UNIQUENESS AND VALUE-SHARING OF MEROMORPHIC
FUNCTIONS IN CLASS $\mathcal{A}$

HARINA P. WAGHAMORE$^1$* AND V. HUSNA $^2$

ABSTRACT. In this paper, we study the value distribution and the uniqueness of meromorphic function in Class $\mathcal{A}$. We obtain significant result which improve as well as generalize the result of C.C.Yang and Xinhou Hua[8].

1. Introduction and preliminaries

In this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a \in \mathbb{C}$. We say that $f$ and $g$ share the value $a$ CM if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities. We shall use the standard notations of value distribution theory $T(r,f), m(r,f), N(r,f), \overline{N}(r,f), ...(Hayman[5], Laine[12], Navanlinna[13] and Yang[7]).$ We denote by $S(r,f)$ any function satisfying $S(r,f) = o\{T(r,f)\},$ as $r \to +\infty,$ possibly outside of finite measure.

Let $f(z)$ and $g(z)$ are non-constant meromorphic functions and $a$ be a finite complex number. We denote by $\overline{N}_L(r,f)$ the counting function for the poles of both $f$ and $g$ about which $f$ has larger multiplicity than $g$, where multiplicity is not counted. Similarly, we have the notation for $\overline{N}_L(r,g)$.

We denote by $\mathcal{A}$ the class of meromorphic functions $f$ in $\mathbb{C}$ which satisfy the condition $\overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) = S(r,f).$ Clearly all functions in $\mathcal{A}$ are transcendental meromorphic functions.

In the 1920’s, R.Nevanlinna [13] proved the following result (the Nevanlinna four-value theorem).

Theorem A. Let $f$ and $g$ be two non-constant meromorphic functions. If $f$ and $g$ share four distinct values CM, then $f$ is a Mobius transformation of $g$. For instance, $f = e^z, g = e^{-z}$ share $0, \pm 1, \infty$ and $f = \frac{1}{g}.$

In 1997 Chung-Chun Yang and Xinhou Hua proved the following result.
Theorem B. Let $f$ and $g$ be two non-constant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value $a$ CM, then either $f = dg$ for some $(n + 1)^{th}$ root of unity $d$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where $c, c_1$ and $c_2$ are constants and satisfy $(c_1c_2)^{n+1}c_2 = -a^2$.

Theorem B, motivate us to think that, whether there exists a similar result, if $f^n f'$ is replaced in Theorem B by $f^{d(P)} f^{(k)}$. In this paper we prove significant results which improve as well as generalize Theorem B in Class $A$.

2. Main results

Theorem 1. If $f, g \in A$, $d(P) \geq 2$ and $k$ be a positive integer. Then $f^{d(P)} f^{(k)} = 1$ has infinitely many zeros.

Theorem 2. Let $f, g \in A$, $d(P) \geq 5$ and $k$ be a positive integer. If $f^{d(P)} f^{(k)}$ and $g^{d(P)} g^{(k)}$ share $1$ CM, then either $f \equiv t g$ for a constant $t$ such that $t^{d(P)+1} = 1$ or $f = c_3 e^{pz}$, $g = c_4 e^{-pz}$ where $c_3, c_4$ and $p$ are constants such that $(-1)^k (c_3 c_4)^{d(P)+1} p^{2k} = 1$.

We now give the following definition and notations which are used in the paper.

Definition 3. Any expression of the type

$$P(f) = \sum_{i=1}^{n} \alpha_i(z) f^{n_{i_0}} (f')^{n_{i_1}} (f'')^{n_{i_2}} ...(f^{(m)})^{n_{i_m}}$$

is called a differential polynomial in $f$ of degree $\overline{d}(P)$, lower degree $d(P)$ and weight $\Gamma_p$, where for each $i = 1, 2, ... n$, $n_{i_0}, n_{i_1}, ... n_{i_m}$ are non-negative integers, $\alpha_i = \alpha_i(z)$ are meromorphic functions satisfying $T(r, \alpha_i) = S(r, f)$ and

$$\overline{d}(P) = \max \left\{ \sum_{j=0}^{m} n_{i_j} : 1 \leq i \leq n \right\}, \quad d(P) = \min \left\{ \sum_{j=0}^{m} n_{i_j} : 1 \leq i \leq n \right\}$$

and

$$\Gamma_p = \max \left\{ \sum_{j=0}^{m} (j+1)n_{i_j} : 1 \leq i \leq n \right\}.$$ 

If $\overline{d}(P) = d(P) = n$ (fixed integer), then $P(f)$ is called homogeneous differential polynomial of degree $n$.

