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EXISTENCE OF WEIGHTED PSEUDO ALMOST PERIODIC MILD SOLUTIONS FOR NONLOCAL SEMILINEAR EVOLUTION EQUATIONS

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Abstract. In this paper, we are concerned with new weighted pseudo almost periodic solutions of the semilinear evolution equations with nonlocal conditions

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad x(0) = x_0 + g(x), \ t \in \mathbb{R}^+.$$

By applying the Banach fixed point theorem, the theory of the measure theory, the theory of semigroups of operators to evolution families and the properties of a class of new weighted pseudo almost periodic functions which proposed by Blot et al.[1] recently, some reasonable conditions for the existence and uniqueness of new weighted pseudo almost periodic mild solutions are established.

1. Introduction

In this paper, we study the existence and uniqueness of new weighted pseudo almost periodic mild solution of the semilinear evolution equations in a Banach space X

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in \mathbb{R}^+, \tag{1.1}$$

with nonlocal conditions

$$x(0) = x_0 + g(x), \tag{1.2}$$

where the family $\{A(t) : t \in \mathbb{R}\}$ of operators in X generates an exponentially stable evolution family $\{U(t, s) : t \geq s\}$, $f : \mathbb{R}^+ \times X \to X$ and $g : X \to X$ are appropriate functions. Inspired by Blot, Cieutat and Ezzinbi[1], under some suitable assumptions, the existence and uniqueness of new weighted pseudo almost periodic mild solutions for equation (1.1) with nonlocal conditions (1.2) are derived.

The weighted pseudo almost periodic solution of semilinear evolution equations becomes a very active topic. Some important properties and applications to several classes of differential equations were established, see, e.g., Blot et al. [2], Diagana [3, 4] and Cao et al. [5]. Diagana [6] first proposed the concept of weighted pseudo almost periodic function, which is the generalization of the

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classical almost periodic functions of Bohr, the vector-valued almost periodic functions of Bochner (e.g.,[6, 7]), the asymptotically almost periodic functions of Fréchet [8, 9] as well as the pseudo almost periodic functions of Zhang [10, 11, 12].

Very recently, Blot et al. [1] presents a new approach to investigate weighted pseudo almost periodic functions using the measure theory. They present a new concept of weighted ergodic function which is more general than the classical one and establish many interesting results on the functional space of weighted ergodic functions like completeness and composition theorems. With these preliminary settings they introduced a new concept called μ-pseudo almost periodic function. The theory of their work generalizes the classical results on weighted pseudo almost periodic functions. Furthermore, in their work, through many counterexamples, they explain the reason why it is false about many results developed in the literature of weighted pseudo almost periodic functions before, for example uniqueness of the decomposition of weighted pseudo almost periodic function is not unique, the proof of the completeness of the space of weighted pseudo almost periodic functions based on the uniqueness is not correct. At last, Blot, Cieutat and Ezzinbi [1] correct many results obtained in the literature of weighted pseudo almost periodic functions and present new more general results based on measure theory, and for illustration, they provide some applications for evolution equations which include reaction diffusion systems and partial functional differential equations. As is known to all, it is more complicated and challengeable to study the properties of weighted pseudo almost periodic functions than the almost periodic and the pseudo almost periodic functions because the influence of the weight ρ is very strong sometimes. Therefore, it is worthy to study deeply the theory of weighted pseudo almost periodic functions by new ideas.

Motivated by the above discussion, this paper studies new weighted pseudo almost periodic solutions for the semilinear evolution equation with nonlocal conditions. The primary contribution of this paper is to derive some reasonable sufficient conditions for the existence and uniqueness of μ-pseudo almost periodic solutions to the class of the semilinear evolution equations (1.1) with nonlocal conditions (1.2).

The rest of this paper is organized as follows. Some concepts and preliminary results are introduced in Section 2. The main results of this paper with the existence and uniqueness theorems are presented in section 3. Finally, an illustrative example is given to verify the theoretical results.

2. Preliminary

Some remarks on the notation are given as follows. $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{C}$ respectively denote the set of nonnegative real numbers, the set of real numbers and the set of complex numbers. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two different Banach spaces. $C(\mathbb{R}, X)$ stands for the space of continuous functions from $\mathbb{R}$ to X, and $BC(\mathbb{R}, X)$ ($BC(\mathbb{R}\times Y, X)$) represents the space of bounded continuous functions with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in \mathbb{R}\},$$

(the space of jointly bounded continuous functions). Let $\mathcal{L}(Y,X)$ be the all bounded linear operators from Y to X over Banach space. Especially, we denote $\mathcal{L}(Y,X)$ by $\mathcal{L}(X)$ when $Y=X$. $L(\mathbb{R},\mathcal{L}(X))$ denotes the Lebesgue space with respect to the Lebesgue measure on $\mathbb{R}$. If L is a linear operator on X, then $D(L)$, $R(L)$, $\rho(L)$ respectively represent the domain, range and resolvent of operator L. If there exist parameters $M\geq 1$ and $\delta>0$ satisfy

$$\|U(t,s)\|\leq Me^{-\delta(t-s)},\quad \text{for}\quad t\geq s,$$

then, we call $\{U(t,s):t\geq s\}$ is an exponentially stable evolution family. Let $\mathcal{B}$ be the Lebesgue σ-field of $\mathbb{R}$ and $\mathcal{M}$ be the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R})=+\infty$, and $\mu([a,b])<+\infty$, for all $a,b\in\mathbb{R}$ $(a\leq b)$, respectively. From $\mu\in\mathcal{M}$, we making the following hypotheses.

