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ON THE NON-SPECIAL PART OF THE WEIERSTRASS SEMIGROUP OF n-POINTS OF A SMOOTH CURVE (ITS MINIMAL NUMBER OF GENERATORS)

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Abstract. We study spanned vector bundles on the blowing up of $\mathbb{P}^3$ at a few points. We give a full classification for very low c_1 and study some more general case.

1. Introduction

Let X be a smooth curve of genus $g \geq 2$. Fix an integer $n > 0$ and $P_1, \dots, P_n \in X$ with $P_i \neq P_j$ for all $i \neq j$. Let $H(P_1, \dots, P_n) \subset \mathbb{N}^n$ be the set of all $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned, i.e. such that there is rational function on X with $a_1P_1 + \cdots + a_nP_n$ as its divisors of poles. The set $H(P_1, \dots, P_n)$ is a semigroup for the componentwise addition $+ : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$ and $G(P_1, \dots, P_n) := \mathbb{N}^n \setminus H(P_1, \dots, P_n)$ is a finite set. The semigroup $H(P_1, \dots, P_n)$ is called the Weierstrass semigroup of the points $P_1, \dots, P_n$, while the elements of $G(P_1, \dots, P_n)$ are called the gaps of $P_1, \dots, P_n$ ([2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15]). We recall some terminology from [1]. As in [1] we consider the following partial ordering $\preccurlyeq$ on $\mathbb{N}^n$. Write $(a_1, \dots, a_n) \preccurlyeq (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$ for all i. Set $H(P_1, \dots, P_n)^+ := \{(a_1, \dots, a_n) \in H(P_1, \dots, P_n) \mid h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = 0\}$. We have $a \in H(P_1, \dots, P_n)^+$ if and only if $b \in H(P_1, \dots, P_n)$ for all $b \in \mathbb{N}^n$ such that $b \succcurlyeq a$. Therefore $H(P_1, \dots, P_n)^+$ is a set closed for $\preccurlyeq$. The set $\mathbb{N}^n \setminus H(P_1, \dots, P_n)^+$ is finite, because every line bundle on X with degree at least $2g$ is spanned and non-special. If $n = 1$, then $H(P_1)^+$ is the set of all integers $\geq n_g + 1$, where n_g is the largest element of $G(P_1)$. A non-empty subset A is closed for $\preccurlyeq$ if and only $a + b \in A$ for every $a \in A$ and every $b \in \mathbb{N}^n$. Hence the union of $\{0\}$ and a set closed with respect to $\preccurlyeq$ is a semigroup. For each $S \subseteq \mathbb{N}^n$, $S \neq \emptyset$, there is a minimal subset $A[S]$ containing S and closed with respect to $\preccurlyeq$. Let $A \subseteq \mathbb{N}^n$ be a non-empty subset closed with respect to $\preccurlyeq$ and with $\mathbb{N}^n \setminus A$ finite. Let $\mathcal{C}(A)$ be a minimal set $S \subset A$ such that $A = A[S]$. The set $\mathcal{C}(A)$ exists, it is finite and it is unique, because the intersection of any family of subsets of $\mathbb{N}^n$ closed with respect to $\preccurlyeq$ is either empty or a set closed with respect to $\preccurlyeq$. We will say

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that $\mathcal{C}(A)$ is the set of all *corners* of A. If $A = H(P_1,\dots,P_n)^+$ for some curve X and some $P_1,\dots,P_n \in X$, then set $\mathcal{C}(P_1,\dots,P_n,^+) := \mathcal{C}(H(P_1,\dots,P_n)^+)$. Let $\Delta(n,g)$ (resp. $\eta(n,g)$) denote the maximal (resp. minimal) cardinality of $\mathcal{C}(P_1,\dots,P_n,^+)$ among all $X, P_1,\dots,P_n$ with X a smooth curve of genus g and $P_1,\dots,P_n$ distinct points of X. In [1] we studied the integer $\Delta(n,g)$.

In the first part of this paper we consider the integer $\eta(n,g)$ of $\mathcal{C}(P_1,\dots,P_n,^+)$ and prove the following result.

