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ANDREWS' SINGULAR OVERPARTITIONS WITHOUT MULTIPLES OF k

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ABSTRACT. In a recent work G. E. Andrews gave definition of combinatorial objects which he called singular overpartitions and proved that these singular overpartitions, which depend on two parameters δ and i, can be enumerated by function $\overline{C}_{\delta,i}(n)$, the number of overpartitions of n such that no part divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. In this paper, we establish several infinite families of congruences $\overline{C}^{k}_{\delta,i}(n)$, the number of singular overpartitions of n without multiples of k such that no part divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. For example, for all $n \geq 0$ and $\alpha \geq 0$, $\overline{C}^{5}_{4,1}\left(2^{2\alpha+5}n + \frac{7\cdot 2^{2\alpha+3}+1}{3}\right) \equiv 0 \pmod{2^2}$.

1. INTRODUCTION

Recently, G. E. Andrews [2] defined combinatorial objects which he called singular overpartitions and proved that these singular overpartitions which depends on two parameters δ and i can be enumerated by the function $\overline{C}_{\delta,i}(n)$ which gives the number of overpartitions of n in which no part divisible by δ and parts $\equiv \pm i \pmod{\delta}$ may be overlined. The generating function of $\overline{C}_{\delta,i}(n)$ is

$$\sum_{n=0}^{\infty} \overline{C}_{\delta,i}(n)q^n = \frac{(q^\delta;q^\delta)_\infty(-q^i;q^\delta)_\infty(-q^{\delta-i};q^\delta)_\infty}{(q;q)_\infty}. \tag{1.1}$$

He also proved that

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}. \tag{1.2}$$

S. C. Chen, M. D. Hirschhorn and J. A. Sellers [4] have found some infinite families of congruences modulo 3 for $\overline{C}_{3,1}(n)$, $\overline{C}_{6,1}(n)$, $\overline{C}_{6,2}(n)$ and modulo powers of 2 for $\overline{C}_{4,1}(n)$. The authors Z. Ahmed and N. D. Baruah [1] have found some new congruences for $\overline{C}_{3,1}(n)$ modulo 18 and 36 and $\overline{C}_{8,2}(n)$, $\overline{C}_{12,4}(n)$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$ modulo 2. Chen [5] has also found some new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{4,1}(n)$ modulo powers of 2. X. M. Yao [12] has proved some congruences modulo 16, 32, 14 for $\overline{C}_{3,1}(n)$. M. S. Mahadeva Naika et al. [8] have found some modulo 6, 12, 16, 18, 24, 48, 72 for $\overline{C}_{3,1}(n)$.

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In this paper, we define the function $\overline{C}^{k}_{\delta,i}(n)$, the number of singular overpartitions of n without multiples of k in which no part divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. The generating function of $\overline{C}^{k}_{\delta,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}^{k}_{\delta,i}(n) q^n = \frac{f(q^i;q^{\delta-i})(q^k;q^k)_\infty}{f(q^{ki};q^{k(\delta-i)})(q;q)_\infty}, \tag{1.3}$$

where $0 < i < \delta$.

Throughout this paper, we use the standard q-series notation, and f_k is defined as

$$f_k := (q^k;q^k)_\infty = \lim_{n\to\infty} \prod_{j=1}^{n} (1-q^{jk}).$$

For $|ab| < 1$, Ramanujan's general theta function $f\left(a,b\right)$ is defined as

$$f\left(a,b\right) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{1.4}$$

Using Jacobi's triple product identity [3, Entry 19, p. 35], the equation (1.4) becomes

$$f\left(a,b\right) = \left(-a,ab\right)_\infty \left(-b,ab\right)_\infty \left(ab,ab\right)_\infty.$$

The most important special cases of $f\left(a,b\right)$ are

$$\varphi\left(q\right) := f\left(q,q\right) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)^2_\infty (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.5}$$

$$\psi\left(q\right) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{1.6}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty = f_1. \tag{1.7}$$

2. Preliminaries

We list few dissection formulas to prove our main results.

