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RATIONALIZATION OF TOEPLITZ-HANKEL OPERATORS

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Abstract. In this paper, we introduce and study the rationalization of all kinds of Toeplitz and Hankel operators on the space $L^2(T)$, T being the unit circle. Firstly, we define the Rationalized (Toeplitz-Hankel) operator of order (k_1, k_2) on the space L^2, for $\varphi \in L^\infty$, as

$$R_\varphi(TH)(f) = W_{k_1} M_\varphi W^*_{k_2}(f), \quad \forall\, f \in L^2$$

where k_1 and k_2 are nonzero integers and $M_\varphi : L^2 \to L^2$ is the multiplication operator. For nonzero $k \in \mathbb{Z}$, W_k is an operator on L^2 defined as

$$W_k(z^i) = \begin{cases} z^{\frac{i}{k}}, & i \text{ is divisible by } k \\ 0 & \text{otherwise} \end{cases}$$

We get some properties and spectral values of the operator W_k. It is proved that if k_1 and k_2 are relatively prime integers then a bounded operator R on L^2 is Rationalized (Toeplitz-Hankel) operator of order (k_1, k_2) on the space L^2 if only if $M_{z^{k_2}} R = R M_{z^{k_1}}$. Further if k_1 or k_2 are not relatively prime then also we give a characterization.

1. Introduction

The study of Toeplitz operators began almost in the beginning of 20th century by O. Toeplitz. A lot of work on Toeplitz operators has been done by different mathematicians in the world. Toeplitz operators and Hankel operators became a subject of investigations for the researchers. Motivated by these, M. C. Ho [7] in the year 1995 introduced slant Toeplitz operators and later the notion of slant Hankel operators was introduced. Further, these notions have been generalized [2] to kth order slant Toeplitz and slant Hankel operators and studied simultaneously. Let us recall the following:

The place $L^2(T) = L^2$ stands for the collection of all complex valued Lebesgue measurable functions on the unit circle that are square integrable. Thus

$$L^2 = \left\{ f : f \text{ complex valued measurable function on } T : \int_T |f|^2 < \infty \right\}$$

L^2 in a Hilbert space with $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g}(e^{i\theta})d\theta,\ f, g \in L^2$, where $\overline{g}$ in complex conjugate of g.

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The set $\{z^n = e^{in\theta} : n \in \mathbb{Z}\}$ is an orthonormal basis of L^2. If $f \in L^2$, the complex numbers

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\, d\theta$$

are called Fourier Coefficients of f and $\sum\limits_{n=-\infty}^{\infty} a_n e^{in\theta}$ is called Fourier expansion of f. We define the space $H^2(T)$ as

$$H^2 = \left\{ f \in L^2 : \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\, d\theta = 0 \text{ for } n < 0 \right\}$$

Elements of H^2 are called analytic functions.

The space $L^\infty = L^\infty(T)$ is the collection of all essentially bounded complex valued measurable functions on the unit circle.

The multiplication operator M_φ, $\varphi \in L^\infty$ on L^2 is defined as $M_\varphi f = \varphi f$ $\forall$ $f \in L^2$. In [7] M. C. Ho defined the slant Toeplitz operator on the space L^2 as $A_\phi : L^2 \to L^2$ as

$$A_\varphi f = W M_\varphi f \quad \forall\ f \in L^2$$

where

$W : L^2 \to L^2$ is the operator defined as
$W(z^{2n}) = z^n$ $\forall$ $n \in \mathbb{Z}$
$W(z^{2n+1}) = 0$

In [4] the slant Hankel operator on L^2 is defined as $K_\varphi : L^2 \to L^2$ as

$$K_\varphi f = J A_\varphi f$$

where J is the reflection operator on L^2.

The notion of Generalized slant Toeplitz operator on L^2 [2] was introduced as

$$U_\varphi : L^2 \to L^2$$
$$U_\varphi(f) = W_k M_\varphi f \quad \forall\ f \in L^2$$

where for $k \geq 2$, $W_k(z^n) = \begin{cases} z^{n/k} & \text{if } n \text{ is divisible by } k, \\ 0 & \text{otherwise.} \end{cases}$

In this paper, the rationalization of all such operators is given and a new notion of Rationalized (Toeplitz-Hankel) operator of order (k_1, k_2) on the space L^2 is introduced.