Theorem 1.1. *We have* $\eta(n,g) \ge \lceil \binom{2g+n-1}{n-1}/\binom{g+n}{n} \rceil$ *and* $\eta(2,g) = 3$

In [1, Proposition 2.3] we proved that $\eta(2,2) = 3$ and that for any smooth curve X of genus 3 and any $P_1, P_2 \in X$ with $P_1 \ne P_2$ we have $\sharp(\mathcal{C}(P_1,,P_2,^+)) = 3$ if any only if P_1, P_2 are not Weierstrass points of X and $P_1 + P_2 \in |\omega_X|$. We generalize this example to prove the second part of Theorem 1.1.

Theorem 1.1 works verbatim for integral curves with arithmetic genus g if we assume that all points $P-1,\dots,P_n$ are smooth points of X (without this assumption not even the semigroup $H(P_1,\dots,P_n)$ may be defined in a well-behaved way).

In the last section we discuss some properties of $H(P_1,\dots,P_n)^+$ when $P_1,\dots,P_n$ are smooth points of a reducible (but connected) curve X with arithmetic genus g. Without some conditions we may even have $H(P_1,\dots,P_n)^+ = \emptyset$. We give a condition which implies that $H(P_1,\dots,P_n)^+ \ne \emptyset$ and, under this conditions, give an upper bound for the number of corners of $H(P_1,\dots,P_n)^+$.

2. Proof of Theorem 1.1

Fix a smooth curve X We first prove the inequality $\eta(n,g) \ge \lceil \binom{2g+n-1}{n-1}/\binom{g+n}{n} \rceil$. Take any $a = (a_1,\dots,a_n) \in \mathbb{N}^n$ with $\|a\| = 2g$. Since $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned and non-special, we have $a \in H(P_1,\dots,P_n)^+$. Hence if we have $b \in \mathcal{C}(P_1,\dots,P_n,^+)$, then $\|b\| \le 2g$. If $b \in H(P_1,\dots,P_n)^+$, then $\|b\| \ge g+1$. Hence each $a \in \mathcal{C}(P_1,\dots,P_n,^+)$ satisfies the inequalities $g+1 \le \|a\| \le 2g$. Take any $a \in \mathcal{C}(P_1,\dots,P_n,^+)$ and set $c := \|a\|$. There are $\binom{2g+n-1-c}{n-1}$ elements $u \in \mathbb{N}^n$ with $\|u\| = 2g$ and $a \preccurlyeq u$. Hence $\eta(n,g) \ge \lceil \binom{2g+n-1}{n-1}/\binom{g+n}{n} \rceil$. If $n = 2$ we get $\eta(2,g) \ge 3$. To conclude the proof of Theorem 1.1 it is sufficient to check the following example.

Example 2.1. Let X be a hyperelliptic curve of genus $g \ge 2$ and let $\sigma : X \to X$ be the hyperelliptic involution. Fix $P_1 \in X$ such that P_1 is not a Weierstrass point and $P_1 - \sigma(P_1)$ is not a torsion point of order k with $1 \le i \le g-1$. Set $P_2 := \sigma(P_1)$. Since P_1 is not a Weierstrass point, $P_2 \ne P_1$.

Claim : We have $\mathcal{C}(P_1,P_2,^+) = \{(g+1,0),(g,g),(0,g+1)\}$.

Proof of the Claim: We have $(g,g) \in H(P_1,\dots,P_n)^+$. Since P_1 (resp. P_2) is not a Weierstrass point of X we have $(k,0) \in H(P_1,\dots,P_n)^+$ (resp. $(k,0) \in H(P_1,\dots,P_n)^+$ if and only if $k \ge g+1$. Hence to conclude the proof of the Claim it is sufficient to prove that if $(a_1,a_2) \in H(P_1,\dots,P_n)^+$, then either $a_1 + a_2 \ge 2g$ or $a_i \ge g+1$ for some i. Assume $a_1 + a_2 < 2g$. Since $\sigma(P_2) = P_1$, it is sufficient to do the case $a_1 \ge a_2$. Since $a_1 + a_2 < 2g$, we have $a_2 \le g-1$. If $a_1 = a_2$ we get $h^1(\mathcal{O}_X(a_1P_1 + a_2P_2)) = g - a_2 > 0$, a contradiction. Hence $a_1 > a_2$. Since

