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REVERSIBLE GEODESICS AND T-TENSOR OF FINSLER SPACE WITH GENERALIZED (α, β)-METRIC

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ABSTRACT. The present article is designed to deal with the presence of reversible geodesics of Finsler space endowed with generalized (α, β)-metric. We have deduce the criteria for a metric F defined on M to be a reversible geodesics. We investigate the various geometrical characteristics of Finsler metric F using reversible geodesics, and show that F generates d_F to be a weighted quasi metric defined on a manifold M. We also examined the T-tensor for the generalized (α, β)-metric.

1. Introduction

In Finsler geometry, the investigation of Finsler metric with reversible geodesics is a fascinating concept. Not just in Finsler geometry but also in physics, the investigation of Finsler space with reversible geodesic is a unique concept of study in the Finslerian geometry. As far as we know, a Finsler space has reversible geodesics if every oriented geodesic path is likewise a geodesic when traveled in the opposite direction. Crampin [2], discussed the Randers space with reversible geodesics. Masca et al. [8, 9] have discussed the reversible geodesics of two dimensional (α, β)-metric and (α, β)-metric with reversible geodesics, respectively. Sabau and Shimada [14], are investigated Finsler manifolds with reversible geodesics. However, we know that the T-tensor is not crucial just in differential geometry but also in a theory of general relativity. Matsumoto presented the concept of T-tensor in [10].

In [13, 15], the study of reversible geodesic of Finsler spaces with special (α, β)-metric are discussed. Many authors [1, 5, 11, 18], have studied the Finsler space with (α, β)-metric on different topics. Pradeep kumar et al. [6] have studied the Ricci tensors of Finsler space with special (α, β)-metric. In recent years, Qiaoling Xia gave an identical characterization for a general (α, β)-metric with reversible geodesics, when $n \geq 0$(dimension) in [19]. Elgendi and Kozma [3], are discussed the (α, β)-metric satisfies the T-tensor and the σ_T-condition in their work. Piscoran et al. [12] have studied the reversible geodesics equipped with a deformed

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special (α, β)-metric.

In this article, we have investigated the reversible geodesics of Finsler space equipped with generalized (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta}$, where μ_1, μ_2 and μ_3 are constants. In addition, we have examined the T-tensor attributed with a generalized (α, β)-metric. Firstly, we give an introduction for reversible geodesics of Finsler space in Section 1. In Section 2, we go through some basic preliminaries of Finsler space and existence of reversible geodesics. In Section 3, we discuss the reversible geodesics of Finsler space endowed with a generalized (α, β)-metric. Further, we investigate the relationship between the (α, β)-metric and the weighted quasi-metric in Section 4. Finally, in Section 5 we have determined the T-tensor attributed with a generalized (α, β)-metric.

2. Preliminaries

Let M be an n-dimensional smooth manifold then we denotes TM_0 be a slit tangent bundle of a manifold M and the local co-ordinates $(x, y) = (x^i, y^i)$ defined on TM. A Finsler metric on M is a function defined as $F : TM \to [0, \infty)$, then

(1) Regularity: F is C^∞ smooth curves on TM_0,
(2) F is absolutely one-homogeneous on TM: $F(x, \psi y) = \psi F(x, y)$, for $\psi > 0$,
(3) Strong-convexity: The Hessian matrix(HM) of F^2, $2\ g_{ij}(x, y) = \left(\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}\right)$, is non-negative at every point $v = (x, y) \in TM_0$. If $\rho : [0, 1] \to M$ is a piece wise curve C^∞ defined on M, moreover F^*(length) of ρ is described as [17]

$$F^*_{F(\rho)} := \int_0^1 F\left(\rho(t), \dot{\rho}(t)\right) dt. \tag{2.1}$$

From the definition of Finsler space, a requirement on F is imposed in order for the above equation must be regular under re-parametrization of curve ρ, that is, $F(x, \Psi y) = \Psi F(x, y)\ \forall\ \Psi > 0$. Length structures obtained from this type of constructions are called Finslerian length structure. Now, since we have a length structure therefore we can measure any distance on the manifold M. As a result, using length structure the distance function d_F defined as $d_F : M \times M \to [0, \infty)$ is also called the Finslerian distance function, which is determined as $d_F(a, b) = \inf_\rho F^*$. In this case, infimum is calculated across all piece-wise curves C^∞ connecting the points $a = \rho(0)$ to $b = \rho(1)$. In general, d_F is irreversible, i.e., $d_F(a, b) \neq d_F(b, a)$.

If a curve $\rho : [0, 1] \to M$ minimizes, equation (2.1) defines the Finslerian length structures for all piece wise curves C^∞ with fixed end-points, it is said to be geodesic of (M, F). Assign $\bar{F} : TM \to [0, \infty)$ be a smooth-function to the reversible Finsler metric of F, defined by $\bar{F}(x, y) := F(x, -y)$. It clears that $\bar{F}$ is likewise a Finsler metric. Remember the basic definitions and important lemmas listed below for further reference.