Lemma 2.1. [10, p. 212] *We have the following* 5*-dissection*

$$f_1 = f_{25}\left(a(q) - q - q^2 b(q)\right), \tag{2.1}$$

where

$$a(q) := \frac{(q^{10},q^{15};q^{25})_\infty}{(q^5,q^{20};q^{25})_\infty} \text{ and } b(q) := \frac{(q^5,q^{20};q^{25})_\infty}{(q^{10},q^{15};q^{25})_\infty}. \tag{2.2}$$

Lemma 2.2. [3, Entry 25 p. 40] *The following 2-dissection formulas hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.3}$$

and

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}. \tag{2.4}$$

Lemma 2.3. [11] *The following 2-dissection formula holds:*

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}. \tag{2.5}$$

Lemma 2.4. *The following* 2*-dissections hold:*

$$\frac{f_5}{f_1}=\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}, \tag{2.6}$$

$$\frac{f_1}{f_5}=\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}. \tag{2.7}$$

Equation (2.6) was proved by Hirschhorn and Sellers in [7]. Replacing q by $-q$ in (2.6), we obtain (2.7).

Lemma 2.5. [9, Lemma 2.3] *The following* 2*-dissection formulas hold:*

$$f_1f_5^3=2q^2f_4f_{20}^3+f_2^3f_{10}-2q^3\frac{f_4^4f_{40}^2f_{10}}{f_2f_8^2}-q\frac{f_2^2f_{10}^2f_{20}}{f_4}, \tag{2.8}$$

$$f_1^3f_5=2q^2\frac{f_4^6f_{40}^2f_{10}}{f_2f_8^2f_{20}^2}+\frac{f_2^2f_4f_{10}^2}{f_{20}}+2qf_4^3f_{20}-5qf_2f_{10}^3. \tag{2.9}$$

Lemma 2.6. [6, Theorem 2.2] *For any prime* $p\geq 5$*,*

$$f_1=\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}}(-1)^kq^{\frac{3k^2+k}{2}}f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}f_{p^2}. \tag{2.10}$$

Furthermore, for $-(p-1)/2\leq k\leq (p-1)/2$ *and* $k\neq(\pm p-1)/6$*,*

$$\frac{3k^2+k}{2}\not\equiv\frac{p^2-1}{24}\pmod{p}.$$

3. Main results

4. Congruences for $\overline{C}^3_{4,1}(n)$.

Theorem 4.1. *For each integer* $n\geq 0$*,*

$$\overline{C}^3_{4,1}(4n+3)\equiv 0\pmod{2^2}, \tag{4.1}$$

$$\sum_{n=0}^{\infty}\overline{C}^3_{4,1}(4n+1)q^n\equiv 2f_2f_3\pmod{2^2}. \tag{4.2}$$

Proof. Setting $\delta=4$, $i=1$ and $k=3$ in (1.3), we find that

$$\sum_{n=0}^{\infty}\overline{C}^3_{4,1}(n)q^n=\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^2}{(q;q)_\infty^2(q^6;q^6)_\infty^2}. \tag{4.3}$$

Substituting (2.5) into (4.3), we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{3}_{4,1}(n)q^n = \frac{f_4^4 f_{12}^2}{f_2^3 f_6 f_8 f_{24}} + 2q\frac{f_4 f_8 f_{24}}{f_2^2 f_{12}}, \tag{4.4}$$

which yields, for each $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}^{3}_{4,1}(2n+1)q^n = 2\frac{f_2 f_4 f_{12}}{f_1^2 f_6}. \tag{4.5}$$

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{2k}^m \equiv f_k^{2m} \pmod{2}, \tag{4.6}$$

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod{4}. \tag{4.7}$$

Using (4.6) in (4.5), we find that

$$\sum_{n=0}^{\infty} \overline{C}^{3}_{4,1}(2n+1)q^n \equiv 2f_4 f_6 \pmod{2^2}. \tag{4.8}$$

Congruences (4.1) follows by extracting the terms involving q^{2n+1} from (4.8).

Collecting the terms involving q^{2n} from (4.8) and replacing q^2 by q, we get (4.2) □

Theorem 4.2. *For any prime $p \geq 5$ with $(\frac{-6}{p}) = -1$, $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{C}^{3}_{4,1}\left(4 \cdot p^{2\alpha} n + \frac{5 \cdot p^{2\alpha} + 1}{6}\right) q^n \equiv 2f_2 f_3 \pmod{2^2}. \tag{4.9}$$

Proof. Define

$$\sum_{n=0}^{\infty} f(n)q^n = f_2 f_3 \pmod{2^2}. \tag{4.10}$$

Combining (4.2) and (4.10), we find that

$$\sum_{n=0}^{\infty} \overline{C}^{3}_{4,1}(4n+1)q^n \equiv 2\sum_{n=0}^{\infty} f(n)q^n \pmod{2^2}. \tag{4.11}$$

