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PERFECT POWERS INCLUDING DUPLICATIONS

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Abstract. Let $N(x)$ be the number of perfect powers not exceeding x and let $M(x)$ be the number of perfect powers with duplications not exceeding x. We prove the formulae $M(x) \sim N(x)$, $(M(x) - N(x)) \sim \sqrt{N(x)}$. Let $A(x)$ be the number of duplicated perfect powers not exceeding x. We prove the formula $A(x) \sim \sqrt{N(x)}$. The number of ways that a duplicated perfect power can be expressed as perfect power we call multiplicity of the perfect power. Let $A_2(x)$ be the number of duplicated perfect powers with multiplicity 2 not exceeding x and let $A_3(x)$ be the number of duplicated perfect powers with multiplicity greater than 2 not exceeding x. We prove the formulae $A_2(x) \sim \sqrt{N(x)}$, $A_3(x) \sim \sqrt[3]{N(x)}$. We also prove more precise asymptotic formulae for $M(x)$, $M(x) - N(x)$, $A(x)$, $A_2(x)$ and $A_3(x)$.

1. Introduction and Main Results

A natural number of the form m^n where $m \geq 2$ and $n \geq 2$ are positive integers is called a perfect power. The first few terms of the integer sequence of perfect powers are (see [2])

$$4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128 \ldots$$

Including duplications the first few terms of the integer sequence of perfect powers are (see [2])

$$4, 8, 9, 16, 16, 25, 27, 32, 36, 49, 64, 64, 64, 81, 81, 100, 121, 125, 128 \ldots$$

Here 16 is duplicated since $16 = 2^4 = 4^2$, 64 is also duplicated since $64 = 2^6 = 4^3 = 8^2$, 81 is also duplicated since $81 = 3^4 = 9^2$, etc.

The first few numbers duplicated (that is, they are perfect powers in more than one way) are (see [2])

$$16, 64, 81, 256, 512, 625, 729, 1024$$

In this article $N(x)$ denotes the number of perfect powers not exceeding x and $M(x)$ denotes the number of perfect powers including duplications not exceeding x. For example (see above) $N(128) = 15$ and $M(128) = 19$.

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A quadratfrei number or square-free number is a number without square factors, a product of different primes. The first few terms of the integer sequence of quadratfrei numbers are

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, \ldots$$

On the other hand, the Möbius function $\mu(n)$ is defined as follows: $\mu(1) = 1$, if n is the product of r different primes, then $\mu(n) = (-1)^r$, if the square of a prime divides n, then $\mu(n) = 0$.

Let p_n be the n-th prime. We have the following theorem (see [1])

Theorem 1.1. *Let $h \geq 2$ an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$N(x) = \left(\sum_{2 \leq q \leq p_h} -\mu(q) x^{1/q} \right) + o\left(x^{1/p_h}\right),$$

where q denotes a quadratfrei number.

If $h = 2$ then Theorem 1.1 becomes,

$$N(x) = \sqrt{x} + \sqrt[3]{x} + o\left(\sqrt[3]{x}\right).$$

Therefore

$$N(x) \sim \sqrt{x}. \tag{1.1}$$

If $h = 3$ then Theorem 1.1 becomes,

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} + o\left(\sqrt[5]{x}\right).$$

If $h = 4$ then Theorem 1.1 becomes,

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x} + \sqrt[7]{x} + o\left(\sqrt[7]{x}\right).$$

Note that Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2. *Let $m \geq 2$ an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$N(x) = \left(\sum_{n=2}^{m} -\mu(n) x^{1/n} \right) + o\left(x^{1/m}\right).$$

For example, if $m = 2$ then Theorem 1.2 becomes

$$N(x) = x^{1/2} + o\left(x^{1/2}\right).$$

If $m = 3$ then Theorem 1.2 becomes

$$N(x) = x^{1/2} + x^{1/3} + o\left(x^{1/3}\right).$$

If $m = 4$ then Theorem 1.2 becomes

$$N(x) = x^{1/2} + x^{1/3} + o\left(x^{1/4}\right). \tag{1.2}$$

In the following theorem we establish exact formulae for $M(x)$. $\lfloor . \rfloor$ denotes (as usual) the integer-part function.