Here we present some lemmas which will be needed in the sequel.

Lemma 4. [1] Let $f$ be a meromorphic function of finite order and $P$ a homogeneous differential polynomial in $f$ of degree $n$. If $\Theta(0, f) = \Theta(\infty, f) = 1$, then $T(r, P) \sim nT(r, f)$.

Lemma 5. [1] Let $f_j$ $(j = 1, 2, 3)$ be meromorphic functions that satisfy

$$\sum_{j=1}^{3} f_j = 1.$$
Assume that $f_1$ is not a constant, and
\[
\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} N(r, f_j) < (\lambda + o(1)) T(r), r \in I
\]
where $\lambda < 1$, $T(r) = \max \{ T(r, f_1), T(r, f_2), T(r, f_3) \}$, $N_2(r, \frac{1}{f_j})$ is the counting function of zeros of $f_j$ ($j = 1, 2, 3$), where a multiple zero is counted two times and a simple zero is counted once. Then $f_2 = 1$ or $f_3 = 1$.

**Lemma 6.** [6] Let $f$ be a non-constant meromorphic function. Then
\[
N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k N(r, f) + S(r, f)
\]
where $k$ is a positive integer.

**Lemma 7.** [6] Let $F$ and $G$ be two distinct non-constant meromorphic functions, and let $c$ be a complex number such that $c \neq 0, 1$. If $F$ and $G$ share 1 and $c$ IM, and if $N(r, \frac{1}{F}) + N(r, F) = S(r, F)$ and $N(r, \frac{1}{G}) + N(r, G) = S(r, G)$, then $F$ and $G$ share $0, 1, c, \infty$ CM.

**Lemma 8.** [3] If $f$ and $g$ are distinct non-constant meromorphic functions that share four values $a_1, a_2, a_3, a_4$ CM, then $f$ is Möbius transformation of $g$, two of the shared values, say $a_1$ and $a_2$ are Picard exceptional values and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.

**Lemma 9.** [6] If $f(z) \in A$ and $k$ is a positive integer, then
\[
T(r, \frac{f^{(k)}}{f}) = S(r, f).
\]

**Lemma 10.** [5] Let $f$ be a non-constant meromorphic functions and $a_1, a_2, a_3$ be three distinct small meromorphic functions of $f$, then
\[
T(r, f) \leq \sum_{j=1}^{3} N(r, \frac{1}{f - a_j}) + S(r, f).
\]

**Lemma 11.** [5] Suppose that $f$ is a non-constant meromorphic function, $k \geq 2$ is an integer. If
\[
N(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) = S(r, \frac{f''}{f})
\]
then $f = e^{az + b}$, where $a \neq 0, b$ are constants.

**Lemma 12.** Let $f, g \in A$, $\bar{d}(P) \geq 2$ and $k$ be a positive integer. If $f^{\bar{d}(P)} f^{(k)}$ and $g^{\bar{d}(P)} g^{(k)}$ share 1 CM, then
\[
T(r, g) \leq \left( \frac{\bar{d}(P) + 1}{\bar{d}(P) - 1} \right) T(r, f) + S(r, g).
\]

**Proof.** Let $G = g^{\bar{d}(P)} g^{(k)}$. Then it is a differential polynomial of degree $(\bar{d}(P) + 1)$. By Lemma 10 we have
\[
(\bar{d}(P) + 1)T(r, g) \sim T(r, G)
\] (2.1)
Applying Lemma 10 to $T(r, G)$, we get

$$(\bar{d}(P) + 1)T(r, g) \leq N(r, G) + \bar{N}\left( r, \frac{1}{G} \right) + \bar{N}\left( r, \frac{1}{G - 1} \right) + S(r, G)$$

$$= \bar{N}(r, g^{\bar{d}(P)} g^{(k)}) + \bar{N}\left( r, \frac{1}{g^{\bar{d}(P)} g^{(k)}} \right) + \bar{N}\left( r, \frac{1}{g^{\bar{d}(P)} g^{(k)}} \right) + S(r, g^{\bar{d}(P)} g^{(k)}).$$

Noting that

$$N(r, g^{\bar{d}(P)} g^{(k)}) \leq N(r, g^{\bar{d}(P)}) + N(r, g^{(k)})$$

$$\leq N(r, g) + N(r, g) + kN(r, g)$$

$$= N(r, g) + (k + 1)N(r, g)$$

and $S(r, G) = S(r, g)$, (by (2.1))