Definition 2.1. [14]: Let Γ be a general vector field defined on TM_0 is known as spray on M, if Γ endorses the below conditions

i. If a conventional co-ordinate system (x,y) defined on TM, then Γ can be written as $\Gamma = -2G^i(x,y)\dfrac{\partial}{\partial y^i} + y^i\dfrac{\partial}{\partial x^i}$, where $\mathrm{G^i}(x,\phi y) = \phi\ \ \mathrm{G^i}(x,y)$, $\forall\ \ \phi > 0$. $\mathrm{G^i}$ are said to be spray co-efficients,

ii. $\mathrm{G^i}$ are flat at $(x,y) \neq 0 \in \mathrm{TM}$.

We are occasionally using Γ rather than $\mathrm{G^i}$ to represent the spray.

Consider ρ be a constant speed geodesic of (M,F) and defined as $\rho : [0,1] \to M$ be locally a flat curve if and only if it assuages the following ODE

$$2G^i(\rho(t),\dot{\rho}(t)) + \ddot{\rho}^i(t) = 0,$$

with $i = 1,\cdots,n$, here $\dot{\rho} = \dot{\rho}^i\dfrac{\partial}{\partial x^i}$. We also recall that the famous Euler Lagrange equation of Finsler space F defined on M is connected with Γ be a geodesic spray, is provided by $\Gamma\left(\dfrac{\partial F}{\partial y^i}\right) = \dfrac{\partial F}{\partial x^i}$.

Definition 2.2. [14]: Let the two vivid Finsler manifolds F and $\bar{F}$ defined on the same differentiable manifold M. Then, if their geodesics as a group of points synergise, then F is projectively related to $\bar{F}$.

The preceding definition is comparable to $\bar{\Gamma}\left(\dfrac{\partial F}{\partial y^i}\right) - \dfrac{\partial F}{\partial x^i} = 0$ using geodesic sprays, where $\bar{\Gamma}$ denoted as a geodesic spray of $(M,\bar{F})$.

Lemma 2.3. [9] *Let Finsler space F defined on M be a connected smooth manifold, Finsler metric with $d_F : M\times M \to [0,\infty)$ as the attributed distance function. If F is absolute homogeneous, then $\mathrm{d_F}$ is said to be a symmetric distance function on $M \times M$, i.e., $F(x,y) := F(x,-y) = \bar{F}(x,y)$.*

Lemma 2.4. [16] *If Finsler metric F and $\bar{\mathrm{F}}$ be reversible function of F are projectively related, then F is a reversible geodesics defined on M. i.e., as a set of points, the geodesics of Finsler metrics F and $\bar{\mathrm{F}}$ synergise.*

If Finsler space F has reversible geodesic, then $\bar{\Gamma}$ denotes the reversible geodesic spray if and only if $\bar{\Gamma}\left(\dfrac{\partial F}{\partial y^i}\right) = \dfrac{\partial F}{\partial x^i}$.

Lemma 2.5. [3] *The components $C_{ijk} = \dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial y^k}$ of the Cartan tensor of an general (α,β)-metric are*

$$C_{ijk} = \frac{p_2}{2\alpha}\left(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j\right) + \frac{p_1^{'}}{2\alpha}m_i m_j m_k,$$

where $h_{ij} = a_{ij} - \alpha_i\alpha_j$, $m_i = b_i - s\alpha_i$.

Lemma 2.6. [3] *The T-tensor of an general (α,β)-metric, takes of the form* $T_{ijk}=\Omega\, m_hm_im_jm_k+\Psi(h_{hk}m_im_j+h_{hj}m_im_k+h_{hi}m_jm_k+h_{ij}m_hm_k+h_{jk}m_im_h+h_{ik}m_jm_h)+\Phi(h_{hi}h_{jk}+h_{hj}h_{ik}+h_{hk}h_{ij})$, *where,*

$$\begin{aligned}
\Omega &= \frac{p_1^{''}\phi}{2\alpha}+\frac{2p_1^{'}\phi^{'}}{\alpha}-3\phi\left(K_2\left(p_2+\frac{p_1^{'}t^2}{2}\right)+\frac{p_2p_1^{'}}{2\alpha p}\right),\\
\Psi &= \frac{p_2\phi^{'}}{\alpha}-\frac{p_1^2\phi}{\alpha p}-\frac{sp_1^{'}\phi}{2\alpha}-\frac{p_2\phi t^2K_2}{2},\\
\Phi &= -\frac{p_2\phi}{2\alpha}\left(s+\alpha K_1t^2\right),\\
K_1 &= \frac{p_2}{2\alpha\left(p+t^2\phi\phi''\right)},\\
K_2 &= \frac{pp_1^{'}-2p_2\phi\phi^{''}}{2\alpha p\left(p+t^2\phi\phi''\right)}.
\end{aligned}\tag{2.2}$$