Theorem 1.3. *Let $x \geq 4$. The following formulae hold*

$$M(x) = \sum_{n=2}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} \left(\left\lfloor x^{1/n} \right\rfloor - 1 \right) \tag{1.3}$$

$$M(x) = \sum_{n=2}^{\lfloor \sqrt{x} \rfloor} \left(\left\lfloor \frac{\log x}{\log n} \right\rfloor - 1 \right) \tag{1.4}$$

Proof. Let $c \geq 4$. Note that $M(c)$ is the number of lattice points (x, y) (that is, points such that x and y are integers) in the region D limited by the curves $x^y = c$, $x = 2$ and $y = 2$.

From $x^y = c$ we obtain

$$y = f(x) = \frac{\log c}{\log x},$$

where $f(\sqrt{c}) = 2$. Therefore the region D can be written in the form

$$D = \left\{ (x, y) : 2 \leq x \leq \sqrt{c},\ 2 \leq y \leq f(x) = \frac{\log c}{\log x} \right\},$$

and consequently the number of lattice points $M(c)$ in the region D will be

$$M(c) = \sum_{n=2}^{\lfloor \sqrt{c} \rfloor} \left(\left\lfloor \frac{\log c}{\log n} \right\rfloor - 1 \right).$$

That is, equation (1.4) if we put $c = x$.

On the other hand, from $x^y = c$ we obtain

$$x = g(y) = c^{1/y},$$

where $g(\frac{\log c}{\log 2}) = 2$. Therefore the region D can be written in the form

$$D = \left\{ (x, y) : 2 \leq y \leq \frac{\log c}{\log 2},\ 2 \leq x \leq g(y) = c^{1/y} \right\},$$

and consequently the number of lattice points $M(c)$ in the region D will be

$$M(c) = \sum_{n=2}^{\left\lfloor \frac{\log c}{\log 2} \right\rfloor} \left(\left\lfloor c^{1/n} \right\rfloor - 1 \right).$$

That is, equation (1.3) if we put $c = x$.

Now, we give other proof of equation (1.3).

Let us consider the inequality $n^s \leq x$, where $s \geq 2$ is a positive integer. The solutions to this inequality (except 1) are

$$2, 3, \ldots, \left\lfloor x^{1/s} \right\rfloor.$$

Therefore there are

$$\left\lfloor x^{1/s} \right\rfloor - 1$$

perfect powers with exponent s not exceeding x.

Now, $2^{\frac{\log x}{\log 2}}=x$. Therefore the greatest exponent s is $\left\lfloor \frac{\log x}{\log 2}\right\rfloor$ since $2^{\left(\left\lfloor \frac{\log x}{\log 2}\right\rfloor+1\right)}>x$. Consequently we obtain equation (1.3). □

In the following theorem we establish an asymptotic formula for $M(x)$.

Theorem 1.4. *Let $m\geq 2$ an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$M(x)=\sum_{n=2}^{m} x^{1/n}+o\left(x^{1/m}\right). \tag{1.5}$$

Proof. We have (see equation (1.3))

$$\begin{aligned} M(x) &= \sum_{n=2}^{\left\lfloor \frac{\log x}{\log 2}\right\rfloor}\left(\left\lfloor x^{1/n}\right\rfloor-1\right)=x^{1/2}+x^{1/3}+\cdots+x^{\frac{1}{\left\lfloor \frac{\log x}{\log 2}\right\rfloor}}-\left\lfloor \frac{\log x}{\log 2}\right\rfloor+1\\ &- A(x)=x^{1/2}+x^{1/3}+\cdots+x^{\frac{1}{\left\lfloor \frac{\log x}{\log 2}\right\rfloor}}-\frac{\log x}{\log 2}+\epsilon(x)+1-A(x)\\ &= \sum_{n=2}^{m} x^{1/n}+x^{1/(m+1)}+\cdots+x^{\frac{1}{\left\lfloor \frac{\log x}{\log 2}\right\rfloor}}-\frac{\log x}{\log 2}+\epsilon(x)+1-A(x) \end{aligned} \tag{1.6}$$

