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ON ORTHOGONAL REVERSE SEMIDERIVATIONS ON PRIME SEMIRINGS

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Abstract. In this paper, we introduce the notion of orthogonal reserve semiderivation on semirings. Some characterizations of semiprime semirimgs are obtained by means of orthogonal reverse semiderivations. We also investigate conditions for two reverse semiderivations on semiring to be orthogonal.

1. Introduction

The notion of semiring was first introduced in 1934 by H. S. Vandiver [7]. A semiring is an algebraic structure , consisting of a nonempty set S on which we have defined two associative binary operations addition (usually denoted by $+$) and multiplication (usually $\cdot$) such that the multiplication is distributive over addition. The notion of rings with derivations is quite old and plays a significant role in the integration of analysis, algebraic geometry, and algebra. The study of derivations in rings though initiated long back, but got interested only after Posner [4] who 1957 established two very striking results on derivations in prime rings. The reverse derivations on semiprime rings has been studied by Samman and Alyamani [5]. Here the authors obtain some results of semiprime rings by reverse semiderivations. In this paper, we introduce the notion of orthogonal reserve semiderivation on semirings. Some characterizations of semiprime semirimgs are obtained by means of orthogonal reverse semiderivations. We also investigate conditions for two reverse semiderivations on semiring to be orthogonal.

2. Semirings

. A *semiring* $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations "+" and " $\cdot$ " such that

(S1) $(S, +)$ is a commutative monoid with identity element 0,
(S2) $(S, \cdot)$ is a monoid with identtiy element 1,
(S3) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b \in S$.

Definition 2.1. A semiring $(S, +, \cdot)$ is said to be *additively commutative* if $(S, +)$ is a commutative semigroup. A semiring $(S, +, \cdot)$ is said to be *multiplicatively*

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commutative if $(S, \cdot)$ is a commutative semigroup. It is said to be commutative if both $(S, +)$ and $(S, \cdot)$ are commutative.

Definition 2.2. Let S be a semiring. An element a in S is said to be *additively left cancellative* if $a+b = a+c$ implies $b = c$, for every $b, c \in S$. It is said to be *additively right cancellative* if $b+a = c+a$ implies $b = c$. It is said to be *additively cancellative* if it is both left and right cancellative. A semiring S is said to be *additively cancellative* if all elements in S are additively cancellative.

Definition 2.3. Let S be a semiring. Then

(D1) S is said to be *prime* if $aSb = 0$ implies $a = 0$, or $b = 0$ for all $a, b \in S$.
(D2) S is said to be *semiprime* if $aSa = 0$ implies $a = 0$, for all $a \in S$.
(D3) S is said to be 2-*torsion free* if $2a = 0$ implies $a = 0$ for all $a \in S$.

Definition 2.4. Let S be a semiring. Any nonempty subset I is called a *left ideal* of S if the following conditions are satisfied:

(1) $1 \notin I$,
(2) $a, b \in I \Rightarrow a + b \in I$,
(3) $a \in I$ and $s \in S \Rightarrow sa \in I$.

Similarly, we can define right and two-sided ideal in a semiring.

Definition 2.5. Let S be a semiring. An additive mapping $d : S \to S$ is called a *derivation* if

$$d(xy) = d(x)y + xd(y)$$

for all $x, y \in S$.

Example 2.6. Let S be a semiring and $M_2(S) = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in S\} \right\}$. Then $M_2(S)$ is a semiring. Define a map $d : M_2(S) \to M_2(S)$ by

$$d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

Then d is a derivation on S.

Definition 2.7. Let S be a semiring. An additive mapping $d : S \to S$ is called a *semiderivation* associated with a surjective function $g : S \to S$ if

(1) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$,
(2) $d(g(x)) = g(d(x))$, for all $x, y \in S$.

3. Orthogonal reverse semiderivations of semiprime semirings

Throughout this paper, we assume that S is an additively cancellative semiring.

Definition 3.1. Let S be a semiring. An additive mapping $d : S \to S$ is called a *reverse semiderivation* associated with a function $g : S \to S$ if

(1) $d(xy) = d(y)g(x) + yd(x) = d(y)x + g(y)d(x)$,
(2) $d(g(x)) = g(d(x))$, for all $x, y \in S$.