So,

$$(\bar{d}(P) + 1)T(r, g) \leq N(r, g) + (k + 1)N(r, g) + N(r, \frac{1}{g}) + N\left( r, \frac{1}{g^{(k)}} \right) + S(r, g).$$

Since $f^{\bar{d}(P)} f^{(k)}$ and $g^{\bar{d}(P)} g^{(k)}$ share 1 CM, it implies that $f^{\bar{d}(P)} f^{(k)} - 1$ and $g^{\bar{d}(P)} g^{(k)} - 1$ have same zeros with the same multiplicities, using this with Lemma 6, we obtain that

$$(\bar{d}(P) + 1)T(r, g) \leq N(r, g) + (k + 1)N(r, g) + N\left( r, \frac{1}{g} \right) + N\left( r, \frac{1}{f^{\bar{d}(P)} f^{(k)} - 1} \right) + S(r, g).$$

By hypothesis, we have

$$N(r, f) + \bar{N}\left( r, \frac{1}{f} \right) = S(r, f)$$

$$N(r, g) + \bar{N}\left( r, \frac{1}{g} \right) = S(r, g)$$

Using Nevanlinna’s first fundamental theorem and Lemma 4, we have

$$\bar{N}\left( r, \frac{1}{f^{\bar{d}(P)} f^{(k)} - 1} \right) \leq T\left( r, \frac{1}{f^{\bar{d}(P)} f^{(k)} - 1} \right) = T(r, f^{\bar{d}(P)} f^{(k)}) + O(1)$$

$$\sim (\bar{d}(P) + 1)T(r, f) + O(1)$$

So

$$\bar{N}\left( r, \frac{1}{f^{\bar{d}(P)} f^{(k)} - 1} \right) \leq (\bar{d}(P) + 1)T(r, f) + O(1).$$

(2.3)
Using (2.3), (2.2) becomes
\[(\bar{d}(P) + 1)T(r, g) \leq N(r, g) + N\left(\frac{r}{g}\right) + (\bar{d}(P) + 1)T(r, f) + S(r, g)\]
\[\leq 2T(r, g) + (\bar{d}(P) + 1)T(r, f) + S(r, g)\]
\[(\bar{d}(P) - 1)T(r, g) \leq (\bar{d}(P) + 1)T(r, f) + S(r, g)\]
\[T(r, g) \leq \frac{(\bar{d}(P) + 1)}{(\bar{d}(P) - 1)}T(r, f) + S(r, g).\]

This completes the proof of Lemma. □

**Lemma 13.** Let \(f, g \in \mathcal{A}, \bar{d}(P) \geq 2\) and \(k\) be a positive integer. If \(f^{\bar{d}(P)} f^{(k)}\) and \(g^{\bar{d}(P)} g^{(k)}\) share 1 CM, then \(S(r, f) = S(r, g)\).

**Proof.** Proceeding as in the proof of Lemma 12 we have
\[T(r, g) \leq \left(\frac{\bar{d}(P) + 1}{\bar{d}(P) - 1}\right) T(r, f) + S(r, g).\]
Similarly, we have
\[T(r, f) \leq \left(\frac{\bar{d}(P) + 1}{\bar{d}(P) - 1}\right) T(r, g) + S(r, f).\]

using above two inequalities we easily obtain \(S(r, f) = S(r, g)\).

This completes the proof of Lemma.

Using the method in [14], we prove the following lemma. □

**Lemma 14.** Let \(f, g \in \mathcal{A}, \bar{d}(P) \geq 2\) and \(k\) be a positive integer. If \(f^{\bar{d}(P)} f^{(k)}\) and \(g^{\bar{d}(P)} g^{(k)}\) = 1 then \(f = c_3 e^{pz}\) and \(g = c_4 e^{-pz}\) where \(c_3, c_4\) and \(p\) are constants such that \((-1)^k (c_3 c_4)^{\bar{d}(P) + 1} p^{2k} = 1\).

**Proof.** Let
\[F = f^{\bar{d}(P)} f^{(k)}\] and \(G = g^{\bar{d}(P)} g^{(k)}\) (2.4)

By Lemma 4, we have
\[T(r, F) \sim (\bar{d}(P) + 1)T(r, f), \quad T(r, G) \sim (\bar{d}(P) + 1)T(r, g)\]
Clearly \(S(r, F) = S(r, f)\) and \(S(r, G) = S(r, g)\).

By Lemma 13, we have \(S(r, f) = S(r, g)\).