$(a_1, a_2) \in H(P_1, \dots, P_n)^+$, we have $h^1(\mathcal{O}_X((a_1-1)P_1 + a_2P_2)) = 0$. Since $\omega_X = (g-1)g^1_2$ and $P_1 + P_2 \in g^1_2$, we get $g - a_2 \le a_1 - a_2$. Hence it remains to check the case $a_1 = g$. Since kP_1 and kP_2 are not linearly equivalent for $k = 1, \dots g-1$, we have $h^1(\mathcal{O}_X(gP_1)) = 0$ and so $h^0(\mathcal{O}_X(gP_1 + a_2P_2)) = a_2 + 1 = h^0(\mathcal{O}_X(a_2(P_1+P_2)))$. Hence P_1 is in the base locus of $\mathcal{O}_X(gP_1 + a_2P_2)$, a contradiction. Note that over an arbitrary algebraically closed base field we may take as P_1 a general point of X.

3. Reducible curves

Let X be a reduced and connected projective curve. Fix an ordering $X_1, \dots, X_k$, $k \ge 2$, of the irreducible component of X such that $Y_i := \cup_{h=1}^{i} X_h$ is connected for all i. We have $X_1 = Y_1$ and each Y_i is connected. For all $i = 2, \dots, k$ set $e_i := \deg(X_i \cap (X_1 \cup Y_{i-1})$. Set $e_1 := 0$. Fix $P_1, \dots, P_n \in X_{\mathrm{reg}}$ such that $P_i \ne P_j$ for all $i \ne j$. Set $S := \{P_1, \dots, P_s\}$, $S_i := S \cap X_i$ and $s_i := \sharp(S_i)$. Since $S \subset X_{\mathrm{reg}}$, we have $S_i \cap S_j = \emptyset$ for all $i \ne j$. Set $q_i := p_a(X_i)$. and $e_i := \deg(X_i \cap (\cup_{j \ne i} X_j))$. Since X is connected, we have $e_i > 0$ for all i. For any $\underline{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ set $\|\underline{a}\|_i := \deg(\mathcal{O}_{X_i}(a_1P_1 + \cdots + a_nP_n)$. This integer is the sum of all a_j with $P_j \in X_i$. Set $Y_i := \cup_{h=1}^{i} X_h$.

Lemma 3.1. *Assume $s_i > 0$ for all i. Fix $a = (a_1, \dots, a_n) \in \mathbb{N}^n$. If $\|a\|_i \ge 2q_i - 1 + e_i$ for all i, then $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = 0$.*

Proof. We have $h^1(\mathcal{O}_{X_1}(a_1P_1 + \cdots + a_kP_k)) = 0$, because X_1 is integral and $\|\underline{a}\|_1 \ge 2q_1 - 1$. Fix an integer $i \ge 2$ and assume $h^1(\mathcal{O}_{Y_{i-1}}((a_1P_1 + \cdots + a_kP_k)) = 0$. Set $Z := Y_{i-1} \cap X_i$ (seen as a zero-dimensional scheme of X_i. The sheaf $\mathcal{O}_{X_i}(a_1P_1 + \cdots + a_kP_k) \otimes \mathcal{I}_{Z,X_i}$ is a rank 1 torsion free sheaf on X_i with degree at least $2q_i - 1$. Hence $h^1(\mathcal{O}_{X_i}(a_1P_1 + \cdots + a_kP_k) \otimes \mathcal{I}_{Z,X_i} \otimes \mathcal{I}_{Z,X_i}) = 0$. Since $\mathcal{L} := \mathcal{O}_X(a_1P_1 + \cdots + a_mP_n)$ is a line bundle, the Mayer-Vietoris exact sequence

$$0 \to \mathcal{L}_{Y_i} \to \mathcal{L}_{|Y_{i-1}} \oplus \mathcal{L}_{|X_i} \to \mathcal{L}_{|Z} \to 0$$

gives $h^1(\mathcal{O}_{Y_i}(a_1P_1 + \cdots + a_kP_k)) = 0$. By induction we get the lemma. $\square$