As we know that for (α,β)-metric $F=\alpha\phi(s)$. This components $g_{ij}=\frac{1}{2}\frac{\partial F^2}{\partial y^i\partial y^j}$, the formula can be used to compute the fundamental tensor is

$$g_{ij}=pa_{ij}+p_1b_ib_j+p_2(b_i\alpha_j+b_j\alpha_i)+p_2\alpha_i\alpha_j,$$

where $\alpha_i=\frac{\partial\alpha}{\partial y^i}$ and

$$\begin{aligned}
p &= -s\phi\phi^{'}+\phi^2,\\
p_1 &= \phi\phi^{''}+\phi^{'2},\\
p_2 &= \phi\phi^{'}-s(\phi\phi^{''}+\phi^{'2}),\\
p_3 &= s^2(\phi^{'2}+\phi\phi^{''})-s\phi\phi^{'},
\end{aligned}\tag{2.3}$$

where $b^i=a^{ij}b_j$.
For $F=\alpha\phi(s)$ be a Finsler metric, the reciprocal (g^{ij}) of the Riemannian metric (g_{ij}) is

$$g^{ij}=qa^{ij}+q_1b^ib^j+q_2(b^i\alpha^j+b^j\alpha^i)+q_2\alpha^i\alpha^j,$$

where

$$\begin{aligned}
q &= \frac{1}{p},\\
q_1 &= \frac{\phi\phi''}{p(p+\phi\phi''t^2)},\\
q_2 &= -\frac{p_2}{p(p+\phi\phi''t^2)},\\
q_3 &= \frac{p_1(sp+(p_2+s\phi\phi^{''})t^2)}{p^2(p+\phi\phi''t^2)},
\end{aligned}\tag{2.4}$$

and $t^2=b^2-s^2$.

3. Reversible geodesics of Finsler space with a generalized (α,β)-metric

Let F be a Finsler space defined on a manifold M with generalized (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\alpha^2}{\beta}$, $\bar{F}=\mu_1\alpha+\mu_3\frac{\alpha^2}{\beta}$ can be represented by metric F. Then $\bar{F}$ be an absolutely homogeneous which can be seen as $\bar{F}(x,y):=\bar{F}(x,-y)$.

It is generally known fact that if Finsler space F is a non Riemannian n-dimensional $(n>2)$ defined on M with (α,β)-metric, it is not absolutely homogeneous, subsequently $F(\alpha,\beta)=F_0(\alpha,\beta)+\nu\beta$ if and only if F has reversible geodesics, here F_0 is absolutely homogeneous metric, ν $(\neq 0)$ is a constant and one-form β is a closed.

In this work, $F_0=\bar{F}$ is absolute homogeneous and $\nu=\mu_2$ is 1-form on M. Assume that if β is closed, at that time F has reversible geodesics. Now, consider the necessary condition for F as well as sufficient condition to show the reversible geodesics is,

$$\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right)=\frac{\partial F}{\partial x^i},$$

where $\bar{\Gamma}$ is the reversible of Γ. Here Γ and $\bar{\Gamma}$ are resemble as geodesic spray of F and $\bar{F}$ respectively. Now we have

$$F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\alpha^2}{\beta}\quad \textit{and}\quad \bar{F}=\mu_1\alpha+\mu_3\frac{\alpha^2}{\beta}.$$

Consequently, $F=\bar{F}+\mu_2\beta$. We have

$$\begin{aligned}
\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right)-\frac{\partial F}{\partial x^i}=&\bar{\Gamma}\left(F_\alpha\frac{\partial\alpha}{\partial y^i}+F_\beta\frac{\partial\beta}{\partial y^i}\right)-F_\alpha\frac{\partial\alpha}{\partial x^i}-F_\beta\frac{\partial\beta}{\partial x^i}\\
=&\bar{\Gamma}\left(F_\alpha\right)\frac{\partial\alpha}{\partial y^i}+F_\alpha\bar{\Gamma}\left(\frac{\partial\alpha}{\partial y^i}\right)+\bar{\Gamma}\left(F_\beta\right)\left(\frac{\partial\beta}{\partial y^i}\right)+F_\beta\bar{\Gamma}\left(\frac{\partial\beta}{\partial y^i}\right)\\
&-F_\alpha\frac{\partial\alpha}{\partial x^i}-F_\beta\frac{\partial\beta}{\partial x^i}\\
\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right)-\frac{\partial F}{\partial x^i}=&\bar{\Gamma}\left(F_\alpha\right)\frac{\partial\alpha}{\partial y^i}+\bar{\Gamma}\left(F_\beta\right)\frac{\partial\beta}{\partial y^i}+F_\alpha\left[\bar{\Gamma}\left(\frac{\partial\alpha}{\partial y^i}\right)-\frac{\partial\alpha}{\partial x^i}\right]\\
&+F_\beta\left[\bar{\Gamma}\left(\frac{\partial\beta}{\partial y^i}\right)-\frac{\partial\beta}{\partial x^i}\right].
\end{aligned}\tag{3.1}$$