where

$$0\leq A(x)\leq \frac{\log x}{\log 2},$$

and

$$0\leq \epsilon(x)<1.$$

On the other hand, we have

$$\begin{aligned} 0 &\leq x^{1/(m+1)}+\cdots+x^{\frac{1}{\left\lfloor \frac{\log x}{\log 2}\right\rfloor}}-\frac{\log x}{\log 2}+\epsilon(x)+1-A(x)\\ &\leq x^{1/(m+1)}+\cdots+x^{\frac{1}{\left\lfloor \frac{\log x}{\log 2}\right\rfloor}}+2\leq x^{1/(m+1)}\frac{\log x}{\log 2}+2=o\left(x^{1/m}\right) \end{aligned}$$

Therefore

$$x^{1/(m+1)}+\cdots+x^{\frac{1}{\left\lfloor \frac{\log x}{\log 2}\right\rfloor}}-\frac{\log x}{\log 2}+\epsilon(x)+1-A(x)=o\left(x^{1/m}\right) \tag{1.7}$$

Equations (1.6) and (1.7) give equation (1.5). □

For example, if $m=2$ then Theorem 1.4 becomes (compare with equation (1.1))

$$M(x)=x^{1/2}+o\left(x^{1/2}\right).$$

That is,

$$M(x)\sim\sqrt{x},$$

and hence

$$M(x)\sim N(x).$$

If $m=4$ then Theorem 1.4 becomes

$$M(x)=x^{1/2}+x^{1/3}+x^{1/4}+o\left(x^{1/4}\right). \tag{1.8}$$

In the following theorem we establish asymptotic formulae for $M(x) - N(x)$.

Theorem 1.5. *Let $m \geq 4$ an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$M(x) - N(x) = \sum_{n=4}^{m} (1 + \mu(n))\, x^{1/n} + o\left(x^{1/m}\right).$$

Proof. It is an immediate consequence of Theorem 1.4 and Theorem 1.2. □

For example, if $m = 4$ then Theorem 1.5 becomes (see equations (1.8) and (1.2))

$$M(x) - N(x) = x^{1/4} + o\left(x^{1/4}\right). \tag{1.9}$$

If $m = 10$ then Theorem 1.5 becomes

$$M(x) - N(x) = x^{1/4} + 2x^{1/6} + x^{1/8} + x^{1/9} + 2x^{1/10} + o\left(x^{1/10}\right).$$

Corollary 1.6. *The following asymptotic formulae hold*

$$(M(x) - N(x)) \sim x^{1/4} \tag{1.10}$$

$$(M(x) - N(x)) \sim \sqrt{N(x)} \tag{1.11}$$

$$\lim_{x\to\infty} \frac{M(x) - N(x)}{N(x)} = 0 \tag{1.12}$$

Proof. Equation (1.10) is equation (1.9). Equation (1.11) is an immediate consequence of equation (1.10) and equation (1.1). Equation (1.12) is an immediate consequence of equation (1.11) and equation (1.1). □

Let $A(x)$ be the number of duplicated perfect powers not exceeding x. For example (see above)$A(1024) = 8$. We have the following theorem.

Theorem 1.7. *The following asymptotic formulae hold*

$$A(x) \sim x^{1/4} \tag{1.13}$$

$$A(x) \sim \sqrt{N(x)} \tag{1.14}$$

$$\lim_{x\to\infty} \frac{A(x)}{N(x)} = 0 \tag{1.15}$$

Note that equation (1.15) say us that almost all perfect powers not exceeding x are not duplicated.

Proof. Clearly we have (see (1.10))

$$A(x) \leq (M(x) - N(x)) \sim x^{1/4}. \tag{1.16}$$

On the other hand $n^4 = (n^2)^2$ is duplicated. The number of these duplicated not exceeding x is $\left\lfloor x^{1/4} \right\rfloor - 1$. Therefore

$$A(x) \geq \left(\left\lfloor x^{1/4} \right\rfloor - 1\right) \sim x^{1/4}. \tag{1.17}$$

Equations (1.16) and (1.17) give equation (1.13). Equation (1.14) is an immediate consequence of equation (1.13) and equation (1.1). Equation (1.15) is an immediate consequence of equation (1.14) and equation (1.1). □

The number of ways that a perfect power can be expressed as perfect power we shall call multiplicity of the perfect power. Therefore a perfect power is duplicated if and only if its multiplicity is greater than 1. For example $8 = 2^3$ has multiplicity 1, $16 = 2^4 = 4^2$ has multiplicity 2 and $64 = 2^6 = 4^3 = 8^2$ has multiplicity 3. Hence, using this definition, limit (1.15) say us that almost all perfect powers not exceeding x have multiplicity 1.