(H_1) For all a, b and $c\in\mathbb{R}$, such that $0\leq a<b\leq c$, there exist $\tau_0\geq 0$ and $\alpha_0\geq 0$ such that

$$|\tau|\geq\tau_0\geq 0\Rightarrow\mu([a+\tau,b+\tau])\geq\alpha_0\mu([\tau,c+\tau]).$$

(H_2) For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that

$$\mu(\{a+\tau,a\in A\})\leq\beta\mu(A),\ \text{when}\ A\in\mathcal{B}\ \text{satisfies}\ A\cap I=\emptyset.$$

Remark 2.1. Hypothesis (H_2) implies (H_1), but the reciprocal assertion does not hold.

Firstly, we shall introduce some basic definitions and results on almost periodic functions.

Definition 2.2. [13] A continuous function $f:\mathbb{R}\to X$ is said to be (in Bohr sense) almost periodic in $t\in\mathbb{R}$ if for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau)-f(t)\|\leq\varepsilon,\quad\text{for each}\quad t\in\mathbb{R}.$$

The number τ is called an ε-translation number of f, and the sequel $AP(\mathbb{R},X)$ denotes the space of the almost periodic X-valued functions. It is well-known that a continuous function $f:\mathbb{R}\to X$ is almost periodic if and only if the set $\{f_\tau:\tau\in\mathbb{R}\}$ is relatively compact in $BC(\mathbb{R},X)$, where the function f_τ is defined by

$$f_\tau(s):=f(\tau+s),\quad\text{for}\quad s\in\mathbb{R}.$$

Definition 2.3. [13] A continuous function $f:\mathbb{R}\times Y\to X$ is said to be (in Bohr sense) almost periodic in $t\in\mathbb{R}$ uniformly with respect to $y\in Y$ if for every $\varepsilon>0$ and any compact $K\subset Y$ there exists $l(\varepsilon)>0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t+\tau,y)-f(t,y)\|\leq\varepsilon,\quad\text{for each}\quad t\in\mathbb{R}\quad\text{and}\quad y\in K.$$

The collection of those functions is denoted by $AP(\mathbb{R}\times Y,X)$.

Let U be the set of all functions $\rho : \mathbb{R} \to [0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $r > 0$ and each $\rho \in U$, set

$$m(r, \rho) := \int_{-r}^{r} \rho(x)\mathrm{d}x,$$

and the notations U_∞ stands for the set of weight functions

$$U_\infty := \{\rho \in U : \lim_{r\to\infty} m(r, \rho) = \infty\}.$$

For $\rho \in U_\infty$, define the weighted ergodic space

$$PAP_0(X, \rho) := \Big\{\varphi \in BC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|\varphi(t)\|\rho(t)\mathrm{d}t = 0\Big\}.$$

Definition 2.4. [6] Let $\rho \in U_\infty$. A continuous function $f : \mathbb{R} \to X$ is said to be weighted pseudo almost periodic if it can be decomposed as

$$f = g + \varphi,$$

where $g \in AP(\mathbb{R}, X)$ and $\varphi \in PAP_0(X, \rho)$.

The collection of those functions is denoted by $WPAP(\mathbb{R}, X, \rho)$.

Secondly, we will present some definitions and basic results on μ-pseudo almost periodic functions (for more details, see [1]).

Definition 2.5. [1] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is called μ-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t)\|\mathrm{d}\mu(t) = 0.$$

Denote the space of all such functions by $\mathcal{E}(\mathbb{R}, X, \mu)$.

Lemma 2.6. [1]. *Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $I \neq \emptyset$). Assume that $f \in BC(\mathbb{R}, X)$. Then the following assertions are equivalent:*

(i) $f \in \mathcal{E}(\mathbb{R}, X, \mu)$.

(ii)

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r] \setminus I)} \int_{[-r,r]\setminus I} \|f(t)\| d\mu(t) = 0.$$

(iii) For any $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{\mu(\{t \in [-r, r] \setminus I : \|f(t)\| > \varepsilon\})}{\mu([-r, r] \setminus I)} = 0.$$

Definition 2.7. [1] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be μ-ergodic in t uniformly with respect to y in Y if the two following conditions are true:

(i) for all $y \in Y$, $f(t, y) \in \mathcal{E}(\mathbb{R}, X, \mu)$,

(ii) f is uniformly continuous on each compact set K in Y with respect to the second variable y.

Denote the space of all such functions by $\mathcal{E}(\mathbb{R} \times Y, X, \mu)$.

Definition 2.8. [1] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is said to be μ-pseudo almost periodic if f is written in the form:

$$f = g + \varphi,$$

where $g \in AP(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$.

Denote by $PAP(\mathbb{R}, X, \mu)$ the set of all such functions.

Remark 2.9. One can easily prove that

$$AP(X) \subset PAP(\mathbb{R}, X, \mu) \subset BC(\mathbb{R}, X).$$

Remark 2.10. Following [6], a function f is the so-called weighted pseudo almost periodic if f is a μ-pseudo almost periodic function in the particular case where the measure μ satisfies the conditions: μ is the positive measure defined by

$$\mu(A) = \int_A \rho(t)\mathrm{d}t, \quad \text{for} \quad A \in \mathcal{B}. \tag{2.1}$$

where ρ is a nonnegative $\mathcal{B}$-measurable function, $\mathrm{d}t$ denotes the Lebesgue measure on $\mathbb{R}$.