Now, we consider the congruence equation

$$2 \cdot \frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod{p}, \tag{4.12}$$

which is equivalent to

$$(2 \cdot (6k+1))^2 + 6 \cdot (6m+1)^2 \equiv 0 \pmod{p}.$$

where $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$ and p is a prime such that $(\frac{-6}{p}) = -1$. Since $(\frac{-6}{p}) = -1$ for $p \geq 5$, the congruence relation (4.12) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute (2.10) into (4.10) and then extracting the terms in which the powers of q are congruent to $5 \cdot \frac{p^2-1}{24}$ modulo p and then divide by $q^{5\cdot\frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} f\left(pn + 5 \cdot \frac{p^2-1}{24}\right) q^{pn} = f_{2p} f_{3p}, \tag{4.13}$$

which implies that

$$\sum_{n=0}^{\infty} f\left(p^2 n + 5 \cdot \frac{p^2-1}{24}\right) q^{n} = f_2 f_3 \tag{4.14}$$

and for $n \geq 0$,

$$f\left(p^2 n + pi + 5 \cdot \frac{p^2-1}{24}\right) = 0, \tag{4.15}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha} n + 5 \cdot \frac{p^{2\alpha}-1}{24}\right) = f(n). \tag{4.16}$$

Replacing n by $p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha}-1}{24}$ in (4.11), we arrive at (4.9). □

Theorem 4.3. *For any prime $p \geq 5$ with $\left(\frac{-6}{p}\right) = -1$, $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{C}^{3}_{4,1}\left(4 \cdot p^{2\alpha+2} n + 4 \cdot p^{2\alpha+1} i + \frac{5 \cdot p^{2\alpha+2}+1}{6}\right) \equiv 0 \pmod{2^2}.$$

where $i = 1, 2, ..., p-1$.

Proof. Replacing n by $p^2 n + pi + \frac{5\cdot p^2-5}{24}$ in (4.16) and using (4.15), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{5 \cdot p^{2\alpha+2}-5}{24}\right) = 0. \tag{4.17}$$

Comparing the coefficients of q^n from the both sides of (4.11), we see that for $n \geq 0$,

$$\overline{C}^{3}_{4,1}(4n+1) \equiv 2f(n) \pmod{2^2}. \tag{4.18}$$

The result follows from (4.17) and (4.18). □

5. Congruences for $\overline{C}^{5}_{4,1}(n)$.

Theorem 5.1. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{C}^{5}_{4,1}\left(2^{2\alpha+5} n + \frac{7 \cdot 2^{2\alpha+3}+1}{3}\right) \equiv 0 \pmod{2^2}, \tag{5.1}$$

$$\overline{C}^{5}_{4,1}(8n+5) \equiv 0 \pmod{2^2}, \tag{5.2}$$

$$\overline{C}^{5}_{4,1}(16n+9) \equiv 0 \pmod{2^2}, \tag{5.3}$$

$$\overline{C}^{5}_{4,1}(8(4n+i)+7) \equiv 0 \pmod{2^2}, \tag{5.4}$$

where i=1, 2, 3.

$$\overline{C}^5_{4,1}(32(5n+i)+15) \equiv 0 \pmod{2^2}, \tag{5.5}$$

where i=1, 2, 3, 4.

$$\overline{C}^5_{4,1}(160n+7) \equiv \overline{C}^5_{4,1}(16n+1) \pmod{2^2}. \tag{5.6}$$

Proof. Setting $\delta = 4$, $i = 1$ and $k = 5$ in (1.3), we find that

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(n)q^n = \frac{(q^2;q^2)^2_\infty (q^5;q^5)^2_\infty}{(q;q)^2_\infty (q^{10};q^{10})^2_\infty}. \tag{5.7}$$

Employing (2.6) into (5.7), we get

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(n)q^n = \frac{f_8^2 f_{20}^4}{f_2^2 f_{10}^2 f_{40}^2} + q^2 \frac{f_4^6 f_{40}^2}{f_2^4 f_8^2 f_{20}^2} + 2q \frac{f_4^3 f_{20}}{f_2^3 f_{10}}. \tag{5.8}$$

Extracting the terms involving q^{2n+1} from (5.8), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(2n+1)q^n = 2\frac{f_2^3 f_{10}}{f_1^3 f_5}. \tag{5.9}$$