Proposition 3.2. *Assume $s_i > 0$ for all i. If $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ satisfies $\|a\|_i \ge 2q_i + e_i$ for all i, then $a \in H(P_1, \dots, P_n)^+$. Moreover, $\sharp(\mathcal{C}(P_1, \dots, P_n,^+)) \le \prod_{i=1}^{k} \binom{2q_i+e_i+s_i}{s_i}$.*

Proof. The line bundle $\mathcal{O}_X(a_1P_1 + \cdots + a_kP_k)$ is spanned outside S. Lemma 3.1 and Riemann-Roch show that $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_kP_k) = 0$ and that $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_kP_k)(-P_i)) = h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_kP_k)) - 1$ for all i. Hence $a \in H(P_1, \dots, P_n)^+$. The assertion just proven shows that if $a \in \mathcal{C}(P_1, \dots, P_n,^+)$, then $\|a\|_i \le 2q + e_i$ for all i. Thus $\sharp(\mathcal{C}(P_1, \dots, P_n,^+)) \le \prod_{i=1}^{k} \binom{2q_i+e_i+s_i}{s_i}$. $\square$

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References

1. E. Ballico, *On the non-special part of the Weierstrass semigroups of n-points of a smooth curve*, Gulf J. Math **3** (2015), no. 3, 1–8.

2. E. Ballico and S. J. Kim, *Weierstrass multiple loci of n-pointed algebraic curves*, J. Algebra **199** (1998), no. 2, 455–471.
3. C. Carvalho and T. Kato, *On Weierstrass semigroups and sets: a review with new results*, Geom. Dedicata **139** (2009), 195–210.
4. C. Carvalho and F. Torres, *On Goppa codes and Weierstrass gaps at several points*, Des. Codes Cryptogr. **35** (2005), no. 2, 211–225.
5. M. Homma and S. J. Kim, *Toward the determination of the minimum distance of two-point codes on a Hermitian curve*, Des. Codes Cryptogr. **37** (2005), no. 1, 111–132.
6. M. Homma and S. J. Kim, *The two-point codes on a Hermitian curve with the designed minimum distance*, Des. Codes Cryptogr. **38** (2006), no. 1, 55–81.
7. M. Homma and S. J. Kim, *The Two-Point Codes with the Designed Distance on a Hermitian Curve in Even Characteristic*, Des. Codes Cryptogr. **39** (2006), no. 3, 375–386.
8. M. Homma and S. J. Kim, *The complete determination of the minimum distance of two-point codes on a Hermitian curve*,Des. Codes Cryptogr. **40** (2006), no. 1, 5–24.
9. M. Homma and S. J. Kim, *The second generalized Hamming weight for two-point codes on a Hermitian curve*, Des. Codes Cryptogr. **50** (2009), no. 1, 1–40.
10. N. Ishii, *A certain graph obtained from a set of several points on a Riemann surface*, Tsukuba J. Math. **23** (1999), no. 1, 55–89.
11. N. Ishii, *Weierstrass gap sets for quadruples of points on compact Riemann surfaces*, J. Algebra **250** (2002), no. 1, 44–66.
12. N. Ishii, *The Weierstrass gap sets for quadruples II.* Bull. Braz. Math. Soc. (N.S.) **42** (2011), no. 2, 243–258.
13. S. J. Kim, *On the index of the Weierstrass semigroup of a pair of points on a curve*, Arch. Math. **62** (1994), 73–82.
14. G. L. Matthews, *The Weierstrass semigroup of an m-tuple of collinear points on a Hermitian curve*, Lecture Notes in Computer Science, Finite fields and applications, pp. 12–24, Springer Berlin, 2004.
15. S. Park, *Minimum distance of Hermitian two-point codes*, Des. Codes Cryptogr. **57** (2010), 195–213.

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