The function ρ which occurs in (2.1) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$. In this case $\mu \in \mathcal{M}$ if and only if its Radon-Nikodym derivative ρ is locally Lebesgue-integrable on $\mathbb{R}$ and it satisfies

$$\int_{-\infty}^{+\infty} \rho(t)\mathrm{d}t = +\infty.$$

By using Radon-Nikodym's theorem, one can prove that the weighted pseudo almost periodic function in the sense of [6] is exactly the μ-pseudo almost periodic function in the sense of Definition 2.8 when the μ-measure zero sets are exactly the Lebesgue measure zero sets.

One can observe that the notion of μ-pseudo almost periodic function is a generalization of the weighted pseudo almost periodic function which is due to Diagana [6], that is a μ-pseudo almost periodic function, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$\mathrm{d}\mu(t) = \rho(t)\mathrm{d}t.$$

Lemma 2.11. [1] *Let $\mu \in \mathcal{M}$ satisfy (H_1). Then the decomposition of a μ-pseudo almost periodic function in the form $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$, is unique.*

Remark 2.12. Since the decomposition of a μ-pseudo almost periodic function f is unique, the functions g and φ in Definition 2.8 are, respectively, called the almost periodic component and the μ-ergodic perturbation of the μ-pseudo almost periodic function f.

Lemma 2.13. [1] *Let $\mu \in \mathcal{M}$ satisfy (H_1). Then $(PAP(\mathbb{R}, X, \mu), \|\cdot\|_\infty)$, is a Banach space.*

Lemma 2.14. [1] *Let $\mu \in \mathcal{M}$ satisfy (H_2). Then $PAP(\mathbb{R}, X, \mu)$ is invariant by translation, that is $f \in PAP(\mathbb{R}, X, \mu)$ implies $f_\tau \in PAP(\mathbb{R}, X, \mu)$ for all $\tau \in \mathbb{R}$.*

Definition 2.15. [1] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times Y \to X$ is said to be μ-pseudo almost periodic in t uniformly with respect to y in Y if f is written in the form:

$$f = g + \varphi,$$

where $g \in AP(\mathbb{R} \times Y, X)$ and $\varphi \in \mathcal{E}(\mathbb{R} \times Y, X, \mu)$.

Denote by $PAP(\mathbb{R} \times Y, X, \mu)$ the set of all such functions.

Finally, we will give some definitions and basic properties of the semigroups of operators to evolution families which are used further in this paper.

Lemma 2.16. [14, 15] *Let $\sum_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \theta\} \cup \{0\} \subset \rho(A(t))$, $\theta \in (\frac{\pi}{2}, \pi)$ if there exist a constant k_0 and a set of real numbers $\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k$ with $0 \leq \beta_i < \alpha_i \leq 2$, $i = 1, 2, \cdots, k$ such that*

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\| \leq k_0 \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1},$$

for $t, s \in \mathbb{R}$, $\lambda \in \sum_\theta \backslash \{0\}$ and there exists a constant $M \geq 0$ such that

$$\|(\lambda - A(t))^{-1}\| \leq \frac{M}{1 + |\lambda|}, \ \lambda \in \sum_\theta .$$

Then there exists a unique evolution family $\{U(t, s) : t \geq s > -\infty\}$.

3. μ-pseudo almost periodic mild solutions

In order to investigate the existence and uniqueness of μ-pseudo almost periodic solutions to Eqs.(1.1) and (1.2), we first introduce the notion of mild solution.

Definition 3.1. A mild solution to the equations (1.1) and (1.2) is a continuous function $x(t) : \mathbb{R}^+ \to X$ satisfying

$$x(t) = U(t, 0)(x_0 + g(x)) + \int_0^t U(t, s) f(s, x(s)) \mathrm{d}s,$$

for $t \geq 0$.

To study the existence and uniqueness of μ-pseudo almost periodic mild solutions to equations (1.1) and (1.2), we need the following assumptions:

(H_3) The function $f(t, x) \in PAP(\mathbb{R} \times X, X, \mu)$ and there exists a constant L_f such that

$$\|f(t, x) - f(t, y)\| \leq L_f \|x - y\|,$$

for all $t \in \mathbb{R}^+$ and $x, y \in PAP(\mathbb{R}, X, \mu)$.

(H_4) The function $g : X \to X$ is continuous and bounded, moreover satisfies that there exists a constant L_g such that

$$\|g(x) - g(y)\| \leq L_g \|x - y\|,$$

for all $t \in \mathbb{R}^+$ and $x, y \in X$.

(H_5) $A(t)$, $t \in \mathbb{R}$, satisfies Lemma 2.16 and $\{U(t,s), t \geq s\}$ is exponentially stable;

(H_6) $U(t,s)$, $t \geq s$ satisfy the condition that, for each $\varepsilon_1 > 0$, there exist positive numbers M, l_{ε_1}, such that each interval of length $l_{\varepsilon_1} > 0$, contains a number τ with the property that

$$\|U(t+\tau, s+\tau) - U(t,s)\| < Me^{-\frac{\delta}{2}(t-s)}\varepsilon_1.$$

In what follows, We giving some characterization lemmas which play an important role in obtain our main results.