Using (4.6) in (5.9), we find

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(2n+1)q^n \equiv 2f_1^3 f_5 \pmod{2^2}. \tag{5.10}$$

Employing (2.9) into (5.10), we obtain

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(2n+1)q^n \equiv 2\frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_2 f_{10}^3 \pmod{2^2}. \tag{5.11}$$

Which implies that

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(4n+3)q^n \equiv 2f_1 f_5^3 \pmod{2^2}. \tag{5.12}$$

Substituting (2.8) into (5.12), we get

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(4n+3)q^n \equiv 2f_2^3 f_{10} + 2q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} \pmod{2^2}. \tag{5.13}$$

Extracting the terms involving q^{2n} from (5.13) and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(8n+3)q^n \equiv 2f_1^3 f_5 \pmod{2^2}. \tag{5.14}$$

In view of congruences (5.10) and (5.14), we arrive at

$$\overline{C}^5_{4,1}(8n+3) \equiv \overline{C}^5_{4,1}(2n+1) \pmod{2^2}. \tag{5.15}$$

Utilizing (5.15) and by mathematical induction on α, we get

$$\overline{C}^{5}_{4,1}\left(2\cdot 4^{\alpha+1}n+\frac{2\cdot 4^{\alpha+1}+1}{3}\right)\equiv \overline{C}^{5}_{4,1}(2n+1) \pmod{2^2}. \tag{5.16}$$

Utilizing (5.2) and (5.16), we obtain (5.1).

Collecting the terms involving q^{2n} from (5.11) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(4n+1)q^n \equiv 2\frac{f_1^2f_2f_5^2}{f_{10}} \pmod{2^2}. \tag{5.17}$$

Using (4.6) in (5.17), we find

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(4n+1)q^n \equiv 2f_4 \pmod{2^2}, \tag{5.18}$$

Congruence (5.2) and (5.3) follows by extracting the terms involving q^{2n+1} and q^{4n+2} from both sides of (5.18).

Collecting the terms involving q^{4n} from (5.18) and replacing q^4 by q, we have

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(16n+1)q^n \equiv 2f_1 \pmod{2^2}, \tag{5.19}$$

Extracting the terms involving q^{2n+1} from (5.13), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(8n+7)q^n \equiv 2\frac{f_1^2f_5^2f_{10}}{f_2} \pmod{2^2}. \tag{5.20}$$

Using (4.6) in (5.20), we get

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(8n+7)q^n \equiv 2f_5^2f_{10} \pmod{2^2}, \tag{5.21}$$

which implies that

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(8n+7)q^n \equiv 2f_{20} \pmod{2^2}. \tag{5.22}$$

Congruence (5.4) follows by extracting the terms involving q^{4n+i} from both sides of (5.22).

Extracting the terms involving q^{4n} from (5.22) and then replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(32n+7)q^n \equiv 2f_5 \pmod{2^2}. \tag{5.23}$$

Congruence (5.5) follows by extracting the terms involving q^{5n+i} from both sides of (5.23).

Extracting the terms involving q^{5n} from (5.23) and then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty}\overline{C}^{5}_{4,1}(160n+7)q^n \equiv 2f_1 \pmod{2^2}. \tag{5.24}$$

In view of congruences (5.19) and (5.24), we obtain (5.6). □

Theorem 5.2. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{C}^5_{4,1}\left(2 \cdot 5^{2\alpha+3}n + \frac{14 \cdot 5^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^2}, \tag{5.25}$$

$$\overline{C}^5_{4,1}(10n+5) \equiv 0 \pmod{2^2}, \tag{5.26}$$

$$\overline{C}^5_{4,1}(10n+9) \equiv 0 \pmod{2^2}, \tag{5.27}$$

$$\overline{C}^5_{4,1}(10(5n+i)+7) \equiv 0 \pmod{2^2}, \tag{5.28}$$

where i=1, 2, 3, 4.

Proof. We prove the equation (5.25) by induction on α. Note that (5.10) is the case when $\alpha = 0$. Suppose that the congruence holds for α. Utilizing (2.1) in (5.10), we get

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(2n+1)q^n \equiv 2f_5 f_{25}^3 \left(a^3 - 3a^2q + 5q^3 - 3\frac{q^5}{a^2} - \frac{q^6}{a^3}\right) \pmod{2^2}. \tag{5.29}$$

Congruence (5.26) and (5.27) follows by extracting the terms involving q^{5n+2} and q^{5n+4} from both sides of (5.29).

