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UNIQUENESS AND VALUE-SHARING OF MEROMORPHIC FUNCTIONS IN CLASS $\mathcal{A}$

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Abstract. In this paper, we study the value distribution and the uniqueness of meromorphic function in Class $\mathcal{A}$. We obtain significant result which improve as well as generalize the result of C.C.Yang and Xinhou Hua[8].

1. Introduction and preliminaries

In this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a \in \overline{\mathbb{C}}$. We say that f and g share the value a CM if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities. We shall use the standard notations of value distribution theory $T(r,f), m(r,f), N(r,f), \overline{N}(r,f)$,...(Hay man[5], Laine[12], Navanlinna[13] and Yang[7]). We denote by $S(r,f)$ any function satisfying $S(r,f) = o\{T(r,f)\}$, as $r \to +\infty$, possibly outside of finite measure.

Let $f(z)$ and $g(z)$ are non-constant meromorphic functions and a be a finite complex number. We denote by $\overline{N}_L(r,f)$ the counting function for the poles of both f and g about which f has larger multiplicity than g, where multiplicity is not counted. Similarly, we have the notation for $\overline{N}_L(r,g)$.

We denote by $\mathcal{A}$ the class of meromorphic functions f in $\mathbb{C}$ which satisfy the condition $\overline{N}(r,f)+\overline{N}(r,\frac{1}{f}) = S(r,f)$. Clearly all functions in $\mathcal{A}$ are transcendental meromorphic functions.

In the 1920's, R.Nevanlinna [13] proved the following result(the Nevanlinna four-value theorem).

Theorem A. *Let f and g be two non-constant meromorphic functions. If f and g share four distinct values CM, then f is a Mobius transformation of g.*
For instance, $f = e^z, g = e^{-z}$ share $0, \pm 1, \infty$ and $f = \frac{1}{g}$.

In 1997 Chung-Chun Yang and Xinhou Hua proved the following result.

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Theorem B. *Let f and g be two non-constant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)^{th}$ root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Theorem B, motivate us to think that, whether there exists a similar result, if $f^n f'$ is replaced in Theorem B by $f^{\bar{d}(P)} f^{(k)}$. In this paper we prove significant results which improve as well as generalize Theorem B in Class $\mathcal{A}$.

2. Main results

Theorem 1. *If $f, g \in \mathcal{A}$, $\bar{d}(P) \geq 2$ and k be a positive integer. Then $f^{\bar{d}(P)} f^{(k)} = 1$ has infinitely many zeros.*

Theorem 2. *Let $f, g \in \mathcal{A}$, $\bar{d}(P) \geq 5$ and k be a positive integer. If $f^{\bar{d}(P)} f^{(k)}$ and $g^{\bar{d}(P)} g^{(k)}$ share 1 CM, then either $f \equiv tg$ for a constant t such that $t^{\bar{d}(P)+1} = 1$ or $f = c_3 e^{pz}$, $g = c_4 e^{-pz}$ where c_3, c_4 and p are constants such that $(-1)^k (c_3 c_4)^{\bar{d}(P)+1} p^{2k} = 1$.*

We now give the following definition and notations which are used in the paper.

Definition 3. Any expression of the type

$$P(f) = \sum_{i=1}^{n} \alpha_i(z) f^{n_{i_0}} (f')^{n_{i_1}} (f'')^{n_{i_2}} \ldots (f^{(m)})^{n_{i_m}}$$

is called a differential polynomial in f of degree $\overline{d}(P)$, lower degree $\underline{d}(P)$ and weight Γ_p where for each $i = 1, 2, \ldots n$, $n_{i_0}, n_{i_1}, \ldots n_{i_m}$ are non-negative integers, $\alpha_i = \alpha_i(z)$ are meromorphic functions satisfying $T(r, \alpha_i) = S(r, f)$ and

$$\overline{d}(P) = max\Big\{ \sum_{j=0}^{m} n_{i_j} : 1 \leq i \leq n \Big\} \quad , \quad \underline{d}(P) = min\Big\{ \sum_{j=0}^{m} n_{i_j} : 1 \leq i \leq n \Big\}$$

and

$$\Gamma_p = max\Big\{ \sum_{j=0}^{m} (j+1) n_{i_j} : 1 \leq i \leq n \Big\}.$$

If $\overline{d}(P) = \underline{d}(P) = n$ (fixed integer), then $P(f)$ is called homogeneous differential polynomial of degree n.