Differentiating F with respect to α and β partially, we get

$$F_\alpha=\mu_1+2\mu_3\frac{\alpha}{\beta},\quad F_\beta=\mu_2-\mu_3\frac{\alpha^2}{\beta^2}.\tag{3.2}$$

The Euler-Lagrange equation for Riemannian metric α become $\bar{\Gamma}\left(\frac{\partial\alpha}{\partial y^i}\right)-\frac{\partial\alpha}{\partial x^i}=0$. Also, if Finsler space with (α,β)-metric defined on M, it is known by (see

lemma (2.5)), then $g(x,y)\frac{\partial\alpha}{\partial y^i}+h(x,y)b_i=0,\ \forall\quad i=1,2,\dots,n$, implies that $g=h=0$, here g and h are smooth functions on TM.
Suppose F is projectively related to α, it induces β is closed, then $\bar{\Gamma}\left(F_\alpha\right)\frac{\partial\alpha}{\partial y^i}+\bar{\Gamma}\left(F_\beta\right)b_i=0$. By lemma (2.4), we found that $\bar{\Gamma}\left(F_\alpha\right)=0,\quad \bar{\Gamma}\left(F_\beta\right)=0$.

Now using (3.2), equation (3.1) can be reduced to

$$\begin{aligned}
\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right)-\frac{\partial F}{\partial x^i} &= F_\beta\left[\bar{\Gamma}\left(\frac{\partial\beta}{\partial y^i}\right)-\frac{\partial\beta}{\partial x^i}\right]\\
&=\left(\mu_2-\mu_3\frac{\alpha^2}{\beta^2}\right)\left(\bar{\Gamma}\left(b_i\right)-\frac{\partial b_j}{\partial x_i}y^j\right)\\
\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right)-\frac{\partial F}{\partial x^i} &= \left(\mu_2-\mu_3\frac{\alpha^2}{\beta^2}\right)\left(\frac{\partial b_i}{\partial x^j}-\frac{\partial b_j}{\partial x^i}\right)y^j.
\end{aligned}$$

Here $\left(\mu_2-\mu_3\frac{\alpha^2}{\beta^2}\right)=0$ is not acceptable. As a result, if $\bar{\Gamma}\left(\frac{\partial F}{\partial y^i}\right)-\frac{\partial F}{\partial x^i}=0$, then F is associated with reversible geodesics. Particularly, if F is with reversible geodesics then β is closed one-form.

As a result, we have obtained the following results

Theorem 3.1. *Let Finsler space F defined on manifold M with generalized (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\alpha^2}{\beta}$ is reversible geodesic(RG) if and only if one-form β is closed on a manifold M.*

Further, if all the geodesics in a (M,F) are straight lines, it is called locally projective flat. The spray co-efficient G^i of Finsler space F can be represented by $G^i=P(x,y)y^i$, in an equivalent condition. Where $P(x,y)=\frac{1}{2F}\frac{\partial F}{\partial x^k}y^k$.

The Hamel's relation is a similar characterization of projective flatness [4]

$$\frac{\partial^2 F}{\partial x^m\partial y^k}y^m=\frac{\partial F}{\partial x^k}.$$

Suppose Finsler metric $F=\bar{F}+\mu_2\beta$, then one of the first two following conditions implies β is closed,

i. F is a projectively flat(P-F),
ii. $\bar{F}$ is also a projectively flat,
iii. β is closed on M.

In this situation, $F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\alpha^2}{\beta}=\bar{F}+\mu_2\beta$, whereas $\bar{F}=\mu_1\alpha+\mu_3\frac{\alpha^2}{\beta}$ is an absolutely homogeneous metric.

Theorem 3.2. *Let Finsler space F defined on manifold M with generalized (α,β)-metric, $F=\bar{F}+\mu_2\beta$. If $\bar{F}$ is projective flat then F is projective flat.*

Proof. By assuming F is projective flat, then F will be satisfying the Hamel's equation

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0$$

$$\frac{\partial^2}{\partial x^m \partial y^k}(\bar{F} + \mu_2\beta) y^m - \frac{\partial}{\partial x^k}(\bar{F} + \mu_2\beta) = 0$$

$$\frac{\partial^2 \bar{F}}{\partial x^m \partial y^k} y^m + \frac{\partial^2(\mu_2\beta)}{\partial x^m \partial y^k} y^m - \frac{\partial \bar{F}}{\partial x^k} - \frac{\partial(\mu_2\beta)}{\partial x^k} = 0$$

since β is closed

$$\frac{\partial^2 \bar{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \bar{F}}{\partial x^k} = 0.$$

This implies $\bar{F}$ is projective flat.