Thus,
\[S(r, F) = S(r, f) = S(r, g) = S(r, G).\] (2.5)

By hypothesis, we have
\[f^{\bar{d}(P)} f^{(k)} g^{\bar{d}(P)} g^{(k)} = 1 \quad \text{or} \quad FG = 1.\] (2.6)

From (2.6) and \(f\) and \(g\) are transcendental functions, it follows that
\[N(r, \frac{1}{f}) = 0, \quad N(r, \frac{1}{g}) = 0.\] (2.7)
By hypothesis, we have
\[ N(r, f) + N(r, \frac{1}{f}) = S(r, f), \quad N(r, g) + N(r, \frac{1}{g}) = S(r, g) \] (2.8)

(2.6) can be expressed as
\[ f^{d(P)} f^{(k)} = \frac{1}{g^{d(P)} g^{(k)}} \]
so we deduce that
\[ N(r, f^{d(P)} f^{(k)}) = N\left(r, \frac{1}{g^{d(P)} g^{(k)}}\right) \] (2.9)

Using (2.7), we get
\[ N(r, f^{d(P)} f^{(k)}) = N(r, f^{d(P)}) + N(r, f^{(k)}) = \bar{d}(P) N(r, f) + N(r, f) + kN(r, f) = (\bar{d}(P) + 1) N(r, f) + S(r, f) \]

Using this with Lemma 6 and equation (2.5),(2.7) and (2.8),(2.9) can be written as
\[ (\bar{d}(P) + 1) N(r, f) + S(r, f) \leq N\left(r, \frac{1}{g^{d(P)}}\right) + N\left(\frac{1}{g^{(k)}}\right) \leq (\bar{d}(P) + 1) N(r, \frac{1}{g}) + kN(r, \frac{1}{g}) + S(r, g) = S(r, g) \]

which implies that
\[ N(r, f) = S(r, f) \] (2.10)

Similarly
\[ N(r, g) = S(r, g). \] (2.11)

By (2.7),(2.8) and Lemma 6, we have
\[ \bar{N}(r, \frac{1}{F}) = \bar{N}\left(r, \frac{1}{f^{d(P)} f^{(k)}}\right) \leq \bar{N}(r, \frac{1}{f}) + N\left(r, \frac{1}{f^{(k)}}\right) \leq \bar{N}(r, \frac{1}{f}) + N(r, \frac{1}{f}) + k\bar{N}(r, f) + S(r, f) = S(r, f). \]

Therefore
\[ \bar{N}(r, \frac{1}{F}) = S(r, F) \] (2.12)

Similarly
\[ \bar{N}(r, \frac{1}{G}) = S(r, G). \] (2.13)
Moreover by using (2.8) and (2.10), we have
\[
\bar{N}(r, F) = \bar{N}(r, f^{\tilde{d}(P)}f^{(k)}) \leq \bar{N}(r, f) + N(r, f^{(k)}) \\
\leq \bar{N}(r, f) + N(r, f) + k\bar{N}(r, f) \\
= S(r, f).
\]
Therefore
\[
\bar{N}(r, F) = S(r, F) \tag{2.14}
\]
Similarly
\[
\bar{N}(r, G) = S(r, G). \tag{2.15}
\]
It follows from (2.12)-(2.15) that
\[
\bar{N}(r, \frac{1}{F}) + \bar{N}(r, F) = S(r, F), \quad \bar{N}(r, \frac{1}{G}) + \bar{N}(r, G) = S(r, G). \tag{2.16}
\]
In view of (2.6), we know that \(F\) and \(G\) share 1 and -1 IM, together this with (2.16) and Lemma 7 implies that \(F\) and \(G\) share \(-1, 0, \infty\) CM, thus by Lemma 8, we get that 0 and \(\infty\) are Picard values of \(F\) and \(G\). Thus we deduce from (2.4) that both \(f\) and \(g\) are transcendental entire functions. By (2.7) we have
\[
f(z) = e^{\alpha(z)}, \quad g(z) = e^{\beta(z)} \tag{2.17}
\]
where \(\alpha(z)\) and \(\beta(z)\) are non constant entire functions.
Then
\[
T(r, \frac{f'}{f}) = T(r, \frac{e^{\alpha} \alpha'}{e^{\alpha}}) = T(r, \alpha').
\]
We claim that \(\alpha(z) + \beta(z) = c, c\) is a constant.
From (2.17), we know that either \(\alpha\) and \(\beta\) are transcendental functions or both \(\alpha\) and \(\beta\) are polynomials.
From (2.6), we have
\[
N(r, \frac{1}{f^{(k)}}) = N(r, g^{\tilde{d}(P)}g^{(k)}f^{\tilde{d}(P)}) \\
\leq \tilde{d}(P)N(r, g) + N(r, g^{(k)}) + \tilde{d}(P)N(r, f) \\
= 0.
\]
From this and (2.6), we get
\[
N(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) = 0.
\]
If \(k \geq 2\), suppose that \(\alpha\) is a transcendental entire function. From Lemma 11, we have \(f = e^{\alpha(z)} = e^{az+b}\), it implies that \(\alpha(z) = az + b\), a polynomial, which is a contradiction.
Thus \(\alpha\) and \(\beta\) are polynomials.
We deduce from (2.17) that
\[
f^{(k)} = [(\alpha')^k + P_{k-1}(\alpha')]e^{\alpha}, \quad g^{(k)} = [(\beta')^k + Q_{k-1}(\beta')]e^{\beta}
\]
where \( P_{k-1}(\alpha') \) and \( Q_{k-1}(\beta') \) are differential polynomials in \( \alpha' \) and \( \beta' \) of degree at most \((k - 1)\) respectively.