Extracting the terms involving q^{5n+3} from (5.29), dividing by q^3 and then replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(10n+7)q^n &\equiv 10 f_1 f_5^3 \pmod{2^2} \\ &\equiv 2f_5^3 f_{25}\left(a - q - \frac{q^2}{a}\right)\end{aligned} \tag{5.30}$$

Congruence (5.28) follows by extracting the terms involving q^{5n+i} from both sides of (5.30).

Extracting the terms involving q^{5n+1} from (5.30), dividing by q and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(50n+17)q^n \equiv 2f_1^3 f_5 \pmod{2^2}. \tag{5.31}$$

In view of congruences (5.10) and (5.31), we obtain

$$\overline{C}^5_{4,1}(50n+17)q^n \equiv \overline{C}^5_{4,1}(2n+1) \pmod{2^2}. \tag{5.32}$$

Utilizing (5.32) and by mathematical induction on α, we get

$$\overline{C}^5_{4,1}\left(2 \cdot 25^{\alpha+1}n + \frac{2 \cdot 25^{\alpha+1}+1}{3}\right) \equiv \overline{C}^5_{4,1}(2n+1) \pmod{2^2}. \tag{5.33}$$

Utilizing (5.26) and (5.33), we get (5.25). □

Theorem 5.3. *For all $n \geq 0$,*

$$\overline{C}^{5}_{4,1}(32n+31) \equiv 0 \pmod{2^3}, \tag{5.34}$$

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(32n+15)q^n \equiv 4f_1f_{10} \pmod{2^3}. \tag{5.35}$$

Proof. Using (4.7) in (5.9), we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(2n+1)q^n \equiv 2\frac{f_1f_2f_{10}}{f_5} \pmod{2^3}. \tag{5.36}$$

Substituting (2.7) into (5.36), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(2n+1)q^n \equiv 2\frac{f_2^2f_8f_{20}^3}{f_4f_{10}^2f_{40}} - 2q\frac{f_2f_4^2f_{40}}{f_8f_{10}} \pmod{2^3}. \tag{5.37}$$

Extracting the terms involving q^{2n+1} from (5.37), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(4n+3)q^n \equiv 6\frac{f_1f_2^2f_{20}}{f_4f_5} \pmod{2^3}. \tag{5.38}$$

Employing (2.7) in (5.38), we get

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(4n+3)q^n \equiv 6\frac{f_2^3f_8f_{20}^4}{f_4^2f_{10}^3f_{40}} - 6q\frac{f_2^2f_4f_{20}f_{40}}{f_8f_{10}^2} \pmod{2^3}. \tag{5.39}$$

Extracting the terms involving q^{2n+1} from (5.39), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(8n+7)q^n \equiv 2\frac{f_1^2f_2f_{10}f_{20}}{f_4f_5^2} \pmod{2^3}. \tag{5.40}$$

Again substituting (2.7) in (5.40), we find that

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(16n+15)q^n \equiv 4\frac{f_1^2f_{10}^4}{f_5^4} \pmod{2^3}. \tag{5.41}$$

Using (4.6) in (5.41), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(16n+15)q^n \equiv 4f_2f_{20} \pmod{2^3}, \tag{5.42}$$

Congruence (5.34) follows by extracting the terms involving q^{2n+1} from (5.42).

Collecting the terms involving q^{2n} from (5.42) and replacing q^2 by q, we get (5.35). ☐

Theorem 5.4. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{C}^{5}_{4,1}\left(32 \cdot 5^{2\alpha+2}(5n+i) + \frac{44 \cdot 5^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^3}, \tag{5.43}$$

$$\overline{C}^{5}_{4,1}(32(5n+i)+15) \equiv 0 \pmod{2^3}, \tag{5.44}$$

where i=3, 4.

$$\overline{C}^{5}_{4,1}(160(5n+j)+47) \equiv 0 \pmod{2^3}, \tag{5.45}$$

where j=1, 3.

Proof. We prove the equation (5.43) by induction on α. Note that (5.35) is the case when $\alpha = 0$. Suppose that the congruence holds for α. Utilizing (2.1) in (5.35), we get

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(32n+15)q^n \equiv 4f_{10}f_{25}\left(a - q - \frac{q^2}{a}\right) \pmod{2^3}. \tag{5.46}$$

Congruence (5.44) follows by extracting the terms involving q^{5n+i} from both sides of (5.46).