Lemma 3.2. *Let $\mu \in \mathcal{M}$ satisfy (H_2) and $f : \mathbb{R} \times X \to X$, $(t,x) \to f(t,x)$ be μ-pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X$, and assume that f satisfies the following Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|,$$

for all $x, y \in X$ and $t \in \mathbb{R}$, where $L > 0$ is a constant which is independent of t. Then for any μ-pseudo almost periodic function $x : \mathbb{R} \to X$, the function $F : \mathbb{R} \to X$ given by $F(t) = f(t, x(t))$ is μ-pseudo almost periodic.

Proof. Since $f(t, x(t))$ is μ-pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $x(t) \in X$, then $f(t, x(t))$ can be written in the form:

$$f(t, x(t)) = f_1(t, x(t)) + f_2(t, x(t)),$$

where $f_1(t, x(t)) \in AP(\mathbb{R} \times X, X)$ and $f_2(t, x(t)) \in \mathcal{E}(\mathbb{R} \times X, X, \mu)$. On the other hand, since $x(t)$ is μ-pseudo almost periodic, then $x(t)$ can be written in the form:

$$x(t) = x_1(t) + x_2(t),$$

where $x_1(t) \in AP(\mathbb{R}, X)$ and $x_2(t) \in \mathcal{E}(\mathbb{R}, X, \mu)$. Thus the function $f(t, x(t))$ can be decomposed as

$$\begin{aligned} f(t, x(t)) &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + [f(t, x_1(t)) - f_1(t, x_1(t))] \\ &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + f_2(t, x_1(t)). \end{aligned}$$

As $f_1(t, x(t)) \in AP(\mathbb{R} \times X, X)$, then for every $\varepsilon > 0$ and any compact $K \subset X$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f_1(t+\tau, x(t)) - f_1(t, x(t))\| \leq \frac{\varepsilon}{2}, \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad x \in K.$$

Let $\varepsilon > 0$ be above, the uniform continuity of $f_1(t, x(t))$ on K implies that there exists a number $\delta > 0$ such that

$$\|f_1(t, m(t)) - f_1(t, n(t))\| \leq \frac{\varepsilon}{2}, \quad \text{for} \quad t \in \mathbb{R} \quad m(t), n(t) \in X \quad \text{and} \quad \|n(t) - m(t)\| \leq \delta.$$

Since $x_1(t) \in AP(\mathbb{R}, X)$, then for the above $\delta > 0$ there exists $l(\delta) > 0$ such that every interval of length $l(\delta)$ contains a number τ with the property that

$$\|x_1(t+\tau) - x_1(t)\| \leq \delta, \quad \text{for each} \quad t \in \mathbb{R}.$$

Thus

$$\begin{aligned}&\|f_1(t+\tau,x_1(t+\tau))-f_1(t,x_1(t))\|\\=&\|f_1(t+\tau,x_1(t+\tau))-f_1(t+\tau,x_1(t))+f_1(t+\tau,x_1(t))-f_1(t,x_1(t))\|\\\leq&\|f_1(t+\tau,x_1(t+\tau))-f_1(t+\tau,x_1(t))\|+\|f_1(t+\tau,x_1(t))-f_1(t,x_1(t))\|\\\leq&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

That is $f_1(t,x_1(t))\in AP(\mathbb{R},X)$. On the other hand, denote by $K=\overline{\{x_1(t):t\in\mathbb{R}\}}$. Since f satisfies the Lipschitz condition, we deduce that for given $\varepsilon>0$, there exists $\delta>0$ such that for all $t\in\mathbb{R}$, $\xi_1,\xi_2\in X$, one has

$$\xi_1\in K,\quad\text{and}\quad\|\xi_1-\xi_2\|\leq\delta\Rightarrow\|f(t,\xi_1)-f(t,\xi_2)\|\leq\varepsilon.$$

Since $x(t)=x_1(t)+x_2(t)$ and $x_1(t)\in K$, then

$$t\in\mathbb{R},\quad\text{and}\quad\|x_2(t)\|\leq\delta\Rightarrow\|f(t,x(t))-f(t,x_1(t))\|\leq\varepsilon,$$

thus

$$\frac{\mu\left(t\in[-r,r]:\|f(t,x(t))-f(t,x_1(t))\|>\varepsilon\right)}{\mu([-r,r])}\leq\frac{\mu\left(t\in[-r,r]:\|x_2(t)\|>\delta\right)}{\mu([-r,r])}.$$

Since $x_2(t)\in\mathcal{E}(\mathbb{R},X,\mu)$, Lemma 2.6 yields that for the above δ,

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\mu\left(t\in[-r,r]:\|x_2(t)\|>\delta\right)=0,$$

then

$$\frac{1}{\mu([-r,r])}\mu\left(t\in[-r,r]:\|f(t,x(t))-f(t,x_1(t))\|>\varepsilon\right)=0.$$

It follows from Lemma 2.6 that $t\to[f(t,x(t))-f(t,x_1(t))]\in\mathcal{E}(\mathbb{R},X,\mu)$.