Thus by (2.6) we obtain that
\[
[(\alpha')^k + P_{k-1}(\alpha')][(\beta')^k + Q_{k-1}(\beta')]e^{(\bar{d}(P)+1)(\alpha+\beta)} = 1 \tag{2.18}
\]
we deduce from (2.18) that \( \alpha(z) + \beta(z) = c \), \( c \) is a constant.

If \( k = 1 \), from (2.17) we get,
\[
\alpha' \beta' e^{(\bar{d}(P)+1)(\alpha+\beta)} = 1. \tag{2.19}
\]

Let \( \alpha + \beta = \gamma \). If \( \alpha \) and \( \beta \) are transcendental entire functions, then \( \gamma \) is not a constant and (2.19) implies that
\[
\alpha'(\gamma' - \alpha')e^{(\bar{d}(P)+1)\gamma} = 1. \tag{2.20}
\]

Since
\[
T(r, \gamma') = m(r, \gamma')
= m \left( r, \frac{e^{(\bar{d}(P)+1)\gamma'\gamma}}{e^{(\bar{d}(P)+1)\gamma}} \right)
= m \left( r, \frac{(e^{(\bar{d}(P)+1)\gamma}'\gamma'}{e^{(\bar{d}(P)+1)\gamma}} \right) = S(r, e^{(\bar{d}(P)+1)\gamma})
\]
Thus (2.20) implies that Since
\[
T(r, e^{(\bar{d}(P)+1)\gamma}) = T(r, \frac{1}{\alpha'(\gamma' - \alpha')})
\leq T(r, \alpha'(\gamma' - \alpha')) + O(1)
\leq 2T(r, \alpha') + S(r, e^{(\bar{d}(P)+1)\gamma}).
\]
Which implies that
\[
T(r, e^{(\bar{d}(P)+1)\gamma}) = O(T(r, \alpha')).
\]
Thus
\[
T(r, \gamma') = S(r, \alpha').
\]
In view of (2.20) and by Lemmas 10, we get
\[
T(r, \alpha') \leq N(r, \alpha') + N(r, \frac{1}{\alpha'}) + \bar{N}(r, \frac{1}{\alpha' - \gamma'}) + S(r, \alpha').
\]
Since \( \alpha \) and \( \beta \) are transcendental entire functions and in view of (2.20), we obtain
\[
T(r, \alpha') \leq S(r, \alpha'),
\]
and this implies that \( \alpha' \) is a constant, which is a contradiction. Thus \( \alpha \) and \( \beta \) are both polynomials and \( \alpha(z) + \beta(z) = c \), for a constant \( c \).

Hence from (2.18), we get
\[
(\alpha')^{2k} = 1 + \bar{P}_{2k-1}(\alpha') \tag{2.21}
\]
where \( \tilde{P}_{2k-1}(\alpha') \) is a differential polynomial in \( \alpha' \) of degree at most \( (2k - 1) \).

From (2.21), we have
\[
2kT(r, \alpha') = T(r, (\alpha')^{2k}) = m(r, (\alpha')^{2k}) \leq m(r, \tilde{P}_{2k-1}(\alpha')) + O(1)
\]
\[
= m \left( r, \frac{\tilde{P}_{2k-1}(\alpha')}{(\alpha')^{2k-1}} \right) + O(1)
\]
\[
= m \left( r, \frac{\tilde{P}_{2k-1}(\alpha')}{(\alpha')^{2k-1}} \right) + m(r, (\alpha')^{2k-1}) + O(1)
\]
\[
\leq (2k - 1)T(r, \alpha') + S(r, \alpha').
\]

Therefore \( T(r, \alpha') \leq S(r, \alpha') \).

which implies that \( \alpha' \) is a constant.