As the commutant of the bilateral shift is the set of all multiplications on L^2. M. C. Ho [7] proved that A is slant Toeplitz operator if and only if $M_z A = A M_{z^2}$. That is $A = W_2 M_\varphi$.

Further, in [1] it has been proved that A is a slant Hankel operator if and only if $M_{\overline{z}} A = A M_{z^2}$. That is $A = W_{-2} M_\varphi$.

Also in [3] it has been proved that A is a kth order slant Toeplitz operator if and only if $M_z A = A M_{z^k}$.

Further it has been proved that A is a kth order slant Hankel operator if and only if

$$M_{\overline{z}} A = A M_{z^k}.$$

Motivated by all these, we give rationalization of these operators and generalized the characteristic equations of these operators $M_{z^{k_2}}A = AM_{z^{k_1}}$ where k_1 and k_2 are any non zero integers. In this paper, we have proved that for any non zero integers k_1 and k_2, $M_{z^{k_2}}R = RM_{z^{k_1}}$ if and only if

$$A/\widetilde{N}_i = W_m M_{\widetilde{\varphi}_i}\ W_n^*/\widetilde{N}_i$$

where $L^2 = \widetilde{N}_0 \oplus \widetilde{N}_1 \oplus \cdot \oplus \widetilde{N}_{d-1}$, $k_1 = dm$, $k_2 = dn$, $d =$ greatest common divisor of k_1 and k_2.

If k_1 and k_2 are relatively prime then a bounded operator R an L^2 is Rationalized (Toeplitz-Hankel) operator of order (k_1, k_2) if and only if $R = W_{k_1}M_\varphi W_{k_2}^*$. That is, if and only if

$$M_{z^{k_2}}R = RM_{z^{k_1}}.$$

We also discuss some properties of the operator W_k and find spectral values of W_k. Precisely we prove

$$\begin{aligned}
&\sigma(W_k) = \sigma_a(W_k) = \sigma_e(W_k) = \overline{D},\\
&\sigma_p(W_k) = D \cup \{1\}\\
&\sigma_c(W_k) = \{1\}
\end{aligned}$$

where D, $\overline{D}$, σ, σ_a, σ_e, σ_p, σ_c denote the unit disc, the closed unit disc, spectrum, the approximate point spectrum, the essential spectrum, the point spectrum, the compression spectrum, respectively. Let us start with the following:

Definition 1.1. For any nonzero integer k. Define

$$W_k : L^2 \to L^2 \text{ as}$$

$$W_k(z^i) = \begin{cases} z^{i/k} & i \text{ is divisible by } k\\ 0 & \text{otherwise}\end{cases}$$

2. Properties of W_k

Theorem 2.1. *(i) For any non zero integer k, we have*

$$\|W_k\| = 1$$

Proof. *As*

$$\begin{aligned}
\|W_k f\|^2 &= \left\|\sum_{i=\infty}^{\infty} a_{ki}z^i\right\|^2\\
&= \sum |a_{ki}z^i|^2\\
&\le \sum |a_i|^2 = \|f\|^2
\end{aligned}$$

and $\|W_k z^0\| = 1$, where $f(z) = \sum\limits_{i=\infty}^{\infty} a_i z^i$ is the Fourier expansion of f. Hence $\|W_k\| = 1$

(ii) $W_k^*(z^i) = z^{ki}$ for any $i \in Z$.

Proof. *For $f, g \in L^2$, we have*

$$\langle W_k^* f, g\rangle = \langle f, W_k g\rangle = \langle f, g_0\rangle$$

where $g(z) = g_0(z^k) + zg_1(z^k) + \cdot + z^{k-1}g_{k-1}(z^k)$ and $g_0, g_1, \ldots, g_{k-1}$ are in L^2. So

$$\begin{aligned}\langle W_k^* f, g\rangle &= \langle f(z^k), g_0(z^k)\rangle \\ &= \langle f(z^k), g_0(z^k) + zg_1(z^k) + \cdot + z^{k-1}g_{k-1}(z^k)\rangle \\ &= \langle f(z^k), g\rangle.\end{aligned}$$

Thus $W_k^*(f) = f(z^k)$ and therefore $W_k^*(z^j) = z^{kj}$, for any $j \in \mathbb{Z}$.