Now to complete the proof, it is enough to prove that $t\to f_2(t,x_1(t))\in\mathcal{E}(\mathbb{R},X,\mu)$. Since $f_2(t,x(t))$ is uniformly continuous on the compact set $K=\overline{\{x_1(t):t\in\mathbb{R}\}}$ with respect to the second variable $x(t)$, we deduce that for given $\varepsilon>0$, there exists $\delta>0$ such that for all $t\in\mathbb{R}$, $\xi_1,\xi_2\in K$, one has

$$\|\xi_1-\xi_2\|\leq\delta\Rightarrow\|f_2(t,\xi_1)-f_2(t,\xi_2)\|\leq\varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{x_i\}_{i=1}^{n(\varepsilon)}\subset K$, such that

$$K\subset\bigcup_{i=1}^{n(\varepsilon)}B(x_i,\delta),$$

then

$$\|f_2(t,x_1(t))\|\leq\varepsilon+\sum_{i=1}^{n(\varepsilon)}\|f_2(t,x_i(t))\|.$$

By the fact that

$$\forall i\in\{1,\cdots,n(\varepsilon)\},\quad\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|f_2(t,x_i)\|\mathrm{d}\mu(t)=0,$$

we deduce that

$$\forall \varepsilon > 0, \quad \limsup_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f_2(t, x_1(t))\| \mathrm{d}\mu(t) \leq \varepsilon,$$

that implies

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f_2(t, x_1(t))\| \mathrm{d}\mu(t) = 0,$$

therefore $t \to f_2(t, x(t)) \in \mathcal{E}(\mathbb{R}, X, \mu)$ and the Lemma is proved. □

Lemma 3.3. *Let $\mu \in \mathcal{M}$ satisfy (H_2), $f(t,x) \in PAP(\mathbb{R} \times X, X, \mu)$ and $U(t,s)$ satisfy the conditions of (H_3), (H_5) and (H_6). If $x(s)$ is μ-pseudo almost periodic, then*

$$F(t) = \int_0^t U(t,s) f(t, x(s)) ds,$$

is μ-pseudo almost periodic.

Proof. Note that the function $f(t,x)$ is μ-pseudo almost periodic, from Lemma 3.1, $f(t, x(t)) \in PAP(\mathbb{R}, X, \mu)$. Let

$$f(t, x(t)) = h(t) + \varphi(t),$$

where $h \in AP(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Then

$$\begin{aligned} F(t) &= \int_0^t U(t,s) h(s) \mathrm{d}s + \int_0^t U(t,s) \varphi(s) \mathrm{d}s \\ &= \int_{-\infty}^t U(t,s) h(s) \mathrm{d}s - \int_{-\infty}^0 U(t,s) h(s) \mathrm{d}s + \int_0^t U(t,s) \varphi(s) \mathrm{d}s. \end{aligned}$$

Let

$$G(t) = \int_{-\infty}^t U(t,s) h(s) \mathrm{d}s, \quad H(t) = \int_0^t U(t,s) \varphi(s) \mathrm{d}s - \int_{-\infty}^0 U(t,s) h(s) \mathrm{d}s.$$

Now we show that $G(t)$ is almost periodic with respect to $t \in \mathbb{R}^+$. Note that the function h is almost periodic, so is bounded. That is there exists a number

$M_1 > 0$, such that $\|h\|_\infty \leq M_1$. We have

$$\begin{aligned}
\|G(t+\tau) - G(t)\| =& \left\| \int_{-\infty}^{t+\tau} U(t,s)h(s)\mathrm{d}s - \int_{-\infty}^{t} U(t,s)h(s)\mathrm{d}s \right\| \\
=& \left\| \int_{-\infty}^{t} U(t+\tau,s+\tau)h(s+\tau)\mathrm{d}s - \int_{-\infty}^{t} U(t,s)h(s)\mathrm{d}s \right\| \\
=& \left\| \int_{-\infty}^{t} [U(t+\tau,s+\tau)h(s+\tau) - U(t,s)h(s)]\mathrm{d}s \right\| \\
\leq& \int_{-\infty}^{t} \|U(t+\tau,s+\tau)h(s+\tau) - U(t,s)h(s)\|\mathrm{d}s \\
\leq& \int_{-\infty}^{t} \|U(t+\tau,s+\tau) - U(t,s)\|h(s+\tau)\|\mathrm{d}s \\
&+ \int_{-\infty}^{t} \|U(t,s)\|\|h(s+\tau) - h(s)\|\mathrm{d}s \\
\leq& MM_1\varepsilon_1 \int_{-\infty}^{t} e^{-\frac{\delta}{2}(t-s)}\mathrm{d}s + M\varepsilon_2 \int_{-\infty}^{t} e^{-\delta(t-s)}\mathrm{d}s \\
=& \frac{2MM_1\varepsilon_1}{\delta} + \frac{M\varepsilon_2}{\delta}.
\end{aligned}$$

Hence $G(t)$ is almost periodic with respect to t.

In order to show that $F(t)$ is μ-pseudo almost periodic, we need to show that $H(t) \in \mathcal{E}(\mathbb{R}, X, \mu)$, i.e., we need to show that

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|H(t)\|\mathrm{d}\mu(t) = 0.$$

Let

$$I = \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|H(t)\|\mathrm{d}\mu(t).$$