Conversely, by assuming $\bar{F}$ be projective flat, consequently $\bar{F}$ satisfies the Hamel's equation

$$\frac{\partial^2 \bar{F}}{\partial x^m \partial y^k} y^m - \frac{\partial \bar{F}}{\partial x^k} = 0$$

$$\frac{\partial^2}{\partial x^m \partial y^k}(F - \mu_2\beta) y^m - \frac{\partial}{\partial x^k}(F - \mu_2\beta) = 0$$

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial^2(\mu_2\beta)}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} + \frac{\partial(\mu_2\beta)}{\partial x^k} = 0$$

since β is closed

$$\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0.$$

This implies F is projective flat. □

Theorem 3.3. *Let F be a Finsler space defined on M with generalized (α,β)-metric, $F = \bar{F} + \mu_2\beta$. F is reversible geodesics if it is projectively flat on M.*

Proof. Using Hammel's equations, it is easy to obtain that $\bar{F}$ is projective flat if and only if F is projective flat. As a result, it indicates that F and $\bar{F}$ are projective flat corresponding to a regular Euclidean metric, if F is with reversible geodesic then F should be projective to $\bar{F}$ in accordance with lemma (2.4). And converse is not true. □

Hence the proof.

4. Generalized (α,β)-metric with weighted quasi-metrics

Riemannian spaces can be represented as metric spaces. Indeed, for a Riemannian space (M,α), we can define the induced metric space (M,d_α) with the metric $d_\alpha : M \times M \to [0,\infty)$ is given by

$$d_\alpha(x,y) := \inf_{\rho\in\Gamma_{xy}} \int_p^q \alpha\left(\rho(t),\dot{\rho}(t)\right) dt,$$

where $\Gamma_{xy} = \{\rho : [p,q] \to M$ such that ρ is a smooth piece-wise with $\rho(p) = x,\ \rho(q) = y\}$ is a set of curves linking with two co-ordinate points x and y and $\dot{\rho}$ is the tangent vector to ρ at $\rho(t)$.

It is simple to demonstrate that d_α is a metric defined on a manifold M. Finsler structures are known to be more general than Riemannian structures [17]. As in the Riemannian situation, a metric d_F on (M, F) can be induced as defined by $d_F : M \times M \to [0, \infty)$ and is given by

$$d_F(x,y) := \inf_{\rho \in \Gamma_{xy}} \int_p^q F\left(\rho(t), \dot{\rho}(t)\right) dt. \tag{4.1}$$

However, in the Riemannian case, d_F does not satisfy the symmetry criteria i.e., $d_F(x,y) \neq d_F(y,x)$. In reality, metric d_F is a sub-case of quasi metric. Remember from [7] the function d be quasi metric defined on set S is $d : S \times S \to [0, \infty)$ that meets the below mentioned features

1 Positivity: $d(a,b) > 0$ if $a \neq b$, $d(a,a) = 0$,
2 Symmetric: $d(a,b) = d(b,a) = 0 \Rightarrow a = b$,
3 Triangle-inequality: $d(a,c) \leq d(a,b) + d(b,c)$,

for any $a, b, c \in S$.

Let us define (M, d, ω) be a weighted quasi metric spaces, whereas quasi metric d defined on a manifold M for which there exists a function ω defined as $\omega : M \to [0, \infty)$, termed as weighted of d, are a specific type of quasi-metric spaces, that satisfies

4 Weight-ability: $d(b,a) + \omega(b) = d(a,b) + \omega(a)$, $\forall a, b \in M$.

When weighted function ω is real valued, it is referred to as generalised weight. It's worth noting that if metric space (M, d), it may be considered as a weighted metric space with ω is constant.

Theorem 4.1. *Let a n-dimensional Finsler space F defined on M is smoothly connected Finsler metric with $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\alpha^2}{\beta}$. Then, F induces generalised distance metric d_F on a manifold M.*

Proof. Let F be a Finsler space defined on a manifold M with (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\alpha^2}{\beta}$, this can be re-written as $F = \bar{F} + \mu_2\beta$, where $\bar{F} = \mu_1\alpha + \mu_3\dfrac{\alpha^2}{\beta}$ is an absolutely homogeneous (α, β)-metric defined on a manifold M and 1-form β is an exact.

Let $\rho_{xy} \in \Gamma_{xy}$ be an F-geodesic, which is in the same time $\bar{F}$-geodesic, then from equation (4.1), we get

$$\begin{aligned} d_F(x,y) &= \int_p^q F\left(\rho(t), \dot{\rho}(t)\right) dt, \\ &= \int_p^q \left(\mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta}\right) dt, \\ &= \int_p^q \left(\mu_1\alpha + \mu_3\frac{\alpha^2}{\beta}\right) dt + \int_{\rho_{xy}} \mu_2\beta, \\ &= \int_{\rho_{xy}} \mu_1\alpha + \mu_3\frac{\alpha^2}{\beta} + \int_{\rho_{xy}} \mu_2\beta. \end{aligned} \tag{4.2}$$

Consider the following point $p \in M$ and the function defined by $\omega_p : M \to R$ by $\omega_p = d_F(p,x) - d_F(x,p)$. Then by equation (4.2) it follows that

$$\omega_p(x) = \int_{\rho_{px}} \mu_2\beta - \int_{\rho_{xp}} \mu_2\beta = -2\mu_2 \int_{\rho_{xp}} \beta, \tag{4.3}$$

where we have used the Stokes theorem for the 1-form β on the closed domain D with boundary $\partial D = \rho_{cx} \cup \rho_{xc}$.
It is clear that an ω_p is a anti-derivative of one-form β. This is well defined if and only if the integral in the R.H.S. of equation (4.3) is independent path, that is, one-form β must be exact.