(iii) $M_z W_k = W_k M_{z^k}$

Proof. *If m is divisible by k then $m = ki$ for some i in $\mathbb{Z}$. Then*

$$\begin{aligned}M_z W_k(z^m) &= M_z z^i = z^{i+1}, \\ W_k M_{z^k}(z^m) &= W_k z^{m+k} = z^{i+1}\,.\end{aligned}$$

If m is not divisible by k, then

$$\begin{aligned}M_z W_k(z^m) &= 0, \\ W_k M_{z^k}(z^m) &= W_k(z^{m+k}) = 0\,.\end{aligned}$$

Hence $M_z W_k = W_k M_{z^k}$.

(iv) $W_k M_\varphi W_k^* = M_{W_k\varphi}$ and hence $W_k\varphi \in L^\infty$.

Proof. *For $i, j \in \mathbb{Z}$*

$$\begin{aligned}\langle W_k M_\varphi W_k^* z^j, z^i\rangle &= \langle \varphi z^{kj}, z^{ki}\rangle \\ &= \langle \varphi, z^{k(i-j)}\rangle \\ \langle M_{W_k\varphi} z^j, z^i\rangle &= \langle z^j W_k\varphi, z^i\rangle \\ &= \langle W_k\varphi, z^{i-j}\rangle \\ &= \langle \varphi, z^{k(i-j)}\rangle\end{aligned}$$

Therefore $W_k M_\varphi W_k^* = M_{W_k\varphi}$ and hence [6] $W_k\varphi \in L^\infty$.

Remark 2.2. From above, we note that $W_k^*(z^i) = z^{ki}$ for any $i = \mathbb{Z}$. In fact some other properties are true for all $k \in \mathbb{Z}$, $k \neq 0$.

(v) $W_k W_k^* = I$

$W_k^* W_k = P_k$ where P_k is the projection of L^2 onto the closed linear span of $\{z^{ki} : i \in \mathbb{Z}\}$

(vi) $W_k^*(\varphi g) = (W_k^*\varphi)(W_u^* g)$

(vii) $W_k\varphi g = (W_k\varphi)(W_k g) + W_k(((I - P_k)\varphi)(I - P_k))g)$.

In particular,

$$\begin{aligned}W_k(\varphi(z^k)g) &= \varphi W_k g, \\ W_k(\varphi(z)g(z^k) &= g W_k\varphi\,.\end{aligned}$$

Theorem 2.3. *For any integer k different from $0, 1, -1$ we have the $\sigma_p(W_k) = D \cup \{1\}$.*

Proof. *Since* $W_k(z^i) = \begin{cases} z^{i/k} & \text{if } i \text{ is divisible by } k \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \qquad W_k(z^{k_1 i+1}) = 0 \quad \forall\ i \in \mathbb{Z}.$$

This follows that 0 is an eigen value of W_k and it is of infinite multiplicity.

For $0 < |\lambda| < 1$ and for each n in $\mathbb{Z}$, define a function

$$f_n(z) = \sum_{i=0}^{\infty} \lambda^i z^{k^i(kn+1)}$$

As $|f_n(z)| \le \frac{1}{1-|\lambda|} < \infty$, so for each n, f_n is in L^2 and it is nonzero. Now

$$\begin{aligned} W_k f_n &= \sum_{i=1}^{\infty} \lambda^i z^{k^{i-1}(kn+1)} \\ &= \lambda f_n\,. \end{aligned}$$

This implies $\lambda \in \sigma_p(W_k)$. Since $W_k z^0 = z^0$, we have $1 \in \sigma_p(W_k)$. Therefore $D \cup \{1\} \subseteq \sigma_p(W_k)$.