Here we present some lemmas which will be needed in the sequel.

Lemma 4. [1] *Let f be a meromorphic function of finite order and P a homogeneous differential polynomial in f of degree n. If $\Theta(0, f) = \Theta(\infty, f) = 1$, then $T(r, P) \backsim nT(r, f)$.*

Lemma 5. [1] *Let f_j $(j = 1, 2, 3)$ be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j = 1.$$

Assume that f_1 is not a constant, and

$$\sum_{j=1}^{3} N_2(r,\frac{1}{f_j}) + \sum_{j=1}^{3} \overline{N}(r,f_j) < (\lambda + o(1))T(r), r \in I$$

where $\lambda < 1$, $T(r) = max\Big\{T(r,f_1), T(r,f_2), T(r,f_3)\Big\}$, $N_2(r,\frac{1}{f_j})$ is the counting function of zeros of f_j $(j = 1,2,3)$, where a multiple zero is counted two times and a simple zero is counted once. Then $f_2 = 1$ or $f_3 = 1$.

Lemma 6. [6] *Let f be a non-constant meromorphic function. Then*
$N(r,\frac{1}{f^{(k)}}) \leq N(r,\frac{1}{f}) + k\overline{N}(r,f) + S(r,f)$
where k is a positive integer.

Lemma 7. [6] *Let F and G be two distinct non-constant meromorphic functions, and let c be a complex number such that $c \neq 0,1$. If F and G share 1 and c IM, and if $\overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) = S(r,F)$ and $\overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) = S(r,G)$, then F and G share $0,1,c,\infty$ CM.*

Lemma 8. [3] *If f and g are distinct non-constant meromorphic functions that share four values a_1, a_2, a_3, a_4 CM, then f is Möbius transformation of g, two of the shared values, say a_1 and a_2 are Picard exceptional values and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

Lemma 9. [6] *If $f(z) \in \mathcal{A}$ and k is a positive integer, then*

$$T(r,\frac{f^{(k)}}{f}) = S(r,f).$$

Lemma 10. [5] *Let f be a non-constant meromorphic functions and a_1, a_2, a_3 be three distinct small meromorphic functions of f, then*

$$T(r,f) \leq \sum_{j=1}^{3} \overline{N}(r,\frac{1}{f-a_j}) + S(r,f).$$

Lemma 11. [5] *Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f^{(k)}}) = S(r,\frac{f'}{f})$$

then $f = e^{az+b}$, where $a \neq 0$, b are constants.

Lemma 12. *Let $f, g \in \mathcal{A}$, $\bar{d}(P) \geq 2$ and k be a positive integer. If $f^{\bar{d}(P)}f^{(k)}$ and $g^{\bar{d}(P)}g^{(k)}$ share 1 CM, then*

$$T(r,g) \leq \left(\frac{\bar{d}(P)+1}{\bar{d}(P)-1}\right) T(r,f) + S(r,g).$$

Proof. Let $G = g^{\bar{d}(P)}g^{(k)}$. Then it is a differential polynomial of degree $(\bar{d}(P)+1)$. By Lemma 10 we have

$$(\bar{d}(P)+1)T(r,g) \sim T(r,G) \tag{2.1}$$

Applying Lemma 10 to T(r,G), we get

$$\begin{aligned}(\bar{d}(P)+1)T(r,g) &\leq \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,G)\\ &= \overline{N}(r,g^{\bar{d}(P)}g^{(k)})+\overline{N}\left(r,\frac{1}{g^{\bar{d}(P)}g^{(k)}}\right)\\ &+\overline{N}\left(r,\frac{1}{g^{\bar{d}(P)}g^{(k)}-1}\right)+S(r,g^{\bar{d}(P)}g^{(k)}).\end{aligned}$$

Noting that

$$\begin{aligned}\overline{N}(r,g^{\bar{d}(P)}g^{(k)}) &\leq \overline{N}(r,g^{\bar{d}(P)})+\overline{N}(r,g^{(k)})\\ &\leq \overline{N}(r,g)+N(r,g)+k\overline{N}(r,g)\\ &= N(r,g)+(k+1)\overline{N}(r,g)\end{aligned}$$

and $S(r,G)=S(r,g)$, (by (2.1))
So,

$$\begin{aligned}(\bar{d}(P)+1)T(r,g) &\leq N(r,g)+(k+1)\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+N\left(r,\frac{1}{g^{(k)}}\right)\\ &+N\left(r,\frac{1}{g^{\bar{d}(P)}g^{(k)}-1}\right)+S(r,g).\end{aligned}$$