It is obviously that $I \geq 0$ and

$$\begin{aligned}
I =& \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left\| \int_{0}^{t} U(t,s)\varphi(s)\mathrm{d}s - \int_{-\infty}^{0} U(t,s)h(s)\mathrm{d}s \right\| \mathrm{d}\mu(t) \\
\leq& \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left\| \int_{0}^{t} U(t,s)\varphi(s)\mathrm{d}s \right\| \mathrm{d}\mu(t) \\
&+ \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left\| \int_{-\infty}^{0} U(t,s)h(s)\mathrm{d}s \right\| \mathrm{d}\mu(t) \\
\leq& \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{0}^{t} \|U(t,s)\| \, \|\varphi(s)\| \, \mathrm{d}s\mathrm{d}\mu(t) \\
&+ \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{-\infty}^{0} \|U(t,s)\| \, \|h(s)\| \, \mathrm{d}s\mathrm{d}\mu(t),
\end{aligned}$$

and by using Fubini's theorem, we deduce

$$\begin{aligned}
I \leq & \int_0^t \frac{\|\varphi(s)\|}{\mu([-r,r])} \int_{[-r,r]} \|U(t,s)\| \, \mathrm{d}\mu(t)\mathrm{d}s + \int_{-\infty}^0 \frac{\|h(s)\|}{\mu([-r,r])} \int_{[-r,r]} \|U(t,s)\| \, \mathrm{d}\mu(t)\mathrm{d}s \\
\leq & \int_0^t \frac{\|\varphi(s)\|}{\mu([-r,r])} \int_{[-r,r]} Me^{-\delta(t-s)}\mathrm{d}\mu(t)\mathrm{d}s + \int_{-\infty}^0 \frac{\|h(s)\|}{\mu([-r,r])} \int_{[-r,r]} Me^{-\delta(t-s)}\mathrm{d}\mu(t)\mathrm{d}s \\
= & I_1 + I_2.
\end{aligned}$$

Thus

$$\begin{aligned}
I_1 = & \int_{-\infty}^0 \frac{\|h(s)\| \, e^{\delta s}}{\mu([-r,r])} \int_{[-r,r]} Me^{-\delta t}\mathrm{d}\mu(t)\mathrm{d}s \\
\leq & M \sup_{t\leq 0} \|h(t)\| \int_{-\infty}^0 \frac{e^{\delta s}}{\mu([-r,r])} \int_{[-r,r]} e^{-\delta t}\mathrm{d}\mu(t) \\
\leq & \frac{M \sup\limits_{t\leq 0} \|h(t)\|}{\delta} \frac{1}{\mu([-r,r])} \int_{[-r,r]} e^{-\delta t}\mathrm{d}\mu(t).
\end{aligned}$$

Since $h(t)$ is bounded and in view of the Lebesgue dominated convergence theorem, one has

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{-\infty}^0 Me^{-\delta(t-s)}\|h(s)\|\mathrm{d}s\mathrm{d}\mu(t) = 0.$$

Also

$$\begin{aligned}
I_2 = & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_0^t Me^{-\delta(t-s)}\|\varphi(s)\|\mathrm{d}s\mathrm{d}\mu(t) \\
\leq & \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\varphi(t)\|\mathrm{d}\mu(t) \int_0^t Me^{-\delta(t-s)}\mathrm{d}s \\
\leq & \frac{1}{\mu([-r,r])} \left(\frac{M}{\delta}e^{-\delta(t-s)}\right)\Big|_0^t \int_{[-r,r]} \|\varphi(t)\|\mathrm{d}\mu(t) \\
= & \frac{1}{\mu([-r,r])} \frac{M}{\delta}[1-e^{-\delta t}] \int_{[-r,r]} \|\varphi(t)\|\mathrm{d}\mu(t) \\
\leq & \frac{1}{\mu([-r,r])} \frac{M}{\delta} \int_{[-r,r]} \|\varphi(t)\|\mathrm{d}\mu(t).
\end{aligned}$$

Since $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$, then

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\varphi(t)\|\mathrm{d}\mu(t) = 0,$$

in view of the Lebesgue dominated convergence theorem, we obtain

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_0^t \|U(t,s)\| \, \|\varphi(s)\| \, \mathrm{d}s\mathrm{d}\mu(t) = 0.$$

Thus we obtain that

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|H(t)\|\mathrm{d}\mu(t)=0.$$

This completes the proof. □

In what follows, we shall investigate the μ-pseudo almost periodic mild solutions to equations (1.1) and (1.2), the following is the main result.

Theorem 3.4. *Let $\mu\in\mathcal{M}$ satisfy (H_2), $f(t,x)$, $g(x)$ and $U(t,s)$ satisfy all the conditions of (H_3)-(H_6) and*

$$ML_g+\frac{ML_f}{\delta}<1.$$

Then equations (1.1) and (1.2) has a unique μ-pseudo almost periodic mild solution.

Proof. We define an operator Γ by

$$(\Gamma x)(t)=U(t,0)(x_0+g(x))+\int_0^t U(t,s)f(s,x(s))\mathrm{d}s.$$

Firstly, let us check that Γ maps $PAP(\mathbb{R}\times X,X,\mu)$ into $PAP(\mathbb{R}\times X,X,\mu)$.