Thus \( \alpha = pz + c_1, \beta = -pz + c_2 \).

By (2.17), we represent \( f \) and \( g \) as
\[
f = c_3e^{pz}, \quad g = c_4e^{-pz}
\]

where \( c_3, c_4 \) and \( p \) are constants such that \( (-1)^k(c_3c_4)^{\tilde{d}(P) + 1}p^{2k} = 1 \).

This completes the proof of Lemma. \( \square \)

**Proof of Theorem 1.** Let \( F = f^{\tilde{d}(P)}f^{(k)}. \) By Lemmas 4 and 10, we have
\[
(\tilde{d}(P) + 1)T(r, f) \sim T(r, f^{\tilde{d}(P)}f^{(k)})
\]
\[
\leq \tilde{N}(r, f^{\tilde{d}(P)}f^{(k)}) + \frac{1}{f^{\tilde{d}(P)}f^{(k)}} + \tilde{N}(r, \frac{1}{f^{\tilde{d}(P)}f^{(k)}}) + S(r, f^{\tilde{d}(P)}f^{(k)}) \quad (2.22)
\]

Noting that
\[
\tilde{N}(r, f^{\tilde{d}(P)}f^{(k)}) \leq \tilde{N}(r, f^{\tilde{d}(P)}) + \tilde{N}(r, f^{(k)})
\]
\[
\leq \tilde{N}(r, f) + N(r, f) + k\tilde{N}(r, f)
\]
\[
\leq N(r, f) + (k + 1)\tilde{N}(r, f)
\]
\[
\tilde{N}(r, \frac{1}{f^{\tilde{d}(P)}f^{(k)}}) \leq \tilde{N}(r, \frac{1}{f^{\tilde{d}(P)}}) + \tilde{N}(r, \frac{1}{f^{(k)}})
\]
\[
\leq \tilde{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}})
\]

and \( (\tilde{d}(P) + 1)T(r, f) \sim T(r, f^{\tilde{d}(P)}f^{(k)}). \)

So \( S(r, f^{\tilde{d}(P)}f^{(k)}) = S(r, f). \)

Substituting above inequalities in (2.22), we obtain,
\[
(\tilde{d}(P) + 1)T(r, f) \leq N(r, f) + (k + 1)\tilde{N}(r, f) + \tilde{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}})
\]
\[
+ \tilde{N}(r, \frac{1}{f^{\tilde{d}(P)}f^{(k)}} - 1) + S(r, f).
\]
using Lemma 6, we get

\[
(\overline{d}(P) + 1)T(r, f) \leq N(r, f) + (k + 1)\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f)
\]

\[
+ \overline{N}\left(r, \frac{1}{f^{\overline{d}(P)} f^{(k)} - 1}\right) + S(r, f).
\]

(2.23)

By hypothesis, we have \(\overline{N}(r, f) = S(r, f), \overline{N}(r, \frac{1}{f}) = S(r, f)\).
Therefore (2.23) becomes

\[
(\overline{d}(P) + 1)T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{\overline{d}(P)} f^{(k)} - 1}\right) + S(r, f)
\]

\[
\leq 2T(r, f) + \overline{N}\left(r, \frac{1}{f^{\overline{d}(P)} f^{(k)} - 1}\right) + S(r, f)
\]

\[
(\overline{d}(P) - 1)T(r, f) \leq \overline{N}\left(r, \frac{1}{f^{\overline{d}(P)} f^{(k)} - 1}\right) + S(r, f).
\]

which implies that \(f^{\overline{d}(P)} f^{(k)} - 1\) has infinitely many zeros for \(\overline{d}(P) \geq 2\).
This completes the proof of Theorem. \(\Box\)

**Proof of Theorem 2.** By hypothesis, \(f^{\overline{d}(P)} f^{(k)}\) and \(g^{\overline{d}(P)} g^{(k)}\) share 1 CM.
Let

\[
H(z) = \frac{f^{\overline{d}(P)} f^{(k)} - 1}{g^{\overline{d}(P)} g^{(k)} - 1}.
\]

(2.24)

Then \(H(z)\) is a meromorphic function satisfying \(T(r, H) = O(T(r, f) + T(r, g))\),
by the first fundamental theorem and Lemma 4.
From (2.24), we see that the zeros and poles of \(H(z)\) are multiple and satisfy

\[
\overline{N}(r, H) \leq \overline{N}_L(r, f), \overline{N}(r, \frac{1}{H}) \leq \overline{N}_L(r, g)
\]

(2.25)