Since $f^{\bar{d}(P)}f^{(k)}$ and $g^{\bar{d}(P)}g^{(k)}$ share 1 CM, it implies that $f^{\bar{d}(P)}f^{(k)}-1$ and $g^{\bar{d}(P)}g^{(k)}-1$ have same zeros with the same multiplicities, using this with Lemma 6, we obtain that

$$\begin{aligned}(\bar{d}(P)+1)T(r,g) &\leq N(r,g)+(k+1)\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+N\left(r,\frac{1}{g}\right)\\ &+k\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1}\right)+S(r,g).\end{aligned}\tag{2.2}$$

By hypothesis, we have

$$\begin{aligned}\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right) &= S(r,f)\\ \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right) &= S(r,g)\end{aligned}$$

Using Nevanlinna's first fundamental theorem and Lemma 4, we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1}\right) &\leq T\left(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1}\right)\\ &= T(r,f^{\bar{d}(P)}f^{(k)})+O(1)\\ &\sim (\bar{d}(P)+1)T(r,f)+O(1)\end{aligned}$$

So

$$\overline{N}\left(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1}\right)\leq(\bar{d}(P)+1)T(r,f)+O(1).\tag{2.3}$$

Using (2.3),(2.2) becomes

$$\begin{aligned}(\bar{d}(P)+1)T(r,g) &\leq N(r,g)+N\left(r,\frac{1}{g}\right)+(\bar{d}(P)+1)T(r,f)+S(r,g)\\ &\leq 2T(r,g)+(\bar{d}(P)+1)T(r,f)+S(r,g)\\ (\bar{d}(P)-1)T(r,g) &\leq (\bar{d}(P)+1)T(r,f)+S(r,g)\\ T(r,g) &\leq \frac{(\bar{d}(P)+1)}{(\bar{d}(P)-1)}T(r,f)+S(r,g).\end{aligned}$$

This completes the proof of Lemma. □

Lemma 13. *Let $f,g\in\mathcal{A}$, $\bar{d}(P)\geq 2$ and k be a positive integer. If $f^{\bar{d}(P)}f^{(k)}$ and $g^{\bar{d}(P)}g^{(k)}$ share 1 CM, then $S(r,f)=S(r,g)$.*

Proof. Proceeding as in the proof of Lemma 12 we have

$$T(r,g)\leq\left(\frac{\bar{d}(P)+1}{\bar{d}(P)-1}\right)T(r,f)+S(r,g).$$

Similarly, we have

$$T(r,f)\leq\left(\frac{\bar{d}(P)+1}{\bar{d}(P)-1}\right)T(r,g)+S(r,f).$$

using above two inequalities we easily obtain $S(r,f)=S(r,g)$.
This completes the proof of Lemma.
Using the method in [14], we prove the following lemma. □

Lemma 14. *Let $f,g\in\mathcal{A}$, $\bar{d}(P)\geq 2$ and k be a positive integer. If $f^{\bar{d}(P)}f^{(k)}g^{\bar{d}(P)}g^{(k)}=1$ then $f=c_3e^{pz}$ and $g=c_4e^{-pz}$ where c_3,c_4 and p are constants such that $(-)^k(c_3c_4)^{\bar{d}(P)+1}p^{2k}=1$.*

Proof. Let

$$F=f^{\bar{d}(P)}f^{(k)}\ and\ G=g^{\bar{d}(P)}g^{(k)} \tag{2.4}$$

By Lemma 4, we have

$$T(r,F)\sim(\bar{d}(P)+1)T(r,f),\quad T(r,G)\sim(\bar{d}(P)+1)T(r,g)$$

Clearly $S(r,F)=S(r,f)$ and $S(r,G)=S(r,g)$.
By Lemma 13, we have $S(r,f)=S(r,g)$.
Thus,

$$S(r,F)=S(r,f)=S(r,g)=S(r,G). \tag{2.5}$$

By hypothesis,we have

$$f^{\bar{d}(P)}f^{(k)}g^{\bar{d}(P)}g^{(k)}=1\ \ or\ \ \ FG=1. \tag{2.6}$$

From (2.6) and f and g are transcendental functions, it follows that

$$N(r,\frac{1}{f})=0,\quad N(r,\frac{1}{g})=0. \tag{2.7}$$

By hypothesis, we have

$$\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})=S(r,f),\quad \overline{N}(r,g)+\overline{N}(r,\frac{1}{g})=S(r,g) \tag{2.8}$$