Example 3.2. Let S be a semiring and let $M_2(S) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in S\} \right\}$.
Define $d : M_2(S) \to M_2(S)$ by

$$d\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and $g : M_2(S) \to M_2(S)$ by

$$g\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

Then d_1 is a reverse semiderivation.

Definition 3.3. Let d_1 and d_2 be two reverse semiderivations on S associated with functions $g_1 : S \to S$ and $g_2 : S \to S$, respectively. Then d_1 and d_2 are orthogonal if $d_1(x)sd_2(y) = d_2(y)sd_1(x) = 0$ for all $x, y \in S$.

Example 3.4. Let S be a semiring. Let $S = S_1 \oplus S_1$. Define $d_1 : S \to S$ such that $d_1(a, b) = (a, 0)$ and $g_1(a, b) = (0, ab)$ for all $a, b \in S_1$. Also define $d_2 : S \to S$ such that $d_2(a, b) = (0, b)$ and $g_2 : S \to S$ such that $g_2(a, b) = (ab, 0)$ for all $a, b \in S_1$. Define addition and multiplication on S by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ for all $a_1, a_2, b_1, b_2 \in S_1$. Then it can easily be seen that d_1 and d_2 are reverse semiderivations of S with associated mappings g_1 and g_2, respectively which are not derivations. Also, it is clear that d_1 and d_2 are orthogonal reverse semiderivations on S.

Example 3.5. Let S be a semiring. Let $S = S_1 \oplus S_1$. Let $\alpha_1 : S_1 \to S_1$ be additive map, $\alpha_2 : S_1 \to S_1$ be a left and right S_1 module which is not a derivation. Define $d_1 : S \to S$ such that $d_1(a, b) = (0, \alpha_2(b))$ and $g_1(a, b) = (\alpha_1(a), 0)$ for all $a, b \in S_1$. Also define $d_2 : S \to S$ such that $d_2(a, b) = (\alpha_2(a), 0)$ and $g_2 : S \to S$ such that $g_2(a, b) = (0, \alpha_1(b))$ for all $a, b \in S_1$. Define addition and multiplication on S by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ for all $a_1, a_2, b_1, b_2 \in S_1$. Then it can easily be seen that d_1 and d_2 are reverse semiderivations of S with associated mappings g_1 and g_2, respectively which are not derivations. Also, it is clear that d_1 and d_2 are orthogonal reverse semiderivations on S.

Lemma 3.6. *Let S be a prime semiring and $a \in S$. If d is a reverse semiderivation associated with a function $g : S \to S$. If $ad(x) = 0$ or $d(x)a = 0$ for $a \in S$, then $a = 0$ or $d = 0$.*

Proof. Suppose that $ad(S) = 0$. For arbitrary $x, y \in S$, we have

$$ad(xy) = 0.$$

Hence we obtain

$$a(d(y)g(x) + yd(x)) = ad(y)g(x) + ayd(x) = 0. \tag{1}$$

Replacing y by x in above equation (1), we get $ad(x)g(x) + axd(x) = 0$. Thus, we get $axd(x) = 0$, for all $x \in S$, which implies $asd(x) = 0$ for all $x \in S$. Since S

is prime, $d(x) = 0$ or $a = 0$ for all $x \in S$. Hence $a = 0$ or $d = 0$. Similarly, we can prove for $d(x)a = 0$ for any $x \in S$. $\square$

Lemma 3.7. *Let S be a 2-torsion free prime semiring and $a \in S$. If d is a reverse semiderivation associated with surjective function $g : S \to S$. If $d^2 = 0$, then $d = 0$.*