Let $A_2(x)$ be the number of duplicated perfect powers with multiplicity 2 not exceeding x and let $A_3(x)$ be the number of duplicated perfect powers with multiplicity greater than 2 not exceeding x. We have the following theorem.

Theorem 1.8. *The following asymptotic formulae hold*

$$A_2(x) \sim x^{1/4} \tag{1.18}$$

$$A_2(x) \sim \sqrt{N(x)}$$

$$\lim_{x\to\infty} \frac{A_2(x)}{A(x)} = 1 \tag{1.19}$$

$$A_3(x) = o\left(x^{1/4}\right) \tag{1.20}$$

Note that limit (1.19) *say us that almost all duplicated perfect powers not exceeding* x *have multiplicity* 2.

Proof. Clearly we have the inequality (see equation (1.13))

$$A_2(x) \leq A(x) \sim x^{1/4} \tag{1.21}$$

since $A(x) = A_2(x) + A_3(x)$.

Let $B(x)$ be the number of not perfect powers not exceeding x (except 1). We have $B(x) = \lfloor x \rfloor - 1 - N(x)$, therefore equation (1.1) gives

$$B(x) \sim x. \tag{1.22}$$

If $n > 1$ is not perfect power then the perfect power n^4 has multiplicity 2 since $n^4 = (n^2)^2$. The number of these perfect powers not exceeding x is (see equation (1.22))

$$B(x^{1/4}) \sim x^{1/4}.$$

Consequently we have

$$A_2(x) \geq B(x^{1/4}) \sim x^{1/4}. \tag{1.23}$$

Equationes (1.21) and (1.23) give equation (1.18).

Limit (1.19) is an immediate consequence of equations (1.13) and (1.18).

Equations (1.13) and (1.18) can be written in the form $A(x) = x^{1/4} + o\left(x^{1/4}\right)$ and $A_2(x) = x^{1/4} + o\left(x^{1/4}\right)$. Therefore $A_3(x) = A(x) - A_2(x) = o\left(x^{1/4}\right)$, that is, equation (1.20). □

Now, we shall obtain more precise asymptotic formulae for $A(x)$, $A_2(x)$ and $A_3(x)$.

We need the following lemma.

Lemma 1.9. *Let $m \geq 1$ an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$B(x) = \sum_{n=1}^{m} \mu(n) x^{1/n} + o\left(x^{1/m}\right). \tag{1.24}$$

Proof. We have $B(x) = \lfloor x \rfloor - 1 - N(x)$. Substituting into this equation the equation for $N(x)$ in Theorem 1.2 we obtain (1.24). □

In the following theorem we obtain asymptotic formulae for $A(x)$.

Theorem 1.10. *If n is an arbitrary but fixed composite number then*

$$A(x) = \sum_{4 \leq c \leq n} a(c) x^{1/c} + o\left(x^{1/n}\right), \tag{1.25}$$

where c denotes a composite number and the coefficients $a(c)$ are integers and they are determined in the following way:

a) If c is a quadratfrei, that is,

$$c = p_1 p_2 \cdots p_k \qquad (k \geq 2)$$

where the p_i $(i = 1, 2, \ldots, k)$ are the different primes in the prime factorization of c, then

$$a(c) = (-1)^k (k-1). \tag{1.26}$$

b) If

$$c = p_1^2 p_2 \cdots p_k \qquad (k \geq 1)$$

where the p_i $(i = 1, 2, \ldots, k)$ are the different primes in the prime factorization of c, then