Let

\[
f_1 = f^{\overline{d}(P)} f^{(k)}, \quad f_2 = -H g^{\overline{d}(P)} g^{(k)}, \quad f_3 = H.
\]

(2.26)

Then by using (2.24), we easily see that

\[
f_1 + f_2 + f_3 = f^{\overline{d}(P)} f^{(k)} - H g^{\overline{d}(P)} g^{(k)} + H
\]

\[
= f^{\overline{d}(P)} f^{(k)} - H\left(g^{\overline{d}(P)} g^{(k)} - 1\right)
\]

\[
= f^{\overline{d}(P)} f^{(k)} - \frac{f^{\overline{d}(P)} f^{(k)} - 1}{g^{\overline{d}(P)} g^{(k)} - 1}\left(g^{\overline{d}(P)} g^{(k)} - 1\right) = 1.
\]
Assuming that \( f_1 \) is non-constant and by Lemma 5, we have
\[
\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \tilde{N}(r, f_j) = N_2(r, \frac{1}{f_1}) + N_2(r, \frac{1}{f_2}) + N_2(r, \frac{1}{f_3}) + \tilde{N}(r, f_1)
\]
\[
+ \tilde{N}(r, f_2) + \tilde{N}(r, f_3)
\]
\[
\leq N_2(r, \frac{1}{f^{d(p)} f^{(k)}}) + N_2(r, \frac{1}{g^{d(p)} g^{(k)}}) + N_2(r, \frac{1}{H})
\]
\[
+ \tilde{N}(r, f^{d(p)} f^{(k)}) + \tilde{N}(r, g^{d(p)} g^{(k)}) + \tilde{N}(r, H).
\]
\[(2.27)\]

Noting that
\[
\tilde{N}(r, f^{d(p)} f^{(k)}) \leq N(r, f) + (k + 1)\tilde{N}(r, f)
\]
\[
\tilde{N}(r, g^{d(p)} f^{(k)}) \leq N(r, g) + (k + 1)\tilde{N}(r, g)
\]

using this with (2.25) and Lemma 6, (2.27) becomes
\[
\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \tilde{N}(r, f_j) \leq 2\tilde{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) + 2\tilde{N}(r, \frac{1}{g}) + N(r, \frac{1}{g^{(k)}})
\]
\[
+ 2\tilde{N}(r, \frac{1}{H}) + N(r, f) + (k + 1)\tilde{N}(r, f) + N(r, g) + (k + 1)\tilde{N}(r, g)
\]
\[
+ \tilde{N}(r, H)
\]
\[
\leq 2\tilde{N}(r, \frac{1}{f}) + N(r, \frac{1}{f}) + k\tilde{N}(r, f) + 2\tilde{N}(r, \frac{1}{g})
\]
\[
+ N(r, \frac{1}{g}) + k\tilde{N}(r, g) + 2\tilde{N}_L(r, g) + N(r, f)
\]
\[
+ (k + 1)\tilde{N}(r, f) + N(r, g) + (k + 1)\tilde{N}(r, g)
\]
\[
+ \tilde{N}_L(r, f) + S(r, f) + S(r, g)
\]
\[
= 2(\tilde{N}(r, \frac{1}{f}) + \tilde{N}(r, \frac{1}{g})) + (N(r, \frac{1}{f}) + N(r, \frac{1}{g}))
\]
\[
+ (2k + 1)(\tilde{N}(r, f) + \tilde{N}(r, g)) + (N(r, f) + N(r, g))
\]
\[
+ 2\tilde{N}_L(r, g) + \tilde{N}_L(r, f) + S(r, f) + S(r, g).
\]

Since \( f, g \in \mathcal{A} \), we have
\[
\tilde{N}(r, f) + \tilde{N}(r, \frac{1}{f}) = S(r, f), \quad \tilde{N}(r, g) + \tilde{N}(r, \frac{1}{g}) = S(r, g).
\]

Therefore
\[
\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \tilde{N}(r, f_j) \leq (N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + (N(r, f) + N(r, g)) + 2\tilde{N}_L(r, g)
\]
\[
+ \tilde{N}_L(r, f) + S(r, f) + S(r, g).
\]
\[(2.28)\]

Noting that
\[
2\tilde{N}_L(r, g) + \tilde{N}_L(r, f) \leq 2\tilde{N}(r, f) = S(r, f)
\]
or \( 2\bar{N}_L(r, g) + \bar{N}_L(r, f) \leq 2\bar{N}(r, g) = S(r, g) \)