(2.6) can be expressed as

$$f^{\bar{d}(P)}f^{(k)}=\frac{1}{g^{\bar{d}(P)}g^{(k)}}$$

so we deduce that

$$N(r,f^{\bar{d}(P)}f^{(k)})=N\left(r,\frac{1}{g^{\bar{d}(P)}g^{(k)}}\right) \tag{2.9}$$

Using (2.7), we get

$$\begin{aligned}N(r,f^{\bar{d}(P)}f^{(k)}) &= N(r,f^{\bar{d}(P)})+N(r,f^{(k)})\\ &= \bar{d}(P)N(r,f)+N(r,f)+k\overline{N}(r,f)\\ &= (\bar{d}(P)+1)N(r,f)+k\overline{N}(r,f)\\ &= (\bar{d}(P)+1)N(r,f)+S(r,f)\end{aligned}$$

Using this with Lemma 6 and equation (2.5),(2.7) and (2.8),(2.9) can be written as

$$\begin{aligned}(\bar{d}(P)+1)N(r,f)+S(r,f) &\leq N\left(r,\frac{1}{g^{\bar{d}(P)}}\right)+N\left(\frac{1}{g^{(k)}}\right)\\ &\leq (\bar{d}(P)+1)N(r,\frac{1}{g})+k\overline{N}(r,\frac{1}{g})+S(r,g)\\ &= S(r,g)\end{aligned}$$

which implies that

$$N(r,f)=S(r,f) \tag{2.10}$$

Similarly

$$N(r,g)=S(r,g). \tag{2.11}$$

By (2.7),(2.8) and Lemma 6, we have

$$\begin{aligned}\bar{N}(r,\frac{1}{F}) &= \bar{N}\left(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}}\right)\\ &\leq \bar{N}(r,\frac{1}{f})+N\left(r,\frac{1}{f^{(k)}}\right)\\ &\leq \bar{N}(r,\frac{1}{f})+N(r,\frac{1}{f})+k\bar{N}(r,f)+S(r,f)\\ &= S(r,f).\end{aligned}$$

Therefore

$$\bar{N}(r,\frac{1}{F})=S(r,F) \tag{2.12}$$

$$Similarly\quad \bar{N}(r,\frac{1}{G})=S(r,G). \tag{2.13}$$

Moreover by using (2.8) and (2.10), we have

$$\begin{aligned}\bar{N}(r,F) =&\bar{N}(r,f^{\bar{d}(P)}f^{(k)}) \leq \bar{N}(r,f)+N(r,f^{(k)})\\ &\leq \bar{N}(r,f)+N(r,f)+k\bar{N}(r,f)\\ &= S(r,f).\end{aligned}$$

Therefore

$$\bar{N}(r,F) = S(r,F) \tag{2.14}$$

Similarly

$$\bar{N}(r,G) = S(r,G). \tag{2.15}$$

It follows from (2.12)-(2.15) that

$$\bar{N}(r,\frac{1}{F})+\bar{N}(r,F) = S(r,F), \quad \bar{N}(r,\frac{1}{G})+\bar{N}(r,G) = S(r,G). \tag{2.16}$$

In view of (2.6), we know that F and G share 1 and -1 IM, together this with (2.16) and Lemma 7 implies that F and G share $1,-1,0,\infty$ CM, thus by Lemma 8, we get that 0 and ∞ are Picard values of F and G. Thus we deduce from (2.4) that both f and g are transcendental entire functions. By (2.7) we have

$$f(z) = e^{\alpha(z)}, \quad g(z) = e^{\beta(z)} \tag{2.17}$$

where $\alpha(z)$ and $\beta(z)$ are non constant entire functions.
Then

$$T(r,\frac{f'}{f}) = T(r,\frac{e^{\alpha}\alpha'}{e^{\alpha}}) = T(r,\alpha').$$

We claim that $\alpha(z)+\beta(z) = c$, c is a constant.
From (2.17), we know that either α and β are transcendental functions or both α and β are polynomials.
From (2.6), we have

$$\begin{aligned}N(r,\frac{1}{f^{(k)}}) &= N(r,g^{\bar{d}(P)}g^{(k)}f^{\bar{d}(P)})\\ &\leq \bar{d}(P)N(r,g)+N(r,g^{(k)})+\bar{d}(P)N(r,f)\\ &= 0.\end{aligned}$$