Hence weighted quasi metric d_F with generalised weight ω_c. Following that,

$$\begin{aligned} d_F(x,y) + \omega_c(x) &= \int_{\rho_{xy}} \mu_1\alpha + \mu_3\frac{\alpha^2}{\beta} + \int_{\rho_{xy}} \mu_2\beta + \int_{\rho_{cx}} \mu_2\beta - \int_{\rho_{xc}} \mu_2\beta, \\ &= \int_{\rho_{xy}} \mu_1\alpha + \mu_3\frac{\alpha^2}{\beta} - \int_{\rho_{xc}} \mu_2\beta - \int_{\rho_{yc}} \mu_2\beta, \end{aligned}$$

where we have again used the Stokes theorem for the 1-form β on the closed domain with $\partial D = \rho_{px} \cup \rho_{xy} \cup \rho_{yp}$. We can also find

$$d_F(y,x) + \omega_p(y) = \int_{\rho_{yx}} \mu_1\alpha + \mu_3\frac{\alpha^2}{\beta} - \int_{\rho_{yp}} \mu_2\beta - \int_{\rho_{xp}} \mu_2\beta,$$

as a result, weighted quasi metric d_F with generalised weight ω_p. □

For later use, remember [7] the following lemma.

Lemma 4.2. *Let quasi metric d defined on M. Then d is weightable quai metric if and only if there exists a function defined as $\omega : M \to [0,\infty)$ is*

$$d(x,y) = \frac{1}{2}\left[2f(x,y) - (-\omega(x) + \omega(y))\right], \quad \forall\, x,y \in M. \tag{4.4}$$

The symmetrized distance of d is denoted by f. Furthermore, we have

$$|\omega(y) - \omega(x)| \leq 2f(x,y), \quad \forall\, x,y \in M.$$

Now, it should be noted that if Finsler space F on M with $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta}$, then the symmetrized metric F and the induced quasi-metric d_F impose the same topology on a manifold M. Furthermore, Lemma (4.2) shows that the premise of smoothness is not required. In reality, if d_F is a weighted quasi metric, then function ω is differentiable on a manifold M. As a consequence, the one-form β exists practically everywhere on M.

Following that, we will go over an essential geometric characteristic of geodesic triangles.

Theorem 4.3. *Let F be a Finsler space defined on M with (α,β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta}$, then the perimeter of every geodesic triangle on M is independent, i.e.,*

$$d_F(z,y)+d_F(y,x)+d_F(x,z) = d_F(z,x)+d_F(x,y)+d_F(y,z) \quad \forall\, x,y,z \in M. \tag{4.5}$$

Proof. Since from the Theorem (4.1), the generalized (α,β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta}$ can be regarded as the Randers change, i.e., $F = \bar{F} + \beta$ with $d\beta = 0$. Where $\bar{F} = \mu_1\alpha + \mu_3\frac{\alpha^2}{\beta}$ is a absolute homogeneous Finsler metric. As a result, the quasi metric is weight-able, and relation (4.4) holds true. A simple computation using the relation (4.4) yields the needed result (4.5). □

5. T-tensor associated with a generalized (α,β)-metric

Let us consider a class of generalized (α,β)-metric on a manifold M^n defined by the following form

$$F = \alpha\phi(s), \quad \phi(s) = \mu_1 + \mu_2 s + \mu_3\frac{1}{s}, \tag{5.1}$$

that is,

$$F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta},$$

where Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and 1-form metric $\beta = b_i(x)y^i$ defined on M. From (2.2), we have

$$\mu_1 + \mu_2 s + \mu_3\frac{1}{s} > 0, \quad \mu_1 + 2\mu_3 b^2 - 3\mu_3\frac{1}{s} > 0, \quad |s| \le b < b_0.$$

F is Finsler metric if and only if β satisfies that $b = \|\beta(x)\|_\alpha < b_0$ for any $x \in M$.