We claim that if $|\lambda| = 1$ and $\lambda \ne 1$ then $\lambda \notin \sigma_p(W_k)$. Because if $\lambda \in \sigma_p(W_k)$, then $W_k f = \lambda f$, for some nonzero f in L^2. If $\sum\limits_{i=-\infty}^{\infty} a_i z^i$ is the Fourier expansion of f, then we have

$$W_k\left(\sum_{i=-\infty}^{\infty} a_i z^i\right) = \lambda \sum_{i=-\infty}^{\infty} a_i z^i\,.$$

This implies that

$$a_0 = \lambda a_0 \text{ and } a_{k^i n} = \lambda^i a_n \text{ for } n = \pm 1, \pm 2, \pm 3$$

This implies $a_n \ne 0$ for some n, otherwise f would be zero.

Hence since $a_i \to 0$ as $i \to \infty$, it follows that $|\lambda| < 1$. Consequently

$$\sigma_p(W_k) = D \cup \{1\}.$$

This completes the proof.

Remark 2.4. Since by [6], we have

$$\sigma_c(W_k) = \overline{\sigma_p(W_k^*)}$$

it follows that $\lambda \in \sigma_c(W_k)$ if and only if $|\lambda| = 1$ (as W_k^* is an isometry). Since $W_k^* z^0 = z^0$, we have $1 \in \sigma_p(W_k^*)$. Here we can also see that if $|\lambda| = 1$ and $\lambda \ne 1$ then $\lambda \notin \sigma_p(W_k^*)$.

If $\lambda \in \sigma_p(W_k^*)$. Then $W_k^* f = \lambda f$ for some nonzero f in L^2. If $\sum a_i z^i$ is the Fourier expansion of f then $W_k^* f = \lambda f$ gives as $a_i = \lambda a_{ki}$ and $a_{ki-n} = 0\ \forall$ $n = 1, 2, 3, \ldots, k-1$. So proceeding as above as we get $\lambda \notin \sigma_p(W_k^*)$. This yields to the following

Theorem 2.5. *For $k \neq 0, 1, -1$, $\sigma_c(W_k) = \{1\}$.*

Remark 2.6. Since the boundary of a spectrum is contained in the approximate point spectrum [6], it follows that $\sigma_a(W_k) = \overline{D} = \sigma(W_k)$. Since for $|\lambda| < 1$, $\langle f_k, f_l \rangle = 0$ for $k \neq l$ we get λ is of infinite multiplicity and hence $\lambda \in \sigma_e(W_k)$.

Also as the essential spectrum is closed we have

$$\sigma_e(W_k) = \overline{D}\,.$$

Remark 2.7. For $k = 1$, $W_k = I$. All the spectral parts of the spectrum of $I = \{1\}$. If $k = -1$ then $W_{-1}^2 = I$ and $W_{-1}^* = W_{-1}$

$$\begin{aligned} W_{-1}(z^i + z^{-i}) &= z^i + z^{-i} \\ W_{-1}^*(z^i - z^{-i}) &= -(z^i - z^{-i}) \quad \forall\, i \in \mathbb{Z} \end{aligned}$$

Thus the spectral parts of $W_{-1} = \{1, -1\}$.

3. Rationalized Toeplitz-Hankel operators

We define a Rationalized (Toeplitz-Hankel) operator of order (k_1, k_2) on the space L^2 as follows. For $\varphi \in L^\infty$,

$$\begin{aligned} &R_\varphi : L^2 \to L^2 \text{ as} \\ &R_\varphi(f) = W_{k_1} M_\varphi W_{k_2}^*(f) \end{aligned}$$

for $f \in L^2$, where k_1 and k_2 are nonzero integers and M_φ is the multiplication operator on L^2. We denote the set of all Rationalized Toeplitz Hankel operators on L^2 as $RTHO(L^2)$.

Thus R_φ is the Rationalization of all Toeplitz Hankel operators, due to the following.