From this and (2.6), we get

$$N(r,f)+N(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}}) = 0.$$

If $k \geq 2$, suppose that α is a transcendental entire function. From Lemma 11, we have $f = e^{\alpha(z)} = e^{az+b}$, it implies that $\alpha(z) = az+b$, a polynomial, which is a contradiction.
Thus α and β are polynomials.
We deduce from (2.17) that

$$f^{(k)} = [(\alpha')^k + P_{k-1}(\alpha')]e^{\alpha}\ , \quad g^{(k)} = [(\beta')^k + Q_{k-1}(\beta')]e^{\beta}$$

where $P_{k-1}(\alpha')$and $Q_{k-1}(\beta')$ are differential polynomials in α' and β' of degree at most $(k-1)$ respectively.
Thus by (2.6) we obtain that

$$[(\alpha')^k + P_{k-1}(\alpha')][(\beta')^k + Q_{k-1}(\beta')]e^{(\bar{d}(P)+1)(\alpha+\beta)} = 1 \tag{2.18}$$

we deduce from (2.18) that $\alpha(z) + \beta(z) = c$, c is a constant.
If $k = 1$, from (2.17) we get,

$$\alpha'\beta' e^{(\bar{d}(P)+1)(\alpha+\beta)} = 1. \tag{2.19}$$

Let $\alpha + \beta = \gamma$. If α and β are transcendental entire functions, then γ is not a constant and (2.19) implies that

$$\alpha'(\gamma' - \alpha')e^{(\bar{d}(P)+1)\gamma} = 1. \tag{2.20}$$

Since

$$\begin{aligned} T(r,\gamma') &= m(r,\gamma') \\ &= m\left(r, \frac{e^{(\bar{d}(P)+1)\gamma}\gamma'}{e^{(\bar{d}(P)+1)\gamma}}\right) \\ &= m\left(r, \frac{(e^{(\bar{d}(P)+1)\gamma})'}{e^{(\bar{d}(P)+1)\gamma}}\right) = S(r, e^{(\bar{d}(P)+1)\gamma}) \end{aligned}$$

Thus (2.20) implies that Since

$$\begin{aligned} T(r, e^{(\bar{d}(P)+1)\gamma}) &= T(r, \frac{1}{\alpha'(\gamma' - \alpha')}) \\ &\leq T(r, \alpha'(\gamma' - \alpha')) + O(1) \\ &\leq 2T(r,\alpha') + S(r, e^{(\bar{d}(P)+1)\gamma}). \end{aligned}$$

Which implies that

$$T(r, e^{(\bar{d}(P)+1)\gamma}) = O(T(r,\alpha')).$$

Thus $T(r,\gamma') = S(r,\alpha')$.
In view of (2.20) and by Lemms 10, we get

$$T(r,\alpha') \leq \bar{N}(r,\alpha') + \bar{N}(r,\frac{1}{\alpha'}) + \bar{N}(r,\frac{1}{\alpha' - \gamma'}) + S(r,\alpha').$$

Since α and β are transcendental entire functions and in view of (2.20), we obtain

$$T(r,\alpha') \leq S(r,\alpha'),$$

and this implies that α' is a constant, which is a contradiction. Thus α and β are both polynomials and $\alpha(z) + \beta(z) = c$, for a constant c.
Hence from (2.18), we get

$$(\alpha')^{2k} = 1 + \widetilde{P}_{2k-1}(\alpha') \tag{2.21}$$

where $\widetilde{P}_{2k-1}(\alpha')$ is a differential polynomial in α' of degree at most $(2k-1)$.
From (2.21), we have

$$\begin{aligned}
2kT(r,\alpha') &= T(r,(\alpha')^{2k}) = m(r,(\alpha')^{2k}) \leq m(r,\widetilde{P}_{2k-1}(\alpha')) + O(1)\\
&= m\left(r, \frac{\widetilde{P}_{2k-1}(\alpha')}{(\alpha')^{2k-1}}.(\alpha')^{2k-1}\right) + O(1)\\
&= m\left(r, \frac{\widetilde{P}_{(2k-1)}(\alpha')}{(\alpha')^{2k-1}}\right) + m(r,(\alpha')^{2k-1}) + O(1)\\
&\leq (2k-1)T(r,\alpha') + S(r,\alpha').
\end{aligned}$$