$$a(c) = (-1)^{k+1}. \tag{1.27}$$

c) If

$$c = p_1^s p_2 \cdots p_k \qquad (k \geq 1) \qquad (s \geq 3)$$

where the p_i $(i = 1, 2, \ldots, k)$ are the different primes in the prime factorization of c, then

$$a(c) = 0. \tag{1.28}$$

d) If

$$c = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \qquad (k \geq 2)$$

where the p_i $(i = 1, 2, \ldots, k)$ are the different primes in the prime factorization of c and the a_i $(i = 1, 2, \ldots, k)$ are the exponents, and at least two a_i $(i = 1, 2, \ldots, k)$ are greater than 1, then

$$a(c) = 0. \tag{1.29}$$

Proof. Let us consider a number $m > 1$ and its prime factorization

$$m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where the p_i $(i = 1, 2, \ldots, k)$ $(k \geq 1)$ are the different primes in the prime factorization of m and the a_i $(i = 1, 2, \ldots, k)$ are the exponents.

Then m is a perfect power if and only if $\gcd(a_1, a_2, \ldots, a_k) > 1$ and therefore m is not a perfect power if and only if $\gcd(a_1, a_2, \ldots, a_k) = 1$. Suppose that m is a perfect power, then m is duplicated if and only if $\gcd(a_1, a_2, \ldots, a_k)$ is a composite number and the multiplicity of m is the number of positive divisors of the composite number $\gcd(a_1, a_2, \ldots, a_k)$ minus 1.

Therefore we have (see the proof of Theorem 1.3)

$$A(x) = \sum_{4 \leq c \leq \left\lfloor \frac{\log x}{\log 2} \right\rfloor} B\left(x^{1/c}\right) \qquad (x \geq 16)$$

where c denotes a composite number. Consequently if n is a composite number we have

$$A(x) = \sum_{4 \leq c \leq n} B\left(x^{1/c}\right) + \sum_{n < c \leq \left\lfloor \frac{\log x}{\log 2} \right\rfloor} B\left(x^{1/c}\right), \tag{1.30}$$

where

$$0 \leq \sum_{n < c \leq \left\lfloor \frac{\log x}{\log 2} \right\rfloor} B\left(x^{1/c}\right) \leq \sum_{n < c \leq \left\lfloor \frac{\log x}{\log 2} \right\rfloor} x^{1/c} \leq x^{1/(n+1)} \frac{\log x}{\log 2} = o\left(x^{1/n}\right).$$

Therefore

$$\sum_{n < c \leq \left\lfloor \frac{\log x}{\log 2} \right\rfloor} B\left(x^{1/c}\right) = o\left(x^{1/n}\right). \tag{1.31}$$

Substituting equation (1.31) into (1.30) we obtain

$$A(x) = \sum_{4 \leq c \leq n} B\left(x^{1/c}\right) + o\left(x^{1/n}\right). \tag{1.32}$$

Lemma 1.9 gives

$$B(x) = \sum_{k=1}^{n} \mu(k) x^{1/k} + o\left(x^{1/n}\right). \tag{1.33}$$

Consequently

$$B\left(x^{1/c}\right) = \sum_{k=1}^{m(c)} \mu(k) x^{1/(ck)} + o\left(x^{1/n}\right) = x^{1/c} + \sum_{k=2}^{m(c)} \mu(k) x^{1/(ck)} + o\left(x^{1/n}\right), \tag{1.34}$$

where $4 \leq c \leq n$ and $m(c)$ depends of c.

Substituting (1.34) into (1.32) we obtain (1.25).

Now, we shall prove the formulae for the coefficients $a(c)$. We need the well-known binomial formula

$$(a+b)^s = \sum_{k=0}^{s} \binom{s}{k} a^{s-k} b^k.$$

If $a = 1$ and $b = -1$ the binomial formula is

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k} = 0. \tag{1.35}$$

If $c = p_1p_2$ then the coefficient of $x^{1/c}$ is (see equation (1.34))

$$a(c) = 1 = (-1)^2(2-1)$$

That is, equation (1.26) with $k = 2$.