Extracting the terms involving q^{5n+1} from (5.46), dividing by q and then replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(160n+47)q^n &\equiv 4f_2f_5 \pmod{2^3} \\ &\equiv 2f_5f_{50}\left(a(q^2) - q^2 - \frac{q^4}{a(q^2)}\right)\end{aligned} \tag{5.47}$$

Congruence (5.45) follows by extracting the terms involving q^{5n+j} from both sides of (5.47).

Extracting the terms involving q^{5n+2} from (5.47), dividing by q^2 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(800n+367)q^n \equiv 4f_1f_{10} \pmod{2^3}. \tag{5.48}$$

In view of congruences (5.35) and (5.48), we obtain

$$\overline{C}^{5}_{4,1}(800n+367)q^n \equiv \overline{C}^{5}_{4,1}(32n+15) \pmod{2^3}. \tag{5.49}$$

Utilizing (5.49) and by mathematical induction on α, we get

$$\overline{C}^{5}_{4,1}\left(32 \cdot 25^{\alpha+1}n + \frac{44 \cdot 25^{\alpha+1}+1}{3}\right) \equiv \overline{C}^{5}_{4,1}(32n+15) \pmod{2^3}. \tag{5.50}$$

Utilizing (5.44) and (5.50), we get (5.43). □

Theorem 5.5. *For any prime $p \geq 5$ with $\left(\frac{-10}{p}\right) = -1$, $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}\left(32 \cdot p^{2\alpha}n + \frac{44 \cdot p^{2\alpha}+1}{3}\right) q^n \equiv 4f_1f_{10} \pmod{2^3}. \tag{5.51}$$

Proof. Define

$$\sum_{n=0}^{\infty} g(n)q^n = f_1f_{10} \pmod{2^3}. \tag{5.52}$$

Combining (5.35) and (5.52), we find that

$$\sum_{n=0}^{\infty}\overline{C}_{4,1}^{5}(32n+15)q^n \equiv 4\sum_{n=0}^{\infty} g(n)q^n \pmod{2^3}. \tag{5.53}$$

Now, we consider the congruence equation

$$\frac{3k^2+k}{2}+10\cdot\frac{3m^2+m}{2} \equiv \frac{11p^2-11}{24} \pmod{p}, \tag{5.54}$$

which is equivalent to

$$(6k+1)^2+10\cdot(6m+1)^2 \equiv 0 \pmod{p}.$$

where $\frac{-(p-1)}{2} \le k,m \le \frac{p-1}{2}$ and p is a prime such that $(\frac{-10}{p}) = -1$. Since $(\frac{-10}{p}) = -1$ for $p \ge 5$, the congruence relation (5.54) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute (2.10) into (5.52) and then extracting the terms in which the powers of q are congruent to $11\cdot\frac{p^2-1}{24}$ modulo p and then divide by $q^{11\cdot\frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} g\left(pn+11\cdot\frac{p^2-1}{24}\right)q^{pn} = f_p f_{10p}, \tag{5.55}$$

which implies that

$$\sum_{n=0}^{\infty} g\left(p^2n+11\cdot\frac{p^2-1}{24}\right)q^{n} = f_1 f_{10} \tag{5.56}$$

and for $n \ge 0$,

$$g\left(p^2n+pi+11\cdot\frac{p^2-1}{24}\right) = 0, \tag{5.57}$$

where i is an integer and $1 \le i \le p-1$. By induction, we see that for $n \ge 0$ and $\alpha \ge 0$,

$$g\left(p^{2\alpha}n+11\cdot\frac{p^{2\alpha}-1}{24}\right) = g(n). \tag{5.58}$$

Replacing n by $p^{2\alpha}n+11\cdot\frac{p^{2\alpha}-1}{24}$ in (5.53), we arrive at (5.35). $\square$

Theorem 5.6. *For any prime $p \ge 5$ with $(\frac{-10}{p}) = -1$, $\alpha \ge 1$ and $n \ge 0$, we have*

$$\sum_{n=0}^{\infty}\overline{C}_{4,1}^{5}\left(32\cdot p^{2\alpha+2}n+32\cdot p^{2\alpha+1}i+\frac{44\cdot p^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^3}.$$

where $i = 1,2,...,p-1$.