Therefore $T(r,\alpha') \leq S(r,\alpha')$.
which implies that α' is a constant.
Thus $\alpha = pz + c_1$, $\beta = -pz + c_2$. By (2.17), we represent f and g as

$$f = c_3 e^{pz}, \quad g = c_4 e^{-pz}$$

where c_3, c_4 and p are constants such that $(-1)^k (c_3 c_4)^{\bar{d}(P)+1} p^{2k} = 1$.
This completes the proof of Lemma. □

Proof of Theorem 1. Let $F = f^{\bar{d}(P)} f^{(k)}$. By Lemmas 4 and 10, we have

$$\begin{aligned}
(\bar{d}(P)+1)T(r,f) &\sim T(r, f^{\bar{d}(P)} f^{(k)})\\
&\leq \bar{N}(r, f^{\bar{d}(P)} f^{(k)}) + \bar{N}(r, \frac{1}{f^{\bar{d}(P)} f^{(k)}})\\
&+ \bar{N}(\frac{1}{f^{\bar{d}(P)} f^{(k)} - 1}) + S(r, f^{\bar{d}(P)} f^{(k)})
\end{aligned} \tag{2.22}$$

Noting that

$$\begin{aligned}
\bar{N}(r, f^{\bar{d}(P)} f^{(k)}) &\leq \bar{N}(r, f^{\bar{d}(P)}) + N(r, f^{(k)})\\
&\leq \bar{N}(r,f) + N(r,f) + k\bar{N}(r,f)\\
&\leq N(r,f) + (k+1)\bar{N}(r,f)\\
\bar{N}(r, \frac{1}{f^{\bar{d}(P)} f^{(k)}}) &\leq \bar{N}(r, \frac{1}{f^{\bar{d}(P)}}) + N(r, \frac{1}{f^{(k)}})\\
&\leq \bar{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}})
\end{aligned}$$

and $(\bar{d}(P)+1)T(r,f) \sim T(r, f^{\bar{d}(P)} f^{(k)})$.
So $S(r, f^{\bar{d}(P)} f^{(k)}) = S(r,f)$.
Substituting above inequalities in (2.22),we obtain,

$$\begin{aligned}
(\bar{d}(P)+1)T(r,f) \leq N(r,f) + (k+1)\bar{N}(r,f) + \bar{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}})\\
+ \bar{N}(r, \frac{1}{f^{\bar{d}(P)} f^{(k)} - 1}) + S(r,f).
\end{aligned}$$

using Lemma 6, we get

$$(\bar{d}(P)+1)T(r,f) \leq N(r,f)+(k+1)\bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f}\right)+N(r,\frac{1}{f})+k\bar{N}(r,f) + \bar{N}\left(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1}\right)+S(r,f). \tag{2.23}$$

By hypothesis, we have $\bar{N}(r,f)=S(r,f)$, $\bar{N}(r,\frac{1}{f})=S(r,f)$.
Therefore (2.23) becomes

$$\begin{aligned}(\bar{d}(P)+1)T(r,f) &\leq N(r,f)+N(r,\frac{1}{f})+\bar{N}(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1})+S(r,f)\\ &\leq 2T(r,f)+\bar{N}(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1})+S(r,f)\\ (\bar{d}(P)-1)T(r,f) &\leq \bar{N}(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}-1})+S(r,f).\end{aligned}$$

which implies that $f^{\bar{d}(P)}f^{(k)}-1$ has infinitely many zeros for $\bar{d}(P)\geq 2$.
This completes the proof of Theorem. □

Proof of Theorem 2. By hypothesis, $f^{\bar{d}(P)}f^{(k)}$ and $g^{\bar{d}(P)}g^{(k)}$ share 1 CM.
Let

$$H(z)=\frac{f^{\bar{d}(P)}f^{(k)}-1}{g^{\bar{d}(P)}g^{(k)}-1}. \tag{2.24}$$

Then $H(z)$ is a meromorphic function satisfying $T(r,H)=O(T(r,f)+T(r,g))$, by the first fundamental theorem and Lemma 4.
From (2.24), we see that the zeros and poles of $H(z)$ are multiple and satisfy

$$\bar{N}(r,H)\leq \bar{N}_L(r,f), \bar{N}(r,\frac{1}{H})\leq \bar{N}_L(r,g) \tag{2.25}$$

Let

$$f_1=f^{\bar{d}(P)}f^{(k)}, \quad f_2=-Hg^{\bar{d}(P)}g^{(k)}, \quad f_3=H. \tag{2.26}$$