If $c = p_1p_2\cdots p_k$ $(k \geq 3)$ then c can be written as the product of a composite number and a quadratfrei number in

$$1 + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k-2}$$

ways and consequently the coefficient of $x^{1/c}$ will be (see equations (1.34) and equation (1.35))

$$\begin{aligned} a(c) &= 1 + \sum_{1\leq i_1\leq k}\mu(p_{i_1}) + \sum_{1\leq i_1<i_2\leq k}\mu(p_{i_1}p_{i_2}) \\ &+ \cdots + \sum_{1\leq i_1<i_2<\cdots<i_{k-2}\leq k}\mu(p_{i_1}p_{i_2}\cdots p_{i_{k-2}}) = \sum_{i=0}^{k-2}(-1)^i\binom{k}{i} \\ &= \sum_{i=0}^{k}(-1)^i\binom{k}{i} + (-1)^k\binom{k}{k-1} + (-1)^{k+1}\binom{k}{k} = 0 + (-1)^k\binom{k}{k-1} \\ &+ (-1)^{k+1}\binom{k}{k} = (-1)^k(k-1) \end{aligned}$$

That is, equation (1.26).

If $c = p_1^2$ then the coefficient of $x^{1/c}$ is (see equation (1.34))

$$a(c) = 1 = (-1)^{(1+1)}$$

That is, equation (1.27) with $k = 1$.

If $c = p_1^2p_2\cdots p_k$ $(k \geq 2)$ then c can be written as the product of a composite number and a quadratfrei number in

$$1 + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k-1}$$

ways and consequently the coefficient of $x^{1/c}$ will be (see equations (1.34) and equation (1.35))

$$a(c) = 1 + \sum_{1\leq i_1\leq k} \mu(p_{i_1}) + \sum_{1\leq i_1<i_2\leq k} \mu(p_{i_1}p_{i_2})$$
$$+ \quad \cdots + \sum_{1\leq i_1<i_2<\cdots<i_{k-1}\leq k} \mu(p_{i_1}p_{i_2}\cdots p_{i_{k-1}}) = \sum_{i=0}^{k-1}(-1)^i\binom{k}{i}$$
$$= \quad \sum_{i=0}^{k}(-1)^i\binom{k}{i} + (-1)^{k+1}\binom{k}{k} = 0 + (-1)^{k+1}\binom{k}{k} = (-1)^{k+1}$$

That is, equation (1.27).

If $c = p_1^s p_2\cdots p_k$ $(k \geq 1)$ $(s \geq 3)$ then c can be written as the product of a composite number and a quadratfrei number in

$$1 + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k}$$

ways and consequently the coefficient of $x^{1/c}$ will be (see equations (1.34) and equation (1.35))

$$a(c) = 1 + \sum_{1\leq i_1\leq k} \mu(p_{i_1}) + \sum_{1\leq i_1<i_2\leq k} \mu(p_{i_1}p_{i_2})$$
$$+ \quad \cdots + \mu(p_1p_2\cdots p_k) = \sum_{i=0}^{k}(-1)^i\binom{k}{i} = 0$$

That is, equation (1.28).

In the same way, if $c = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ where at least two a_i $(i = 1, 2, \ldots, k)$ are greater than 1, then $a(c) = 0$. That is, equation (1.29). □

In the following example we do not use equations (1.25), (1.26), (1.27), (1.28) and (1.29). We follow the proof of the theorem.

Example 1.11. Suppose, for example, that $n = 36$. We have (see equation (1.32))

$$A(x) = \sum_{4\leq c\leq 36} B(x^{1/c}) + o\left(x^{1/36}\right). \tag{1.36}$$

Equation (1.33) is

$$B(x) \quad = \quad \sum_{k=1}^{36}\mu(k)x^{1/k} + o\left(x^{1/36}\right) = x - x^{1/2} - x^{1/3} - x^{1/5} + x^{1/6} - x^{1/7} + x^{1/10}$$
$$- \quad \cdots + x^{1/35} + o\left(x^{1/36}\right)$$