Proof. For arbitrary $x, y \in S$, we have

$$d^2(xy) = d(d(xy)) = d(d(y)g(x) + yd(x)) = 0.$$

Hence, we get for any $x, y \in S$,

$$d(g(x))g(d(y)) + g(x)d^2(y) + d(d(x))g(y) + d(x)d(y) = 0$$

By hypothesis,

$$d(g(x))g(d(y)) + d(x)d(y) = 0.$$

Since g is surjective and 2-torsion free, we have

$$d(x)d(y) = 0 \tag{1}$$

for every $x, y \in S$. Replacing y by yz in the above equation (1), we get for all $x, y \in S$,

$$\begin{aligned} 0 &= d(x)d(yz) \\ &= d(x)d(z)g(y) + d(x)zd(y) \\ &= d(x)zg(y) \end{aligned}$$

This implies that $d(x)sd(y) = 0$ for all $x, y \in S$. Since S is prime, we have $d(x) = 0$ or $d(y) = 0$ for all $x, y \in S$. In both cases, we have $d = 0$. $\square$

Lemma 3.8. *Let S be a 2-torsion free semiprime semiring and $a, b \in S$. Then the following conditions are equivalent:*

(1) $asb = 0$,
(2) $bsa = 0$,
(3) $asb + bsa = 0$, *for every* $s \in S$.

If one of these conditions is fulfilled, then $ab = ba = 0$.

Proof. (1) $\Rightarrow$ (2): Let $asb = 0$, for every $a, b, s \in S$. Multiplying by bs on left side and multiplying by sa on right side, we have $(bsa)s(bsa) = 0$. Since S is semiprime, we have $bsa = 0$.

(2) $\Rightarrow$ (3): Let $bsa = 0$. Multiplying by as on left side and multiplying by sb, on right side we have $(asb)s(asb) = 0$. Since S is semiprime, we have $asb = 0$, which implies $asb + bsa = 0$.

(3) $\Rightarrow$ (1): Let $asb + bsa = 0$, for every $s, a, b \in S$. Multiplying by bs on left side, we have $bs(asb) + bs(bsa) = 0$. Also, multiplying by as on left side, we get

$$(asb)s(asb) + (asb)s(bsa) = 0. \tag{1}$$

Multiplying by sa on right side, we have $(asb)sa + (bsa)sa = 0$. Also, multiplying by sb, on right side, we get

$$(asb)s(asb) + (bsa)s(asb) = 0. \tag{2}$$

Adding equation (1) to (2) and using Lemma 3.8 (3), we have $2((asb)s(asb)) = 0$. Since S is 2-torsion free and S is semiprime, we have $asb = 0$, for all $x \in S$.

Let $asb = 0$. Multiplying by b on left side and multiplying by a on right side, we have $(ba)s(ba) = 0$. Since S is semiprime, we have $ba = 0$. Similarly, from $bsa = 0$, we can prove that $ab = 0$. □

Lemma 3.9. *Let S be a 2-torsion free semiprime semiring and let additive mappings of d and g of S into itself satisfying $d(x)sg(x) = 0$, for any $s \in S$. Then $d(x)sg(y) = 0$, for every $x, y \in S$.*

Proof. Suppose that $d(x)sg(x) = 0$, for any $x, s \in S$. Replacing x by $x + y$, we have

$$\begin{aligned}
0 &= d(x+y)sg(x+y) \\
&= d(x)sg(x) + d(x)sg(y) + d(y)sg(x) + d(y)sg(y) \\
&= d(x)sg(y) + d(y)sg(x) \\
&= (d(x)sg(y))s_1(d(x)sg(y)) + d(x)s(g(y)s_1d(y))sg(x).
\end{aligned}$$

By Lemma 3.8, we have $(d(x)sg(y))s_1(d(x)sg(y)) = 0$. Since S is semiprime, $d(x)sg(y) = 0$, for every $x, y \in S$. □

Theorem 3.10. *Let S be a 2-torsion free prime semiring and let d_1 and d_2 be reverse semiderivations of S into S associated with surjective functions $g_1 : S \to S$ and $g_2 : S \to S$, respectively. Then*

$$d_1(x)d_2(y) + d_2(x)d_1(y) = 0, \qquad (1)$$

for all $x, y \in S$ if and only if d_1 and d_2 are orthogonal.