(i) If $k_1 = k_2 = 1$ then

$$\begin{aligned} R_\varphi f &= W_1 M_\varphi W_1^* \\ &= M_\varphi f \quad \forall\ f \in L^2 \\ \Rightarrow \quad R_\varphi &= M_\varphi, \text{ multiplication operator an } L^2 \end{aligned}$$

(ii) If $k_1 = -1$, $k_2 = 1$ then

$$\begin{aligned} R_\varphi f &= W_{-1} M \varphi\ W_1^* f \\ &= W_{-1} M_\varphi f \text{ which is a Hankel operator on } L^2 \\ R_\varphi &= W_{-1} M_\varphi \end{aligned}$$

(iii) If $k_1 = 2$, $k_2 = 1$ then $R_\varphi = W_2 M_\varphi$ which is a slant Toeplitz operator

(iv) If $k_1 = -2$ and $k_2 = 1$ then $R_\varphi = W_2 M_\varphi$ which is a slant Hankel operator

(v) If $k_1 = k$ then R_φ is kth order slant Toeplitz operator

(vi) If $k_1 = -k$ then R_φ is kth order slant Hankel operator

So, for non zero integers k_1, k_2, we give the characterisation of Rationalized (Toeplitz-Hankel) operators as follows:

Theorem 3.1. *Let R be bounded operator an L^2. Then for a given relatively prime integers k_1 and k_2, $M_{z^{k_2}}R = RM_{z^{k_1}}$ if and only if $R = W_{k_1}M_\varphi W_{k_2}^*$. That is R is a Rationalized (Toeplitz-Hankel) operator if and only if $M_{z^{k_2}}R = RM_{z^{k_1}}$.*

Proof. *Suppose that $R = W_{k_1}M_\varphi W_{k_2}^*$. Then from the properties of W_k we get*

$$\begin{aligned} M_{z^{k_2}}(W_{k_1}M_\varphi W_{k_2}^*) &= W_{k_1}M_{z^{k_1k_2}}M_\varphi W_{k_2}^* \\ &= W_{k_1}M_\varphi M_{z^{k_1k_2}}W_{k_2}^* \\ &= (W_{k_1}M_\varphi W_{k_2}^*)M_{z^{k_1}}\ . \end{aligned}$$

So, $M_{z^{k_2}}R = RM_{z^{k_1}}$.

Conversely, let k_1 and k_2 be positive integers and let $f \in L^2$ and $\sum\limits_{i=-\infty}^{\infty} a_i z^i$ be its Fourier expansion, where $\{z^i : \ i \in \mathbb{Z}\}$ is an orthonormal basis of L^2. Then as we have $M_{z^{k_2}}R = RM_{z^{k_1}}$, so we get

$$\begin{aligned} R(f(z^{k_1})) &= R\left(\sum_{i=-\infty}^{\infty} a_i z^{k_1 i}\right) \\ &= \sum_{i=-\infty}^{\infty} a_i\, Rz^{k_1 i} \\ &= \sum_{z=-\infty}^{\infty} a_i z^{k_2 i}R1 \\ &= f(z^{k_2})R1, \qquad R1 = Rz^0 \end{aligned}$$

This implies that

$$\begin{aligned} \|f(z^{k_2})R1\| &= \|R(f(z^{k_1}))\| \\ &\le \|R\|\ \|f(z^{k_1})\| \\ &= \|R\|\ \|f(z^{k_2})\| \end{aligned}$$

Claim. *$R1$ is bounded.*

Let $K_2 = \{f(z^{k_2}) : f \in L^2\}$ and $B : K_2 \to L^2$ be such that

$$B(f(z^{k_2})) = f(z^{k_2})\varphi_0$$

where $\varphi_0 = R1$. Then B is bounded. If possible, let N be a set of positive measure on which $|\varphi_0| > \|B\|$ and $f(z^{k_2})$ is the characteristic function of N. Then

$$\begin{aligned} \|B(f(z^{k_2}))\|^2 &= \int_T |\varphi_0(z)f(z^{k_2})|^2 d\mu \\ &= \int_N |\varphi_0|^2 d\mu \\ &> \|B\|^2\mu(\mathbb{N}) \\ &= \|B\|^2\ \|f(z^{k_2})\|^2 \end{aligned}$$

This contradicts that B is bounded. Hence $\mu(N) = 0$. This implies that $|\varphi_0| \leq \|B\|$ almost everywhere.