Then by using (2.24), we easily see that

$$\begin{aligned}f_1+f_2+f_3 &= f^{\bar{d}(P)}f^{(k)}-Hg^{\bar{d}(P)}g^{(k)}+H\\ &= f^{\bar{d}(P)}f^{(k)}-H(g^{\bar{d}(P)}g^{(k)}-1)\\ &= f^{\bar{d}(P)}f^{(k)}-\left(\frac{f^{\bar{d}(P)}f^{(k)}-1}{g^{\bar{d}(P)}g^{(k)}-1}\right)(g^{\bar{d}(P)}g^{(k)}-1) \quad =1.\end{aligned}$$

Assuming that f_1 is non-constant and by Lemma 5, we have

$$\begin{aligned}\sum_{j=1}^{3} N_2(r,\frac{1}{f_j})+\sum_{j=1}^{3}\bar{N}(r,f_j) &= N_2(r,\frac{1}{f_1})+N_2(r,\frac{1}{f_2})+N_2(r,\frac{1}{f_3})+\bar{N}(r,f_1)\\ &\quad+\bar{N}(r,f_2)+\bar{N}(r,f_3)\\ &\leq N_2(r,\frac{1}{f^{\bar{d}(P)}f^{(k)}})+N_2(r,\frac{1}{g^{\bar{d}(P)}g^{(k)}})+N_2(r,\frac{1}{H})\\ &\quad+\bar{N}(r,f^{\bar{d}(P)}f^{(k)})+\bar{N}(r,g^{\bar{d}(P)}g^{(k)})+\bar{N}(r,H).\end{aligned} \tag{2.27}$$

Noting that
$\bar{N}(r,f^{\bar{d}(P)}f^{(k)})\leq N(r,f)+(k+1)\bar{N}(r,f)$
$\bar{N}(r,g^{\bar{d}(P)}f^{(k)})\leq N(r,g)+(k+1)\bar{N}(r,g)$
using this with (2.25) and Lemma 6, (2.27) becomes

$$\begin{aligned}\sum_{j=1}^{3} N_2(r,\frac{1}{f_j})+\sum_{j=1}^{3}\bar{N}(r,f_j) &\leq 2\bar{N}(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}})+2\bar{N}(r,\frac{1}{g})+N(r,\frac{1}{g^{(k)}})\\ &\quad+2\bar{N}(r,\frac{1}{H})+N(r,f)+(k+1)\bar{N}(r,f)+N(r,g)+(k+1)\bar{N}(r,g)\\ &\quad+\bar{N}(r,H)\\ &\leq 2\bar{N}(r,\frac{1}{f})+N(r,\frac{1}{f})+k\bar{N}(r,f)+2\bar{N}(r,\frac{1}{g})\\ &\quad+N(r,\frac{1}{g})+k\bar{N}(r,g)+2\bar{N}_L(r,g)+N(r,f)\\ &\quad+(k+1)\bar{N}(r,f)+N(r,g)+(k+1)\bar{N}(r,g)\\ &\quad+\bar{N}_L(r,f)+S(r,f)+S(r,g)\\ &= 2(\bar{N}(r,\frac{1}{f})+\bar{N}(r,\frac{1}{g}))+(N(r,\frac{1}{f})+N(r,\frac{1}{g}))\\ &\quad+(2k+1)(\bar{N}(r,f)+\bar{N}(r,g))+(N(r,f)+N(r,g))\\ &\quad+2\bar{N}_L(r,g)+\bar{N}_L(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Since $f,g\in\mathcal{A}$, we have

$$\bar{N}(r,f)+\bar{N}(r,\frac{1}{f})=S(r,f),\quad \bar{N}(r,g)+\bar{N}(r,\frac{1}{g})=S(r,g).$$

Therefore

$$\begin{aligned}\sum_{j=1}^{3} N_2(r,\frac{1}{f_j})+\sum_{j=1}^{3}\bar{N}(r,f_j) &\leq (N(r,\frac{1}{f})+N(r,\frac{1}{g}))+(N(r,f)+N(r,g))+2\bar{N}_L(r,g)\\ &\quad+\bar{N}_L(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{2.28}$$