Proof. Suppose that d_1 and d_2 are orthogonal. Then $d_1(x)sd_2(y) = 0 = d_2(y)sd_1(x)$, for every $x, y \in S$. Since $d_1(x)d_2(y) = 0 = d_2(y)d_1(x)$ for every $x, y \in S$, by Lemma 3.8, we have $d_1(x)d_2(y) = 0$ for every $x, y \in S$. Hence $d_1(x)d_2(y) + d_2(y)d_1(x) = 0$ for every $x, y \in S$. Conversely, assume that

$$d_1(x)d_2(y) + d_2(y)d_1(x) = 0 \qquad (2)$$

for every $x, y \in S$. Replacing y by xy in in the above equation (2), since g_1 and g_2 are surjective, we have

$$\begin{aligned}
0 &= d_1(x)d_2(xy) + d_2(x)d_1(xy) \\
&= d_1(x)(d_2(y)g_2(x) + yd_2(x)) + d_2(x)(d_1(y)g_1(x) + yd_1(x)) \\
&= d_1(x)d_2(y)x + d_1(x)yd_2(x) + d_2(x)d_1(y)x + d_2(x)yd_1(x)
\end{aligned}$$

By hypothesis, $d_1(x)yd_2(x) + d_2(x)yd_1(x) = 0$, and so by Lemma 3.9, we have $d_1(x)d_2(x) = 0 = d_2(x)d_1(x)$, for every $x \in S$. Hence, by Lemma 3.8, we get $d_1(x)yd_2(z) = 0 = d_2(x)yd_1(z)$, for any $x, y, z \in S$. This proves that d_1 and d_2 are orthogonal. □

The following theorem gives a few criteria on the orthogonality of reverse semiderivation.

Theorem 3.11. *Let S be a 2-torsion free semiprime semiring and let d_1 and d_2 be reverse semiderivations of S into S associated with surjective functions $g_1 : S \to S$ and $g_2. : S \to S$, respectively. Then d_1 and d_2 are orthogonal if and only if $d_1 d_2 = 0 = d_2 d_1$.*

Proof. Suppose that $d_1 d_2 = 0 = d_2 d_1$. Then we have for every $x, y \in S$,

$$\begin{aligned}
0 &= d_1 d_2(xy) \\
&= d_1(d_2(y)g_2(x) + y d_2(x)) \\
&= d_1(g_2(x))g_1(d_2(y)) + g_2(x)d_1(d_2(y)) + d_1(d_2(x))g_1(y) + d_2(x)d_1(y) \\
&= d_1(g_2(x))g_1(d_2(y)) + d_2(x)d_1(y) \\
&= d_1(x)d_2(y) + d_2(x)d_1(y).
\end{aligned}$$

Therefore, by Theorem 3.10, d_1 and d_2 are orthogonal. Conversely, assume that d_1 and d_2 are orthogonal. Then we have $d_1(x)sd_2(y) = 0 = d_2(y)sd_1(x)$ for every $x, y, s \in S$. Then by Lemma 3.8, $d_2(x)d_1(y) = 0$ for all $x, y \in S$. Hence, for all $x, y \in S$,

$$\begin{aligned}
0 &= d_1(x)d_2(y) \\
&= d_1(d_1(x)d_2(y)) \\
&= d_1(d_2(y))g_1(d_1(x)) + d_2(y)d_1(d_1(x)) \\
&= d_1(d_2(y))d_1(x) + d_2(y)d_1(d_1(x)) \\
&= d_1(d_2(y))d_1(x),
\end{aligned}$$

since g_1 is surjective. Premultiplying by $d_1(d_2(y))$, we have for all $x, y, s \in S$,

$$\begin{aligned}
0 &= d_1(d_2(y))d_1(x)d_1(d_2(y)) \\
&= d_1(d_2(y))sd_1(d_2(y)) \text{ for all } y \in S.
\end{aligned}$$

Since S is semiprime, we have $d_1 d_2(y) = 0$ for all $y \in S$. Hence $d_1 d_2 = 0$. Similarly, we can prove $d_2 d_1 = 0$. $\square$