Therefore equations (1.34) are

$$B(x^{1/4}) = x^{1/4} - x^{1/8} - x^{1/12} - x^{1/20} + x^{1/24} - x^{1/28} + o\left(x^{1/36}\right)$$
$$B(x^{1/6}) = x^{1/6} - x^{1/12} - x^{1/18} - x^{1/30} + x^{1/36} + o\left(x^{1/36}\right)$$

$$B(x^{1/8}) = x^{1/8} - x^{1/16} - x^{1/24} + o\left(x^{1/36}\right)$$
$$B(x^{1/9}) = x^{1/9} - x^{1/18} - x^{1/27} + o\left(x^{1/36}\right)$$
$$B(x^{1/10}) = x^{1/10} - x^{1/20} - x^{1/30} + o\left(x^{1/36}\right)$$
$$B(x^{1/12}) = x^{1/12} - x^{1/24} - x^{1/36} + o\left(x^{1/36}\right)$$
$$B(x^{1/14}) = x^{1/14} - x^{1/28} + o\left(x^{1/36}\right)$$
$$B(x^{1/15}) = x^{1/15} - x^{1/30} + o\left(x^{1/36}\right)$$
$$B(x^{1/16}) = x^{1/16} - x^{1/32} + o\left(x^{1/36}\right)$$
$$B(x^{1/18}) = x^{1/18} - x^{1/36} + o\left(x^{1/36}\right)$$
$$B(x^{1/c}) = x^{1/c} + o\left(x^{1/36}\right) \qquad (20 \le c \le 36)$$

Substituting these equations into equation (1.36) we obtain

$$\begin{aligned} A(x) &= x^{1/4} + x^{1/6} + x^{1/9} + x^{1/10} - x^{1/12} + x^{1/14} + x^{1/15} - x^{1/18} - x^{1/20} \\ &+ x^{1/21} + x^{1/22} + x^{1/25} + x^{1/26} - x^{1/28} - 2x^{1/30} + x^{1/33} + x^{1/34} + x^{1/35} \\ &+ o\left(x^{1/36}\right) \end{aligned}$$

In the following theorem we obtain asymptotic formulae for $A_2(x)$.

Theorem 1.12. *If n is the square of a prime then*

$$A_2(x) = \sum_{4 \le c' \le n} b(c') x^{1/c'} + o\left(x^{1/n}\right), \tag{1.37}$$

where c' denotes a composite number whose prime factorization is of the forms

$$c' = p_1^2 p_2 \cdots p_k \qquad (k \ge 1)$$
$$c' = p_1^3 p_2 \cdots p_k \qquad (k \ge 1)$$

and where the p_i $(i = 1, 2, \ldots, k)$ are the different primes in the prime factorization of c'.

Besides the coefficient $b(c')$ are integers and they are determined in the following way:

a) If $c' = p_1^2 p_2 \cdots p_k \quad (k \ge 1)$ then $b(c') = (-1)^{k-1}$.
b) If $c' = p_1^3 p_2 \cdots p_k \quad (k \ge 1)$ then $b(c') = (-1)^k$.

Proof. Let us consider a number $m > 1$ and its prime factorization

$$m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where the p_i $(i = 1, 2, \ldots, k)$ $(k \ge 1)$ are the different primes in the prime factorization of m and the a_i $(i = 1, 2, \ldots, k)$ are the exponents. Then (see the proof of Theorem 1.10) m is a perfect power with multiplicity 2 if and only if $\gcd(a_1, a_2, \ldots, a_k)$ is the square of a prime.

Therefore we have

$$A_2(x) = \sum_{4 \le p^2 \le \left\lfloor \frac{\log x}{\log 2} \right\rfloor} B\left(x^{1/(p^2)}\right) \qquad (x \ge 16)$$

where p denotes a positive prime. Consequently if n is the square of a prime we have

$$A_2(x) = \sum_{4\leq p^2\leq n} B\left(x^{1/(p^2)}\right) + \sum_{n<p^2\leq \left\lfloor \frac{\log x}{\log 2}\right\rfloor} B\left(x^{1/(p^2)}\right).$$

From here, as in Theorem 1.12 we obtain

$$A_2(x) = \sum_{4\leq p^2\leq n} B\left(x^{1/(p^2)}\right) + o\left(x^{1/n}\right). \tag{1.38}$$

Lemma 1.9 gives

$$B(x) = \sum_{k=1}^{n} \mu(k) x^{1/k} + o\left(x^{1/n}\right). \tag{1.39}$$

Consequently

$$B\left(x^{1/(p^2)}\right) = \sum_{k=1}^{r(p^2)} \mu(k) x^{1/(p^2k)} + o\left(x^{1/n}\right), \tag{1.40}$$

where $4 \leq p^2 \leq n$ and $r(p^2)$ depends of p^2.