Proceeding with the same arguments, we get

$$R(zf(z^{k_1})) = f(z^{k_2})Rz$$
$$R(z^2 f(z^{k_1})) = f(z^{k_2})Rz^2$$
$$\vdots$$
$$R(z^{k_1-1} f(z^{k_1})) = f(z^{k_2})Rz^{k_1-1}$$

$\Rightarrow Rz, Rz^2, \ldots, Rz^{k_1-1}$ *are all bounded.*

We denote $R1 = \varphi_0$, $Rz = \varphi_1$, $Rz^{k_1-1} = \varphi_{k_1-1}$, *and*

$$\varphi(z) = \varphi_0(z^{k_1}) + z^{-k_2}\varphi_1(z^{k_1}) + z^{-2k_2}\varphi_2(z^{k_1}) + \cdots + z^{-(k_1-1)k_2}\varphi_{k_1-1}(z^{k_1})$$

Then φ is bounded. Also f in L^2 can be written as

$$f(z) = f_0(z^{k_1}) + zf_1(z^{k_1}) + z^2 f_2(z^{k_2}) + \cdots + z^{k_1-1} f_{k_1-1}(z^{k_1})$$

where $f_0, f_1, \ldots, f_{k_1-1}$ are in L^2. This implies that

$$\begin{aligned} W_{k_1} M_\varphi W_{k_2}^* f &= \varphi_0(z) f_0(z^{k_2}) + \varphi_1(z) f_1(z^{k_2}) + \cdots + \varphi_{k_{1-1}}(z^{k_1}) f_{k_1-1}(z^{k_2}) \\ &\qquad\qquad\qquad\qquad \textit{(others are eliminated)} \\ &= f_0(z^{k_2})R1 + f_1(z^{k_2})Rz + \cdots + f_{k_1-1}(z^{k_2})Rz^{k_1-1} \\ &= R(f_0(z^{k_1}) + R(zf_1(z^{k_1})) + \cdots + R(z^{k_1-1} f_{k_1-1}(z^{k_1})) \\ &= Rf \end{aligned}$$

Thus $W_{k_1} M_\varphi W_{k_2}^* f = Rf$, when k_1 and k_2 are positive integers.

Now, when one or both of the integers are negative. We observe that for positive integers k_1 and k_2

$$M_{z^{-k_2}} R = RM_{z^{k_1}} \quad \textit{if and only if} \quad M_{z^{k_2}}\ W_{-1}R = W_{-1}RM_{z^{k_1}}$$

Thus from this and above, we get

$$\begin{aligned} W_{-1}R &= W_{k_1} M_\varphi W_{k_2}^* \\ \Rightarrow \quad R &= W_{-k_1} M_\varphi W_{k_2}^* \\ &= W_{k_1} M_{\varphi(\overline{z})} W_{-k_2}^* \end{aligned}$$

Similarly

$$M_{z^{k_2}} R = RM_{z^{-k_1}} \quad \textit{if and only if} \quad M_{z^{k_2}} RW_{-1} = RW_{-1} M_{z^{k_1}}$$

and

$$M_{z^{-k_2}} R = RM_{z^{-k_1}} \quad \textit{if annd only if} \quad M_{z^{k_2}} W_{-1} RW_{-1} = W_{-1} RW_{-1} M_{z^{k_1}}$$

Therefore by same arguments

$$\begin{aligned} R &= W_{k_1} M_\varphi W_{-k_2}^* \\ &= W_{-k_1} M_{\varphi(\overline{z})} W_{k_2}^* \end{aligned}$$

and

$$R = W_{-k_1} M_\varphi W_{-k_2}^* .$$

This completes the proof.

Remark 3.2. However if the integers k_1 and k_2 are not relatively prime then what would be the line of action? To answer this, we proceed as follows:

If k_1 and k_2 have the greatest common divisor as d. Let $k_1 = dn$ and $k_2 = dm$.

Then if $N_i =$ The closed linear span of $\{z^{k_1 t+i} : t \in \mathbb{Z}\}$ for $i = 0, 1, 2, \dots, k_1 - 1$

$$L^2 = N_0 \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_{k_1-1}$$

Let

$$\widetilde{N}_0 = N_0 \oplus N_1 \oplus \cdots \oplus N_{m-1}$$
$$\widetilde{N}_1 = N_m \oplus N_{m+1} \cdots \oplus N_{2m-1}$$
$$\widetilde{N}_{d-1} = N_{(d-1)m} \oplus N_{(d-1)m+1} \oplus \cdots \oplus N_{dm-1}$$

Then

$$L^2 = \widetilde{N}_0 \oplus \widetilde{N}_1 \oplus \cdots \oplus \widetilde{N}_{d-1}$$

Now we are ready to generalize the above result as follows for any non zero integers.