In virtue of equation (5.1) we obtain the coefficients of a fundamental tensor for the metric g_{ij} and reciprocal metric g^{ij}, equations (2.3) and (2.4) are morphed as

$$\begin{aligned} p &= \frac{(\mu_1 s + 2\mu_3)(\mu_2 s^2 + \mu_1 s + \mu_3)}{s^2}, \\ p_1 &= \frac{\mu_2^2 s^4 + 2\mu_1\mu_3 s + 3\mu_3^2}{s^4}, \\ p_2 &= -\frac{4\mu_3^2 + 3\mu_1\mu_3 s - \mu_1\mu_2 s^3}{s^3}, \\ p_3 &= \frac{4\mu_3^2 + 3\mu_1\mu_3 s - \mu_1\mu_2 s^3}{s^2}, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} q &= \frac{s^2}{(\mu_1 s + 2\mu_3)(\mu_2 s^2 + \mu_1 s + \mu_3)}, \\ q_1 &= -\frac{2\mu_3 s^2}{(\mu_1 s + 2\mu_3)(\mu_1 s^3 + 2\mu_3 s^2 + 2\mu_3 t^2)(\mu_2 s^2 + \mu_1 s + \mu_3)}, \\ q_2 &= -\frac{s^3(4\mu_3^2 + 3\mu_1\mu_3 s - \mu_1\mu_2 s^3)}{(\mu_1 s + 2\mu_3)(\mu_1 s^3 + 2\mu_3 s^2 + 2\mu_3 t^2)(\mu_2 s^2 + \mu_1 s + \mu_3)^2}, \\ q_3 &= \frac{s(\mu_1 s + 2\mu_3)(\mu_2 s^4 + \mu_1 s^3 + \mu_2 s^2 t^2 + \mu_3 s^2 - \mu_3 t^2)}{(\mu_1 s^3 + 2\mu_3 s^2 + 2\mu_3 t^2)(\mu_2 s^2 + \mu_1 s + \mu_3)}. \end{aligned} \tag{5.3}$$

Theorem 5.1. *Let $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\alpha^2}{\beta}$ be a generalized (α,β)-metric. The T-tensor of an generalized (α,β)-metric F takes of the form $T_{ijk} = \Omega m_h m_i m_j m_k + \Phi(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij}) + \Psi(h_{hk}m_i m_j + h_{hj}m_i m_k + h_{hi}m_j m_k + h_{ij}m_h m_k + h_{jk}m_i m_h + h_{ik}m_j m_h)$, where Ω, Φ and Ψ are defined in (2.2).*

Proof. Theorem proof is straightforward, by using the expressions of lemma (2.6) we obtain the coefficients of T_{ijk} and making the calculations by using (5.1), (5.2) and (5.3), we get

$$\begin{aligned} \Omega =& \frac{1}{s^5(\mu_1 s + 2\mu_3)(\mu_1 s^3 + 2\mu_3 s^2 + 2\mu_3 t^2)} \left[6\mu_1^2\mu_2^2\mu_3 s^6 + 18\mu_1^3\mu_2\mu_3 s^5 + (12\mu_1^4\mu_3 + \right. \\ & 48\mu_1^2\mu_2\mu_3^2)s^4 + (48\mu_1^3\mu_3^2 + 48\mu_1\mu_2\mu_3^3)s^3 + (84\mu_1^2\mu_3^3 + 24\mu_2\mu_3^4)s^2 + (12\mu_1\mu_2\mu_3^3 t^2 \\ & \left. + 72\mu_1\mu_3^4 - 3\mu_1^3\mu_3^2 t^2)s + (24\mu_2\mu_3^4 t^2 - 6\mu_1^2\mu_3^3 t^2 + 24\mu_3^5)\right], \end{aligned}$$

$$\begin{aligned} \Phi =& \frac{\alpha}{4(\mu_1 s^5 + 2\mu_3 s^4 + 2\mu_3 t^2 s^2)} \left[-2\mu_1^2\mu_2^2 s^7 - (2\mu_1^3\mu_2 + 4\mu_1\mu_2^2\mu_3)s^6 - (\mu_1^2\mu_2^2 t^2 + \right. \\ & 6\mu_1^2\mu_2\mu_3)s^5 + (6\mu_1^2\mu_2\mu_3 - 8\mu_1\mu_2\mu_3^2 - 4\mu_1\mu_2^2\mu_3 t^2 + 6\mu_1^3\mu_3)s^4 + (26\mu_1^2\mu_3^2 + \\ & 16\mu_2\mu_3^3 + 12\mu_1^2\mu_2\mu_3 t^2)s^3 + (36\mu_1\mu_3^2 + 16\mu_1\mu_2\mu_3^2 t^2)s^2 + (16\mu_2^2\mu_3^2 + 16\mu_2\mu_3^3 t^2 \\ & \left. + 3\mu_1^2\mu_3^2 t^2)s + 4\mu_1\mu_3^3 t^2\right] \end{aligned}$$

and

$$\Psi = \left[\frac{\begin{array}{c}2\mu_1^2\mu_2^2 s^{11} + k_1 s^{10} + k_2 s^9 + k_3 s^8 + k_4 s^7 + k_5 s^6 \\ + k_6 s^5 + k_7 s^4 + k_8 s^3 + k_9 s^2 + k_{10} s + 12\mu_3^5 t^2\end{array}}{2\alpha s^6(\mu_1 s + 2\mu_3)(\mu_1 s^3 + 2\mu_3 s^2 + 2\mu_3 t^2)}\right]$$

