1}(z^{k_1})$$

where $f_0, f_1, \ldots, f_{k_1-1}$ are in $L^2$. This implies that

$$W_{k_1}M_{\varphi}W_{k_2}^*f = \varphi_0(z)f_0(z^{k_2}) + \varphi_1(z)f_1(z^{k_2}) + \cdots + \varphi_{k_1-1}(z)f_{k_1-1}(z^{k_2})$$

(others are eliminated)

$$= f_0(z^{k_2})R1 + f_1(z^{k_2})Rz + \cdots + f_{k_1-1}(z^{k_2})Rz^{k_1-1}$$

$$= R(f_0(z^{k_1}) + R(zf_1(z^{k_1})) + \cdots + R(z^{k_1-1}f_{k_1-1}(z^{k_1}))$$

$$= Rf$$

Thus $W_{k_1}M_{\varphi}W_{k_2}^*f = Rf$, when $k_1$ and $k_2$ are positive integers.

Now, when one or both of the integers are negative. We observe that for positive integers $k_1$ and $k_2$

$$M_{z^{-k_2}}R = RM_{z^{k_1}} \quad \text{if and only if} \quad M_{z^{k_2}}W_{-1}R = W_{-1}RM_{z^{k_1}}$$

Thus from this and above, we get

$$W_{-1}R = W_{k_1}M_{\varphi}W_{k_2}^*$$

$\Rightarrow \quad R = W_{-k_1}M_{\varphi}W_{k_2}^*$$

$$= W_{k_1}M_{\varphi(z)}W_{-k_2}^*$$

Similarly

$$M_{z^{k_2}}R = RM_{z^{-k_1}} \quad \text{if and only if} \quad M_{z^{k_2}}RW_{-1} = RW_{-1}M_{z^{k_1}}$$

and

$$M_{z^{-k_2}}R = RM_{z^{-k_1}} \quad \text{if and only if} \quad M_{z^{k_2}}W_{-1}RW_{-1} = W_{-1}RW_{-1}M_{z^{k_1}}$$

Therefore by same arguments

$$R = W_{k_1}M_{\varphi}W_{-k_2}^*$$

$$= W_{-k_1}M_{\varphi(z)}W_{-k_2}^*$$

and

$$R = W_{-k_1}M_{\varphi}W_{-k_2}^*$$
This completes the proof.

Remark 3.2. However if the integers \( k_1 \) and \( k_2 \) are not relatively prime then what would be the line of action? To answer this, we proceed as follows:

If \( k_1 \) and \( k_2 \) have the greatest common divisor as \( d \). Let \( k_1 = dn \) and \( k_2 = dm \). Then if \( N_i = \) The closed linear span of \( \{z^{k_1t+i} : t \in \mathbb{Z}\} \) for \( i = 0, 1, 2, \ldots, k_1 - 1 \)

\[
L^2 = N_0 \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_{k_1-1}
\]

Let

\[
\tilde{N}_0 = N_0 \oplus N_1 \oplus \cdots \oplus N_{m-1}
\]

\[
\tilde{N}_1 = N_m \oplus N_{m+1} \oplus \cdots \oplus N_{2m-1}
\]

\[
\tilde{N}_{d-1} = N_{(d-1)m} \oplus N_{(d-1)m+1} \oplus \cdots \oplus N_{dm-1}
\]

Then

\[
L^2 = \tilde{N}_0 \oplus \tilde{N}_1 \oplus \cdots \oplus \tilde{N}_{d-1}
\]

Now we are ready to generalize the above result as follows for any non zero integers.

**Theorem 3.3.** Let \( R \) be a bounded operator on \( L^2 \). Given a pair of nonzero integers \( k_1 \) and \( k_2 \), then \( M_{z^{k_2}}R = R M_{z^{k_1}} \) if and only if

\[
R/\tilde{N}_i = W_m M_{\tilde{\varphi}_i} W_n^* / \tilde{N}_i
\]

for same \( \tilde{\varphi}_i \) in \( L^\infty \) and for \( i = 0, 1, 2, \ldots, d - 1 \) where \( k_2 = dn \) and \( k_2 = dm \) and \( d \) is the greatest common divisor of \( k_1 \) and \( k_2 \).

**Proof.** Since

\[
M_{z^{k_2}}(W_m M_{\tilde{\varphi}_i} W_n^*) = W_m M_{z^{k_2m}} M_{\tilde{\varphi}_i} W_n^* \\
= (W_m M_{\tilde{\varphi}_i} W_n^*) M_{z^{k_1}}
\]

Conversely, let \( R \) be a bounded operator on \( L^2 \) such that \( M_{z^{k_2}}R = R M_{z^{k_1}} \). Then for \( f \in L^2 \), we have for \( i = 0, 1, 2, \ldots, k_1 - 1 \)

\[
R(z^i(f(z^{k_1}))) = f(z^{k_2})Rz^i
\]

Let \( \varphi_0 = R1, \varphi_1 = Rz \cdots \varphi_{k_1-1} = Rz^{k_1-1} \). Then by similar arguments as in previous theorem \( \varphi_0, \varphi_1, \ldots, \varphi_{k_1-1} \) are bounded.