Theorem 3.3. *Let R be a bounded operator on L^2. Given a pair of nonzero integers k_1 and k_2, then $M_{z^{k_2}}R = RM_{z^{k_1}}$ if and only if*

$$R/\widetilde{N}_i = W_m M_{\widetilde{\varphi}_i} W_n^*/\widetilde{N}_i$$

for same $\widetilde{\varphi}_i$ in L^∞ and for $i = 0, 1, 2, \dots, d-1$ where $k_2 = dn$ and $k_2 = dm$ and d is the greatest common divisor of k_1 and k_2.

Proof. *Since*

$$\begin{aligned} M_{z^{k_2}}(W_m M_{\widetilde{\varphi}_i} W_n^*) &= W_m M_{z^{k_2 m}} M_{\widetilde{\varphi}_i} W_n^* \\ &= (W_m M_{\widetilde{\varphi}_i} W_n^*) M_{z^{k_1}} \end{aligned}$$

Conversely, let R be a bounded operator on L^2 such that $M_{z^{k_2}}R = RM_{z^{k_1}}$. Then for $f \in L^2$, we have for $i = 0, 1, 2, \dots, k_1 - 1$

$$R(z^i(f(z^{k_1}))) = f(z^{k_2})Rz^i$$

Let $\varphi_0 = R1$, $\varphi_1 = Rz \cdots \varphi_{k_1-1} = Rz^{k_1-1}$.

Then by similar arguments as in previous theorem $\varphi_0, \varphi_1, \dots, \varphi_{k-1}$ are bounded.

Let

$$\widetilde{\varphi}_0(z) = \varphi_0(z^m) + \overline{z}^n \varphi_1(z^m) + \cdots + \overline{z}^{(m-1)n} \varphi_{m-1}(z^m)$$
$$\widetilde{\varphi}_1(z) = \overline{z}^{ml} \varphi_m(z^m) + \overline{z}^{(m+1)l} \varphi_{m+1}(z^m) + \cdots + \overline{z}^{(2m-1)n} \varphi_{2m-1}(z^m)$$
$$\widetilde{\varphi}_{d-1}(z) = \overline{z}^{(d-m)mn} \varphi_{(d-1)m}(z^m) + \cdots + \overline{z}^{(dm-1)n} \varphi_{dm-1}(z^m)$$

Let $f \in \widetilde{N}_0$. Then f can be written as

$$f(z) = f_0(z^{k_1}) + z f_1(z^{k_1}) + \cdots + z^{m-1} f_{m-1}(z^{k_1})$$

where $f_0, f_1 = f_{m-1} \in L^2$. Now

$W_m M_{\widetilde{\varphi}_0} W_n^*(f) = W_m(\widetilde{\varphi}_0(z) f(z^n))$

$$
\begin{aligned}
&= W_m[(\overline{\varphi}_0(z^m) + \overline{z}^n\widetilde{\varphi}_1(z^m) + \cdots + \overline{z}^{(m-1)n}\widetilde{\varphi}_{m-1}(z^m))(f_0(z^{k_1 n}) + z^n f_1(z^{k_1 n}) + \cdots + z^{n(}\\
&= W_m(\varphi_0(z^m) f_0(z^{k_1 n})) + W_m(\varphi_1(z^m) f_1(z^{k_1 n})) + \cdots + W_m(\varphi_{m-1}(z^m) f_{m-1}(z^{k_1 n}))\\
&= f_0(z^{k_2})\varphi_0(z) + f_1(z^{k_2})\varphi_2(z) + \cdots + f_{m-1}(z^{k_2})\varphi_{m-1}(z)\\
&= f_0(z^{k_2})R1 + f_1(z^{k_2})Rz + \cdots + f_{m-1}(z^{k_2})Rz^{m-1}\\
&= R(f_0(z^{k_1})) + R(zf_1(z^{k_1})) + \cdots + R(z^{m-1}f_{m-1}(z^{k_1}))\\
&= Rf
\end{aligned}
$$

Thus

$$A|\widetilde{N}_0 = W_m M_{\widetilde{\varphi_0}} W_n^* |\widetilde{N}_0$$

Similarly for $i = 1, 2, 3, \ldots, d-1$, *we get*

$$A/\widetilde{N}_i = W_m M_{\widetilde{\varphi}_i} W_n^* / \widetilde{N}_i$$

This completes the proof.