Take $x\in PAP(\mathbb{R}\times X,X,\mu)$, it is easy to prove that Γx is continuous. By Lemma 3.3, we have

$$F(t)=\int_0^t U(t,s)f(s,x(s))\mathrm{d}s\in PAP(\mathbb{R}\times X,X,\mu).$$

On the other hand, since $U(t,s)$ is exponentially stable, then

$$\lim_{t\to+\infty}U(t,0)(x_0+g(x))=0.$$

Thus $\Gamma x\in PAP(\mathbb{R}\times X,X,\mu)$

Let $x, y \in PAP(\mathbb{R} \times X, X, \mu)$, and observe

$$\begin{aligned}
&\|(\Gamma x)(t) - (\Gamma y)(t)\| \\
=&\|U(t,0)(x_0 + g(x)) + \int_0^t U(t,s) f(s, x(s)) \mathrm{d}s \\
&- U(t,0)(x_0 + g(y)) + \int_0^t U(t,s) f(s, y(s)) \mathrm{d}s\| \\
=&\|U(t,0)(g(x) - g(y)) + \int_0^t U(t,s)[f(s, x(s)) - f(s, y(s))] \mathrm{d}s\| \\
\leq& ML_g \|x - y\| + \int_0^t \|U(t,s)[f(s, x(s)) - f(s, y(s))]\| \, \mathrm{d}s \\
\leq& ML_g \|x - y\| + \int_0^t Me^{-\delta(t-s)} \|f(s, x(s)) - f(s, y(s))\| \, \mathrm{d}s \\
\leq& ML_g \|x - y\|_\infty + ML_f \|x - y\|_\infty \int_0^t e^{-\delta(t-s)} \mathrm{d}s \\
=& \left[ML_g + \frac{ML_f}{\delta}\right] \|x - y\|_\infty.
\end{aligned}$$

So we have

$$\|(\Gamma x)(t) - (\Gamma y)(t)\|_\infty \leq \left[ML_g + \frac{ML_f}{\delta}\right] \|x - y\|_\infty.$$

For

$$ML_g + \frac{M}{\delta} L_f < 1,$$

then Γ is a contractive mapping. Therefore, by the Banach fixed point theorem, Γ has a unique fixed point $x \in PAP(\mathbb{R} \times X, X, \mu)$ such that $\Gamma x = x$. We conclude that $x(t)$ is the unique mild solution to equations (1.1) and (1.2). □

Remark 3.5. When μ is the positive measure defined by (2.1), we can deal with the existence and uniqueness of a weighted pseudo almost periodic solution for

$$x'(t) = A(t)x(t) + f(t, x(t)), \ t \in \mathbb{R}^+, \ x \in WPAP(\mathbb{R}, X, \rho),$$

with nonlocal conditions $x(0) = x_0 + g(x)$. In this case we have the weighted pseudo almost periodic mild solution given by

$$x(t) = U(t,0)(x_0 + g(x)) + \int_0^t U(t,s) f(s, x(s)) \mathrm{d}s, \ \text{for } t \in \mathbb{R}^+.$$

Remark 3.6. When $U(t,s) = T(t-s)$, we can deal with the existence and uniqueness of a μ-pseudo almost periodic solution for

$$x'(t) = Ax(t) + f(t, x(t)), \ t \in \mathbb{R}^+, \ x \in PAP(\mathbb{R}, X, \mu),$$

with nonlocal conditions $x(0) = x_0 + g(x)$, where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}t_{\geq 0}$. In this case we obtain the μ-pseudo almost periodic mild solution given by

$$x(t) = T(t)(x_0 + g(x)) + \int_0^t T(t-s)f(s, x(s))\mathrm{d}s, \text{ for } t \in \mathbb{R}^+.$$

Remark 3.7. When $U(t,s) = T(t-s)$ and the measure μ satisfies the conditions: μ is the positive measure defined by (2.1), we can also deal with the existence and uniqueness of a weighted pseudo almost periodic solution for

$$x'(t) = Ax(t) + f(t, x(t)), \ t \in \mathbb{R}^+, \ x \in WPAP(\mathbb{R}, X, \rho),$$

with nonlocal conditions $x(0) = x_0 + g(x)$. In this case we have the weighted pseudo almost periodic mild solution given by

$$x(t) = T(t)(x_0 + g(x)) + \int_0^t T(t-s)f(s, x(s))\mathrm{d}s, \text{ for } t \in \mathbb{R}^+.$$

4. Applications

In this section we will give an example to illustrate our main result.

Consider the following one-dimensional heat equation with Dirichlet boundary conditions,

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)[\sin(t) + \sin(\sqrt{3}t)] + F(t, u(t,x)), \\ &u(t,0) = u(t,1) = 0, \quad t \in \mathbb{R}^+, \\ &u(0,x) = \frac{\sin(x)}{3}, \quad x \in [0,1], \end{aligned} \tag{4.1}$$

where

$$F(t, u(t,x)) = \frac{u(t,x)[\sin(t) + \sin(\sqrt{2}t)] + \xi(t)\cos(u(t,x))}{6(\pi^2 - 2)}.$$

and

$$\xi(t) = tI_{[0,1]}(t) + I_{[1,+\infty)}(t),$$

with $I_C(\cdot)$ is a characteristic function on the interval C.

Let $D(A) := \Big\{\varphi \in C^1[0,1] : \varphi' \text{ is absolutely continuous on } [0,1], \varphi'' \in L^2[0,1], \varphi(0) = \varphi(1) = 0\Big\}$. A thus generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^2[0,1]$, which is given by

$$[T(t)x](r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < x, e_n >_{L^2} e_n(r),$$

where

$$e_n(r) = \sqrt{2}\sin(n\pi r) \ \text{ for } \ n = 1, 2, \cdots, \ \text{ and } \ \|T(t)\| \leq e^{-\pi^2 t} \ \text{ for } \ t \geq 0.$$

Define a family of linear operators $A_1(t)$ by

$$\begin{cases} D(A_1(t)) = D(A), & t \in \mathbb{R}^+, \\ A_1(t)x = (A + \sin(t) + \sin(\sqrt{3}t))x, & x \in D(A_1(t)). \end{cases}$$

Hence, $\{A_1(t), t \in \mathbb{R}\}$ generates an evolution family $\{U(t,s)\}_{t\geq s}$ such that

$$U(t,s)x = T(t-s)e^{\int_s^t[\sin(\nu)+\sin(\sqrt{3}\nu)]\mathrm{d}\nu}x. \tag{4.2}$$

In view of the above, it is clear that the system (4.1) can be rewritten as an abstract system of the form equations (1.1), where $x(t) = u(t,\cdot)$.