Let

\[
\tilde{\varphi}_0(z) = \varphi_0(z) + z \varphi_1(z^m) + \cdots + z^{(m-1)n} \varphi_{m-1}(z^m)
\]

\[
\tilde{\varphi}_1(z) = \tilde{\varphi}_0(z) + z^{m+1} \varphi_{m+1}(z^m) + \cdots + z^{(2m-1)n} \varphi_{2m-1}(z^m)
\]

\[
\tilde{\varphi}_{d-1}(z) = \tilde{\varphi}_{d-2}(z) + z^{(d-m)n} \varphi_{(d-1)m}(z^m) + \cdots + z^{(dm-1)n} \varphi_{dm-1}(z^m)
\]

Let \( f \in \tilde{N}_0 \). Then \( f \) can be written as

\[
f(z) = f_0(z^{k_1}) + zf_1(z^{k_1}) + \cdots + z^{m-1}f_{m-1}(z^{k_1})
\]

where \( f_0, f_1 = f_{m-1} \in L^2 \). Now

\[
W_m M_{\tilde{\varphi}_0} W_n^*(f) = W_m(\tilde{\varphi}_0(z)f(z^n))
\]
\[ = W_m[(\varphi_0(z^m) + z^n\varphi_1(z^m) + \cdots + z^{(m-1)n}\varphi_{m-1}(z^m))(f_0(z^{k_1n}) + z^n f_1(z^{k_1n}) + \cdots + z^{n(m-1)})
\]
\[ = W_m(\varphi_0(z^m)f_0(z^{k_1n})) + W_m(\varphi_1(z^m)f_1(z^{k_1n})) + \cdots + W_m(\varphi_{m-1}(z^m)f_{m-1}(z^{k_1n}))
\]
\[ = f_0(z^{k_2})\varphi_0(z) + f_1(z^{k_2})\varphi_2(z) + \cdots + f_{m-1}(z^{k_2})\varphi_{m-1}(z)
\]
\[ = f_0(z^{k_2})R1 + f_1(z^{k_2})Rz + \cdots + f_{m-1}(z^{k_2})Rz^{m-1}
\]
\[ = R(f_0(z^{k_1})) + R(f_1(z^{k_1})) + \cdots + R(z^{m-1}f_{m-1}(z^{k_1}))
\]
\[ = Rf
\]

Thus
\[ A|\tilde{N}_0 = W_m M\tilde{\varphi} W^*_n|\tilde{N}_0
\]

Similarly for \( i = 1, 2, 3, \ldots, d-1 \), we get
\[ A/\tilde{N}_i = W_m M\tilde{\varphi}_i W^*_n/\tilde{N}_i
\]

This completes the proof.

**Remark 3.4.** If \( k_1 \) and \( k_2 \) are relatively prime then \( d = 1 \) and hence \( \tilde{N}_0 = L^2 \) and
\[ \tilde{\varphi}_0(z) = \varphi_0(z^{k_1}) + z^{k_2}\varphi_1(z^{k_1}) + \cdots + z^{(k_1-1)k_2}\varphi_{k_1-1}(z^{k_1})
\] which gives us the previous result. Thus we can define Rationalized (Toeplitz-Hankel) operator for any non zero integers \( k_1 \) and \( k_2 \).

**Remark 3.5.** From here we can see that the matrix of Rationalized (Toeplitz-Hankel) operator of order \((k_1,k_2)\) on the space \( L^2 \) with respect to the basis \( \{z^i : i \in \mathbb{Z}\} \) is a two way infinite matrix \((\alpha_{ij})\) such that
\[ \alpha_{ij} = (i,j)\text{th element of the matrix of } R_{\tilde{\varphi}}
\]
\[ = \langle R_{\tilde{\varphi}}z^j, z^i \rangle
\]
\[ = \langle W_{k_1} M_{\varphi} W_{k_2}^* z^j, z^i \rangle
\]
\[ = \langle W_{k_1} M_{\varphi} z^{k_2 j}, z^i \rangle
\]
\[ = \langle M_{\varphi} z^{k_2 j}, z^{k_1 i} \rangle
\]
\[ = \langle \varphi, z^{k_1 i - k_2 j} \rangle
\]
\[ = a_{k_1 i - k_2 j}
\]

i.e. the \((i,j)\)th element of the matrix of Rationalized (Toeplitz-Hankel) operator is \( a_{k_1 i - k_2 j} \), where \( \sum a_i z^i \) is the Fourier expansion of \( \varphi \).
We also observe the following result

**Theorem 3.6.** For fixed integers $k_1$ and $k_2$ the set $RTHO(L^2)$ of all Rationalized (Toeplitz-Hankel) operators of order $(k_1, k_2)$ is weakly closed and hence strongly closed.

**Proof.** Suppose that for each $\alpha$, $R_\alpha$ is a Rationalized (Toeplitz-Hankel) operator and $R_\alpha \to R$ weakly, where $\{\alpha\}$ being a net. Then for $f, g \in L^2$

$$\langle R_\alpha f, g \rangle \to \langle Rf, g \rangle .$$

This implies that

$$\langle M_{z^{k_2}} R_\alpha M_{z^{k_1}} f, g \rangle = \langle R_\alpha z^{k_1} f, z^{k_2} g \rangle \to \langle R z^{k_1} f, z^{k_2} g \rangle = \langle M_{z^{k_2}} R_\alpha M_{z^{k_1}} f, g \rangle$$

But

$$M_{z^{k_2}} R_\alpha M_{z^{k_1}} = R_\alpha \quad \text{for each } \alpha .$$

Therefore $R = M_{z^{k_2}} R M_{z^{k_1}}$. Consequently $R$ is a Rationalized (Toeplitz-Hankel) operator. This completes the proof.

**References**


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