Proof. Replacing n by $p^2n+pi+\frac{11\cdot p^2-11}{24}$ in (5.58) and using (5.57), we find that for $n \ge 0$ and $\alpha \ge 0$,

$$g\left(p^{2\alpha+2}n+p^{2\alpha+1}i+\frac{11\cdot p^{2\alpha+2}-11}{24}\right) = 0. \tag{5.59}$$

Comparing the coefficients of q^n from the both sides of (5.53), we see that for $n \geq 0$,

$$\overline{C}^{5}_{4,1}(32n+15) \equiv 4g(n) \pmod{2^3}. \tag{5.60}$$

The result follows from (5.59) and (5.60). □

Theorem 5.7. *For all $n \geq 0$*

$$\overline{C}^{5}_{4,1}(16n+13) \equiv 0 \pmod{2^3}, \tag{5.61}$$

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(16n+5)q^n \equiv 4f_2f_5 \pmod{2^3}. \tag{5.62}$$

Proof. Equation (5.9) can be rewritten as

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(2n+1)q^n = 2f_2^3f_{10}\frac{f_1}{f_5f_1^4}. \tag{5.63}$$

Substituting (2.4) and (2.7) into (5.63), we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(2n+1)q^n = 2\frac{f_4^{13}f_{20}^3}{f_2^{10}f_8^3f_{10}^2f_{40}} - 8q^2\frac{f_4^4f_8^3f_{40}}{f_2^7f_{10}} - 2q\frac{f_4^{16}f_{40}}{f_2^{11}f_8^5f_{10}} + 8q\frac{f_4f_8^5f_{20}^3}{f_2^6f_{10}^2f_{40}}. \tag{5.64}$$

Collecting the terms involving q^{2n} from (5.64) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(4n+1)q^n = 2\frac{f_2^{13}f_{10}^3}{f_1^{10}f_4^3f_5^2f_{20}} - 8q\frac{f_2^4f_4^3f_{20}}{f_1^7f_5}. \tag{5.65}$$

Using (4.7) in (5.65), we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(4n+1)q^n \equiv 2\frac{f_2f_4f_5^2f_{10}}{f_1^2f_{20}} \pmod{8}. \tag{5.66}$$

Employing (2.6) into (5.66), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(4n+1)q^n \equiv 2\frac{f_4f_8^2f_{10}f_{20}^3}{f_2^3f_{40}^2} + 2q^2\frac{f_4^7f_{10}^3f_{40}^2}{f_2^5f_8^2f_{20}^3} + 4q\frac{f_4^4f_{10}^2}{f_2^4} \pmod{8}. \tag{5.67}$$

Extracting the terms involving q^{2n+1} from (5.67), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(8n+5)q^n \equiv 4\frac{f_2^4f_5^2}{f_1^4} \pmod{8}. \tag{5.68}$$

Using (4.6) in (5.68), we get

$$\sum_{n=0}^{\infty} \overline{C}^{5}_{4,1}(8n+5)q^n \equiv 4f_4f_{10} \pmod{8}. \tag{5.69}$$

Congruence (5.61) follows by extracting the terms involving q^{2n+1} from (5.69).

Collecting the terms involving q^{2n} from (5.69) and replacing q^2 by q, we get (5.62) □

Theorem 5.8. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{C}_{4,1}^{5}\left(16 \cdot 5^{2\alpha+2}(5n+i) + \frac{14 \cdot 5^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^3}, \tag{5.70}$$

$$\overline{C}_{4,1}^{5}(16(5n+i)+5) \equiv 0 \pmod{2^3}, \tag{5.71}$$

where i=1, 3.

$$\overline{C}_{4,1}^{5}(80(5n+j)+37) \equiv 0 \pmod{2^3}, \tag{5.72}$$

where j=3, 4.

Proof. We prove the equation (5.70) by induction on α. Note that (5.62) is the case when $\alpha = 0$. Suppose that the congruence holds for α. Utilizing (2.1) in (5.62), we get

$$\sum_{n=0}^{\infty} \overline{C}_{4,1}^{5}(16n+5)q^n \equiv 4f_5 f_{50}\left(a(q^2) - q^2 - \frac{q^4}{a(q^2)}\right) \pmod{2^3}. \tag{5.73}$$

Congruence (5.71) follows by extracting the terms involving q^{5n+i} from both sides of (5.73).

Extracting the terms involving q^{5n+2} from (5.73), dividing by q^2 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{4,1}^{5}(80n+37)q^n \equiv 4f_1 f_{10} \pmod{2^3}$$

$$\equiv 4f_{10}f_{25}\left(a - q - \frac{q^2}{a}\right) \tag{5.74}$$

Congruence (5.72) follows by extracting the terms involving q^{5n+j} from both sides of (5.74).