Substituting (1.40) into (1.38) we obtain (1.37).

Now, we shall prove the formulae for the coefficients $b(c')$. Note that c' can be written as the product of the square of a prime and a quadratfrei number in an unique way. Hence

a) If $c' = p_1^2 p_2 \cdots p_k \quad (k \geq 1)$ then $b(c') = \mu(p_2 \cdots p_k) = (-1)^{k-1}$.
b) If $c' = p_1^3 p_2 \cdots p_k \quad (k \geq 1)$ then $b(c') = \mu(p_1 \cdots p_k) = (-1)^{k}$.

□

Example 1.13. For example, if $n = 5^2 = 25$ then we have

$$A_2(x) = x^{1/4} - x^{1/8} + x^{1/9} - x^{1/12} - x^{1/18} - x^{1/20} + x^{1/24} + x^{1/25} + o\left(x^{1/25}\right).$$

In the following theorem we obtain asymptotic formulae for $A_3(x)$.

Theorem 1.14. *If n is the square of a prime then*

$$A_3(x) = \sum_{4\leq c\leq n} d(c) x^{1/c} + o\left(x^{1/n}\right), \tag{1.41}$$

where c denotes a composite number and the coefficients $d(c)$ are integers.

Proof. It is an immediate consequence of Theorem 1.10 and Theorem 1.12, since

$$A_3(x) = A(x) - A_2(x) = \sum_{4\leq c\leq n} a(c) x^{1/c} - \sum_{4\leq c'\leq n} b(c') x^{1/c'} + o\left(x^{1/n}\right).$$

□

Example 1.15. If, for example, $n = 5^2 = 25$ then we have (see Example 1.11)

$$\begin{aligned} A(x) &= x^{1/4} + x^{1/6} + x^{1/9} + x^{1/10} - x^{1/12} + x^{1/14} + x^{1/15} - x^{1/18} - x^{1/20} \\ &+ x^{1/21} + x^{1/22} + x^{1/25} + o\left(x^{1/25}\right) \end{aligned}$$

On the other hand, we have (see Example 1.13)

$$A_2(x) = x^{1/4} - x^{1/8} + x^{1/9} - x^{1/12} - x^{1/18} - x^{1/20} + x^{1/24} + x^{1/25} + o\left(x^{1/25}\right)$$

Therefore

$$A_3(x) = x^{1/6} + x^{1/8} + x^{1/10} + x^{1/14} + x^{1/15} + x^{1/21} + x^{1/22} - x^{1/24} + o\left(x^{1/25}\right)$$

Corollary 1.16. *The following asymptotic formulae hold*

$$A_3(x) \sim x^{1/6}$$

$$A_3(x) \sim \sqrt[3]{N(x)}$$

Let A_n be the n-th duplicated perfect power, $(A_2)_n$ the n-th duplicated perfect power with multiplicity 2 and $(A_3)_n$ the n-th duplicated perfect power with multiplicity greater than 2.

We have the following theorem.

Theorem 1.17. *The following asymptotic formulae hold*

$$A_n \sim n^4 \tag{1.42}$$

$$(A_2)_n \sim n^4 \tag{1.43}$$

$$(A_3)_n \sim n^6 \tag{1.44}$$

Proof. We have $A(x) \sim x^{1/4}$, $A_2(x) \sim x^{1/4}$ and $A_3(x) \sim x^{1/6}$. Therefore $n = A(A_n) \sim A_n^{1/4}$ and consequently $A_n \sim n^4$. That is, equation (1.42). Equations (1.43) and (1.44) can be proved in the same way. □

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References

1. R. Jakimczuk, *On the Distribution of Perfect Powers*, J. Integer Seq. **14** (2011), Article 11.8.5.
2. N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences.

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