$$\begin{aligned}
k_1 =& 2\mu_1^4\mu_2 - 2\mu_1^3\mu_2^2 + 8\mu_1^2\mu_2^2\mu_3,\\
k_2 =& 4\mu_1^3\mu_2\mu_3 - 4\mu_1^2\mu_2^2\mu_3 + 8\mu_1\mu_2^2\mu_3^2,\\
k_3 =& 6\alpha^2\mu_1^3\mu_2\mu_3 - 6\mu_1^4\mu_3 + 12\mu_1^3\mu_2\mu_3 + 4\mu_1^2\mu_2^2\mu_3 t^2 - 16\mu_1^2\mu_2\mu_3^2,\\
k_4 =& 6\alpha^2\mu_1^4\mu_3 + 36\alpha^2\mu_1^2\mu_2\mu_3^2 + 4\mu_1^3\mu_2\mu_3 t^2 - 38\mu_1^3\mu_3^2 - 4\mu_1^2\mu_2^2\mu_3 t^2 + 40\mu_1^2\mu_2\mu_3^2\\
&+ 24\mu_1\mu_2^2\mu_3 t^2 + 8\mu_1\mu_2^2\mu_3^2 t^2 - 48\mu_1\mu_2\mu_3^3,\\
k_5 =& 42\alpha^2\mu_1^3\mu_3^2 + 72\alpha^2\mu_1\mu_2\mu_3^2 - 18\mu_1^3\mu_3^2 + 3\mu_1^2\mu_2^2\mu_3 t^2 - 88\mu_1^2\mu_3^2 + 32\mu_1\mu_2\mu_3^2\\
&- 32\mu_2\mu_3^4,\\
k_6 =& 12\alpha^2\mu_1^2\mu_2\mu_3^2 t^2 + 108\alpha^2\mu_1^2\mu_3^3 + 48\alpha^2\mu_2\mu_3^4 - 12\mu_1^3\mu_3^2 t^2 + 24\mu_1^2\mu_2\mu_3^2 t^2\\
&- 84\mu_1^2\mu_3^2 + 6\mu_1\mu_2^2\mu_3^2 t^2 - 32\mu_1\mu_2\mu_3^3 t^2 - 88\mu_1\mu_3^4,\\
k_7 =& 12\alpha^2\mu_1^3\mu_3^2 t^2 + 48\alpha^2\mu_1\mu_2\mu_3^3 + 120\alpha^2\mu_1\mu_3^4 + 4\mu_1^2\mu_2\mu_3^2 t^2 - 52\mu_1^2\mu_3^2 t^2 +\\
&32\mu_1\mu_2\mu_3^2 t^2 - 128\mu_1\mu_3^4 + 4\mu_2^2\mu_3^3 t^2 - 32\mu_2\mu_3^4 t^2 - 32\mu_3^5,\\
k_8 =& 48\alpha^2\mu_2\mu_3^4 t^2 + 48\alpha^2\mu_3^5 - 24\mu_1^2\mu_3^3 t^2 + 6\mu_1\mu_2\mu_3^2 t^2 - 72\mu_1\mu_3^4 t^2 - 64\mu_3^5,\\
k_9 =& 96\alpha^2\mu_1\mu 3^4 + 21\mu_1^2\mu_3^3 t^2 - 90\mu_1\mu_3^4 t^2 - 32\mu_3^5 t^2,\\
k_{10} =& 48\alpha^2\mu_3^5 t^2 + 26\mu_1\mu_3^4 t^2 - 64\mu_3^5 t^2.
\end{aligned}$$

Also

$$K_1 = -\frac{s\left(4\mu_3^2 + 3\mu_1\mu_3 s - \mu_1\mu_2 s^3\right)}{2\alpha\left(\mu_1 s^3 + 2\mu_3 s^2 + 2\mu_3 t^2\right)\left(\mu_2 s^2 + \mu_1 s + \mu_3\right)},$$

and

$$K_2 = -\frac{\mu_3\left(3\mu_1^2 s^2 + 6\mu_1\mu_3 s + 2\mu_1\mu_2 s^3 + 4\mu_3^2\right)}{\alpha s\left(2\mu_3 + \mu_1 s\right)\left(\mu_1 s^3 + 2\mu_3 s^2 + 2\mu_3 t^2\right)\left(\mu_2 s^2 + \mu_1 s + \mu_3\right)}.$$

□

6. Conclusion

We succeed to explore the existence of reversible geodesics of Finsler space with generalized (α, β)-metric. We have obtained the criteria for Finsler space F on M to be reversible geodesics. Further, we have examined various geometrical characteristics of F using reversible geodesics also, we have proved that the generalized (α, β)-metric F generates a weighted quasi metric d_F on M. Finally, we have obtained the result on T-tensor with generalized (α, β)-metric.

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