Thus (2.28) becomes
\[
\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r, f_j) \leq 2(T(r, f) + T(r, g)) + S(r, f) + S(r, g)
\]

using Lemmas 12 and 13, we get
\[
\sum_{j=1}^{3} N_2(r, \frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r, f_j) \leq 2T(r, f) + \left( 2\frac{(\bar{d}(P) + 1)}{d(P) - 1} \right) T(r, f) + S(r, f)
\]
\[
= \frac{4\bar{d}(P)}{(d(P) - 1)} T(r, f) + S(r, f)
\]
\[
\leq \left( \frac{4\bar{d}(P)}{(d(P) - 1)(d(P) + 1)} \right) T(r) + S(r, f)
\]
\[
\leq \left( \frac{4\bar{d}(P)}{(d(P) - 1)(d(P) + 1) + o(1)} \right) T(r)
\]

Since \( d(P) \geq 5 \), \( \frac{4\bar{d}(P)}{(d(P) - 1)(d(P) + 1)} < 1 \) using Lemma 5, we get \( F_2 = 1 \) or \( F_3 = 1 \).

Next we consider two cases:

**Case 1.** \( F_2 = 1 \) i.e., \(-H\bar{g}(P)g(k) = 1\)

Using (2.24) we have
\[
- \left( \frac{f\bar{d}(P)f(k) - 1}{g\bar{d}(P)g(k) - 1} \right) g\bar{d}(P)g(k) = 1
\]

by simple computing, we get
\[
f\bar{d}(P)g(k)g\bar{d}(P)g(k) = 1
\]

By Lemma 14, we get the conclusion of Theorem 2.

**Case 2.** \( F_3 = 1 \) i.e., \( H = 1 \)

Using (2.24), we have
\[
\frac{f\bar{d}(P)f(k) - 1}{g\bar{d}(P)g(k) - 1} = 1
\]
\[
i.e., \quad f\bar{d}(P)f(k) = g\bar{d}(P)g(k).
\]

By Lemma 4, we have
\[
T(r, f\bar{d}(P)f(k)) = T(r, g\bar{d}(P)g(k))
\]
\[
(\bar{d}(P) + 1)T(r, f) = (\bar{d}(P) + 1)T(r, g)
\]
\[
T(r, f) = T(r, g)
\]
(2.30)

and also
\[
S(r, f) = S(r, g).
\]
(2.31)
Let \( h = \frac{g}{f} \). Then by (2.29), we have

\[
h^{d(P)} = \frac{f^{(k)}}{g^{(k)}}, \quad h^{(d(P)+1)} = \frac{gf^{(k)}}{fg^{(k)}}
\]

Suppose that \( h \) is not a constant. By (2.30), we have

\[
T(r, h) = T(r, \frac{g}{f}) \\
\leq T(r, g) + T(r, f) + O(1) \\
\leq 2T(r, f) + O(1).
\]

Which implies that

\[
S(r, h) = S(r, f)
\]

similarly

\[
S(r, h) = S(r, g)
\]

Thus, by (2.31)

\[
S(r, h) = S(r, f) = S(r, g)
\]

By the first fundamental theorem and Lemma 9, we have

\[
T(r, h^{d(P)+1}) = T \left( r, \frac{gf^{(k)}}{fg^{(k)}} \right)
\]

\[
(d(P) + 1)T(r, h) \leq T(r, \frac{f^{(k)}}{f}) + T(r, \frac{g}{g^{(k)}}) + O(1)
\]

\[
= T(r, \frac{f^{(k)}}{f}) + T(r, \frac{g^{(k)}}{g}) + O(1)
\]

\[
= S(r, f) + S(r, g)
\]

\[
= S(r, h).
\]

which is a contradiction since \( d(P) \geq 5 \). Therefore \( h \) is a constant. Since \( f \) and \( g \) are transcendental meromorphic functions, we have \( h \neq 0 \).

Let \( t = \frac{1}{h} \), which implies that \( f = tg \). From (2.29), we obtain \( t^{d(P)+1} = 1 \). This completes the proof of the Theorem. \( \square \)

Acknowledgement. The author (VH) is grateful to the University Grants Commission (UGC), New Delhi, India for supporting her research work by providing her with a Maulana Azad National Fellowship (MANF).

References


1 Department of Mathematics, Jnanbharathi Campus, Bangalore University, Bangalore-560 056, INDIA
   E-mail address: harinapw@gmail.com , harinapw@bub.ernet.in
2 Department of Mathematics, Jnanbharathi Campus, Bangalore University, Bangalore-560 056, INDIA
   E-mail address: husnav43@gmail.com , husnav@bub.ernet.in