Remark 3.4. If k_1 and k_2 are relatively prime then $d = 1$ and hence $\widetilde{N}_0 = L^2$ and

$$\widetilde{\varphi}_0\left(z\right) = \varphi_0\left(z^{k_1}\right) + \overline{z}^{k_2}\varphi_1\left(z^{k_1}\right) + \cdots + \overline{z}^{(k_1-1)k_2}\varphi_{k_1-1}(z^{k_1})$$

which gives us the previous result. Thus we can define Rationalized (Toeplitz-Hanked) operator for any non zero integers k_1 and k_2.

Remark 3.5. From here we can see that the matrix of Rationalized (Toeplitz-Hankel) operator of order (k_1, k_2) on the space L^2 with respect to the basis $\{z^i : i \in \mathbb{Z}\}$ is a two way infinite matrix (α_{ij}) such that

$$
\begin{aligned}
\alpha_{ij} &= (i,j)\text{th elememnt of the matrix of } R_\varphi\\
&= \langle R_\varphi z^j, z^i \rangle\\
&= \langle W_{k_1} M_\varphi W_{k_2}^* z^j, z^i \rangle\\
&= \langle W_{k_1} M_\varphi z^{k_2 j}, z^i \rangle\\
&= \langle M_\varphi z^{k_2 j}, z^{k_1 i} \rangle\\
&= \langle \varphi, z^{k_1 i - k_2 j} \rangle\\
&= a_{k_1 i - k_2 j}
\end{aligned}
$$

i.e. the (i,j)th element of the matrix of Rationalized (Toeplitz-Hankel) operator is $a_{k_1 i - k_2 j}$, where $\sum a_i z^i$ is the Fourier expansion of φ.

$$
\left(\begin{array}{ccc|cccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & a_{-k_1+2k_2} & a_{-k_1+k_2} & a_{-k_1} & a_{-k_1-k_2} & a_{-k_1-2k_2} & \cdots\\
\hline
\cdots & a_{2k_2} & a_{k_2} & a_0 & a_{-k_2} & a_{-2k_2} & \cdots\\
\cdots & a_{k_1+2k_2} & a_{k_1+k_2} & a_{k_1} & a_{k_1-k_2} & a_{k_1-2k_2} & \cdots\\
\cdots & a_{2k_1+2k_2} & a_{2k_1+k_2} & a_{2k_1} & a_{2k_1-k_2} & a_{2k_1-2k_2} & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)
$$

We also observe the following result

Theorem 3.6. *For fixed integers k_1 and k_2 the set $RTHO(L^2)$ of all Rationalized (Toeplitz-Hankel) operators of order (k_1, k_2) is weakly closed and hence strongly closed.*

Proof. *Suppose that for each α, R_α is a Rationalized (Toeplitz-Hankel) operator and $R_\alpha \to R$ weakly, where $\{\alpha\}$ being a net. Then for $f, g \in L^2$*

$$\langle R_\alpha f, g\rangle \to \langle Rf, g\rangle .$$

This implies that

$$\langle M_{\overline{z}^{k_2}} R_\alpha M_{z^{k_1}} f, g\rangle = \langle R_\alpha z^{k_1} f, z^{k_2} g\rangle \quad \to \quad \langle R z^{k_1 f}, z^{k_2} g\rangle = \langle M_{\overline{z}^{k_2}} R_\alpha M_{z^{k_1}} f, g\rangle$$

But

$$M_{\overline{z}^{k_2}} R_\alpha M_{z^{k_1}} = R_\alpha \quad \textit{for each } \alpha .$$

Therefore $R = M_{\overline{z}^{k_2}} R M_{z^{k_1}}$. Consequently R is a Rationalized (Toeplitz-Hankel) operator. This completes the proof.

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