Extracting the terms involving q^{5n+1} from (5.74), dividing by q and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{4,1}^{5}(400n+117)q^n \equiv 4f_2 f_5 \pmod{2^3}. \tag{5.75}$$

In view of congruences (5.62) and (5.75), we obtain

$$\overline{C}_{4,1}^{5}(400n+117)q^n \equiv \overline{C}_{4,1}^{5}(16n+5) \pmod{2^3}. \tag{5.76}$$

Utilizing (5.76) and by mathematical induction on α, we get

$$\overline{C}_{4,1}^{5}\left(16 \cdot 25^{\alpha+1}n + \frac{14 \cdot 25^{\alpha+1}+1}{3}\right) \equiv \overline{C}_{4,1}^{5}(16n+5) \pmod{2^3}. \tag{5.77}$$

Using (5.71) in (5.77), we get (5.70). $\square$

Theorem 5.9. *For any prime $p \geq 5$ with $(\frac{-10}{p}) = -1$, $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{4,1}^{5}\left(16 \cdot p^{2\alpha}n + \frac{14 \cdot p^{2\alpha}+1}{3}\right) q^n \equiv 4f_2 f_5 \pmod{2^3}. \tag{5.78}$$

Proof. Define

$$\sum_{n=0}^{\infty} h(n)q^n = f_2 f_5 \pmod{2^3}. \tag{5.79}$$

Combining (5.62) and (5.79), we find that

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}(16n+5)q^n \equiv 4\sum_{n=0}^{\infty} h(n)q^n \pmod{2^3}. \tag{5.80}$$

Now, we consider the congruence equation

$$2\cdot\frac{3k^2+k}{2}+5\cdot\frac{3m^2+m}{2}\equiv\frac{7p^2-7}{24} \pmod p, \tag{5.81}$$

which is equivalent to

$$(12k+2)^2+10\cdot(6m+1)^2\equiv 0 \pmod p.$$

where $\frac{-(p-1)}{2}\le k,m\le\frac{p-1}{2}$ and p is a prime such that $(\frac{-10}{p})=-1$. Since $(\frac{-10}{p})=-1$ for $p\ge 5$, the congruence relation (5.81) holds if and only if both $k=m=\frac{\pm p-1}{6}$. Therefore, if we substitute (2.10) into (5.79) and then extracting the terms in which the powers of q are congruent to $7\cdot\frac{p^2-1}{24}$ modulo p and then divide by $q^{7\cdot\frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} h\left(pn+7\cdot\frac{p^2-1}{24}\right)q^{pn}=f_{2p}f_{5p},$$

which implies that

$$\sum_{n=0}^{\infty} h\left(p^2n+7\cdot\frac{p^2-1}{24}\right)q^{n}=f_{2}f_{5} \tag{5.82}$$

and for $n\ge 0$,

$$h\left(p^2n+pi+7\cdot\frac{p^2-1}{24}\right)=0, \tag{5.83}$$

where i is an integer and $1\le i\le p-1$. By induction, we see that for $n\ge 0$ and $\alpha\ge 0$,

$$h\left(p^{2\alpha}n+7\cdot\frac{p^{2\alpha}-1}{24}\right)=h(n). \tag{5.84}$$

Replacing n by $p^{2\alpha}n+7\cdot\frac{p^{2\alpha}-1}{24}$ in (5.80), we arrive at (5.78). □

Theorem 5.10. *For any prime $p\ge 5$ with $(\frac{-10}{p})=-1$, $\alpha\ge 1$ and $n\ge 0$, we have*

$$\sum_{n=0}^{\infty} \overline{C}^5_{4,1}\left(16\cdot p^{2\alpha+2}n+16\cdot p^{2\alpha+1}i+\frac{14\cdot p^{2\alpha+2}+1}{3}\right)\equiv 0 \pmod{2^3}.$$

where $i=1,2,...,p-1$.

Proof. Replacing n by $p^2n + pi + \frac{7\cdot p^2-7}{24}$ in (5.84) and using (5.83), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$h\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{7\cdot p^{2\alpha+2}-7}{24}\right) = 0. \tag{5.85}$$

Comparing the coefficients of q^n from the both sides of (5.80), we see that for $n \geq 0$,

$$\overline{C}^5_{4,1}(16n+5) \equiv 4h(n) \pmod{2^3}. \tag{5.86}$$

The result follows from (5.85) and (5.86). □

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