It follows from (4.2) that

$$\|U(t,s)\| \leq e^{-(\pi^2-2)(t-s)} \quad \text{for} \quad t \geq s,$$

which gives $M = 1$ and $\delta = \pi^2 - 2$. Notice that for any $x \in L^2[0,1]$,

$$\begin{aligned}
&\|U(t+\tau, s+\tau) - U(t,s)\| \\
=&\left\|T(t-s)e^{\int_{s+\tau}^{t+\tau}[\sin(\nu)+\sin(\sqrt{3}\nu)]\mathrm{d}\nu} - T(t-s)e^{\int_s^t[\sin(\nu)+\sin(\sqrt{3}\nu)]\mathrm{d}\nu}\right\| \\
=&\left\|T(t-s)e^{\int_s^t[\sin(\nu+\tau)+\sin(\sqrt{3}(\nu+\tau))]\mathrm{d}\nu} - T(t-s)e^{\int_s^t[\sin(\nu)+\sin(\sqrt{3}\nu)]\mathrm{d}\nu}\right\| \\
<&e^{-\pi^2(t-s)}\left|e^{\int_s^t[\sin(\nu+\tau)+\sin(\sqrt{3}(\nu+\tau))]\mathrm{d}\nu} - e^{\int_s^t[\sin(\nu)+\sin(\sqrt{3}\nu)]\mathrm{d}\nu}\right|.
\end{aligned}$$

From the almost periodic property of $\sin(t)+\sin(\sqrt{3}t)$, it follows that $U(t,s)$ satisfies (H_6). Therefore, it is easy to conclude that $\xi(t)\cos u$ belongs to $\mathcal{E}(\mathbb{R}, X, \mu)$, and

$$F(t,u) = \frac{u[\sin(t)+\sin(\sqrt{2}t)] + \xi(t)\cos(u)}{6(\pi^2-2)}.$$

is μ-pseudo almost periodic whenever $\mu \in \mathcal{M}$ is the Lebsgue measure and satisfies (H_2).

Also, it is clear that F satisfies the global Lipschitz condition with Lipschitz constant $L_f = \frac{1}{2(\pi^2-2)}$, and g satisfies the global Lipschitz condition with Lipschitz constant $L_g = \frac{1}{2}$. By direct computation, it yields

$$ML_g + \frac{ML_f}{\delta} = \frac{5}{6} < 1.$$

Consequently, by Theorems 3.4, we conclude that the system (4.1) has a unique μ-pseudo almost periodic mild solution.

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References

1. J. Blot, P. Cieutat, K. Ezzinbi, New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications, Appl. Anal. **92** (2013) 493-526.
2. J. Blot, G. Mophou, G. N'Guérékata, D. Pennequin, Weighted pseudo almost automorphic functions and applications to abstract differential equations, Nonlinear Anal. Theory Methods Appl. **71** (2009) 903-909.

3. T. Diagana, Existence of weighted pseudo almost periodic solutions to some classes of hyperbolic evolution equations, J. Math. Anal. Appl. **350** (2009) 18-28.
4. T. Diagana, Weighted pseudo almost periodic solutions to some differential equations, Nonlinear Anal. Theory Methods Appl. **68** (2008) 2250-2260.
5. J. Cao, Q. Yang, Z. Huang, Optimal mild solutions and weighted pseudo-almost periodic classical solutions of fractional integro-differential equations, Nonlinear Anal. Theory Methods Appl. **74** (2011) 224-234.
6. T. Diagana, Weighted pseudo almost periodic functions and applications, C. Acad. Sci. Paris Ser. I **343** (2006) 643-646.
7. J. Liang, T. Xiao, J. Zhang, Decomposition of weighted pseudo-almost periodic functions, Nonlinear Anal. Theory Methods Appl. **73** (2010) 3456-3461.
8. G. N'Guérékata, Topics in Almost Automorphy, Springer-Verlag, New York, 2005.
9. S. Zaidman, Topics in Abstract Differential Equations, Pitman Research Notes in Mathematics Ser. II, Wiley, New York, 1995.
10. C. Zhang, Integration of vector-valued pseudo almost periodic functions, Proc. Amer. Math. Soc. **121** (1994) 167-174.
11. C. Zhang, Pseudo almost periodic solutions of some differential equations, J. Math. Anal. Appl. **151** (1994) 62-76.
12. C. Zhang, Pseudo almost periodic solutions of some differential equations II, J. Math. Anal. Appl. **192** (1995) 543-561.
13. A. Fink, Almost Periodic Differential Equations, in: Lecture Notes in Math., vol. 37, Springer-Verlag, Berlin, 1974.
14. P. Acquistapace, B. Terreni, A unified approach to abstract linear parabolic equations, Rend. Sem. Math. Uni. Padova **78** (1987) 47-107.
15. P. Acquistapace, Evolution operators and strong solution of abstract linear parabolic equations, Differ. Integral Eqs. **1** (1998) 433-457.

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