Noting that

$$2\bar{N}_L(r,g)+\bar{N}_L(r,f)\leq 2\bar{N}(r,f)=S(r,f)$$

$$or \quad 2\bar{N}_L(r,g) + \bar{N}_L(r,f) \leq 2\bar{N}(r,g) = S(r,g)$$

Thus (2.28) becomes

$$\sum_{j=1}^{3} N_2(r,\frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r,f_j) \leq 2(T(r,f)+T(r,g)) + S(r,f) + S(r,g)$$

using Lemmas 12 and 13, we get

$$\begin{aligned}\sum_{j=1}^{3} N_2(r,\frac{1}{f_j}) + \sum_{j=1}^{3} \bar{N}(r,f_j) &\leq 2T(r,f) + \left(2\frac{(\bar{d}(P)+1)}{\bar{d}(P)-1}\right)T(r,f) + S(r,f)\\ &= \frac{4\bar{d}(P)}{(\bar{d}(P)-1)}T(r,f) + S(r,f)\\ &\leq \frac{4\bar{d}(P)}{(\bar{d}(P)-1)(\bar{d}(P)+1)}T(r) + S(r,f)\\ &\leq \left(\frac{4\bar{d}(P)}{(\bar{d}(P)-1)(\bar{d}(P)+1)} + o(1)\right)T(r)\end{aligned}$$

Since $\bar{d}(P) \geq 5$,$\dfrac{4\bar{d}(P)}{(\bar{d}(P)-1)(\bar{d}(P)+1)} < 1$ using Lemma 5, we get $F_2 = 1$ or $F_3 = 1$.

Next we consider two cases :

Case 1. $F_2 = 1$ i.e., $-Hg^{\bar{d}(P)}g^{(k)} = 1$

Using (2.24) we have

$$-\left(\frac{f^{\bar{d}(P)}f^{(k)}-1}{g^{\bar{d}(P)}g^{(k)}-1}\right)g^{\bar{d}(P)}g^{(k)} = 1$$

by simple computing, we get

$$f^{\bar{d}(P)}g^{(k)}g^{\bar{d}(P)}g^{(k)} = 1$$

By Lemma 14, we get the conclusion of Theorem 2.

Case 2. $F_3 = 1$ i.e., $H = 1$

Using (2.24), we have

$$\frac{f^{\bar{d}(P)}f^{(k)}-1}{g^{\bar{d}(P)}g^{(k)}-1} = 1$$

$$i.e., \quad f^{\bar{d}(P)}f^{(k)} = g^{\bar{d}(P)}g^{(k)}. \tag{2.29}$$

By Lemma 4, we have

$$\begin{aligned}T(r,f^{\bar{d}(P)}f^{(k)}) &= T(r,g^{\bar{d}(P)}g^{(k)})\\ (\bar{d}(P)+1)T(r,f) &= (\bar{d}(P)+1)T(r,g)\\ T(r,f) &= T(r,g)\end{aligned} \tag{2.30}$$

and also

$$S(r,f) = S(r,g). \tag{2.31}$$

Let $h = \frac{g}{f}$. Then by (2.29), we have

$$h^{\bar{d}(P)} = \frac{f^{(k)}}{g^{(k)}} \quad , \quad h^{(\bar{d}(P)+1)} = \frac{gf^{(k)}}{fg^{(k)}}$$

Suppose that h is not a constant. By (2.30), we have

$$\begin{aligned} T(r,h) &= T(r,\frac{g}{f}) \\ &\leq T(r,g) + T(r,f) + O(1) \\ &\leq 2T(r,f) + O(1). \end{aligned}$$

Which implies that

$$S(r,h) = S(r,f)$$

similarly

$$S(r,h) = S(r,g)$$

Thus, by (2.31)

$$S(r,h) = S(r,f) = S(r,g)$$

By the first fundamental theorem and Lemma 9, we have

$$\begin{aligned} T(r,h^{\bar{d}(P)+1}) &= T\left(r,\frac{gf^{(k)}}{fg^{(k)}}\right) \\ (\bar{d}(P)+1)T(r,h) &\leq T(r,\frac{f^{(k)}}{f}) + T(r,\frac{g}{g^{(k)}}) + O(1) \\ &= T(r,\frac{f^{(k)}}{f}) + T(r,\frac{g^{(k)}}{g}) + O(1) \\ &= S(r,f) + S(r,g) \\ &= S(r,h). \end{aligned}$$

which is a contradiction since $\bar{d}(P) \geq 5$. Therefore h is a constant. Since f and g are transcendental meromorphic functions, we have $h \neq 0$.
Let $t = \frac{1}{h}$, which implies that $f = tg$, From (2.29), we obtain $t^{\bar{d}(P)+1} = 1$. This completes the proof of the Theorem. □

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