Theorem 3.12. *Let S be a 2-torsion free semiprime semiring and let d_1 and d_2 be reverse semiderivations of S into S associated with surjective functions $g_1 : S \to S$ and $g_2 : S \to S$, respectively. Then d_1 and d_2 are orthogonal if and only if $d_1 d_2 + d_2 d_1 = 0$.*

Proof. Suppose that d_1 and d_2 are orthogonal. Then $d_1(x)d_2(y) = 0 = d_2(y)d_1(x)$ for all $x, y \in S$ by Theorem 3.11 Hence $d_1 d_2 + d_2 d_1 = 0$. Conversely, assume that

$d_1d_2 + d_2d_1 = 0$. Then we have, for all $x, y \in S$,

$$\begin{aligned}
0 &= (d_1d_2 + d_2d_1)(xy) \\
&= d_1d_2(xy) + d_2d_1(xy) \\
&= d_1(d_2(y)g_2(x) + yd_2(x)) + d_2(d_1(y)g_1(x) + yd_1(x)) \\
&= d_1(g_2(x))g_1(d_2(y)) + g_2(x)d_1(d_2(y)) + d_1(d_2(x))g_1(y) + d_2(x)d_1(y)) \\
&\quad + d_2(g_1(x)g_2(d_1(y)) + g_1(x)d_2(d_1(y)) + d_2(d_1(x))g_2(y) + d_1(x)d_2(y) \\
&= d_1(x)d_2(y) + xd_1(d_2(y)) + d_1(d_2(x))y + d_2(x)d_1(y) \\
&\quad + d_2(x)d_1(y) + xd_2(d_1(y)) + d_2(d_1(x))y + d_1(x)d_2(y) \\
&= 2(d_1(x)d_2(y) + d_2(x)d_1(y)).
\end{aligned}$$

Since S is 2-torsion free, we have $d_1(x)d_2(y) + d_2(x)d_1(y)$ for all $x, y \in S$. Hence d_1 and d_2 are orthogonal by Theorem 3.10. □

Theorem 3.13. *Let S be a 2-torsion free semiprime semiring and multiplicatively commutative and let d_1 and d_2 be reverse semiderivations of S into S associated with surjective functions $g_1 : S \to S$ and $g_2 : S \to S$, respectively. Then d_1 and d_2 are orthogonal if and only if d_1d_2 or d_2d_1 is a reverse semiderivation associated with the function $g_1g_2 : S \to S$.*

Proof. Suppose that d_1 and d_2 are orthogonal. Then by theorem 3.10, $d_1(x)d_2(y)+ d_2(x)d_1(y) = 0$ for all $x, y \in S$. Also, $d_1d_2 = 0$ by Theorem 3.11. Hence we have

$$\begin{aligned}
0 &= d_1d_2(xy) \\
&= d_1(d_2(y)g_2(x) + yd_1(x)) \\
&= d_1(g_2(x)d_2(y) + d_1(x)y) \\
&= d_1(d_2(y))g_1(g_2(x)) + d_2(y)d_1(g_2(x)) + d_1(y)g_1(d_2(x)) + yd_1(d_2(x)) \\
&= d_1(d_2(y))g_1(g_2(x)) + d_2(y)d_1(x) + d_1(y)d_2(x) + yd_1(d_2(x)).
\end{aligned}$$

Also, we have $g_1g_2(d_1d_2(x)) = d_1d_2(x) = d_1d_2(g_1g_2(x))$ for all $x \in S$. Hence d_1d_2 is a reverse semiderivation associated with the function $g_1g_2 : S \to S$. Similarly, d_2d_1 is a reverse semiderivation associated with the function $g_1g_2 : S \to S$. Conversely, assume that d_1d_2 is a reverse semiderivation associated with the function $g_1g_2 : S \to S$. Then we have for all $x, y \in S$,

$$d_1d_2(xy) = d_1d_2(y)g_1g_2(x) + yd_1d_2(x) \tag{1}$$

Also, we get

$$\begin{aligned}
d_1d_2(xy) &= d_1(d_2(y)g_2(x) + yd_2(x)) \\
&= d_1(g_2(x)d_2(y) + d_2(x)y) \\
&= d_1(d_2(y))g_1(g_2(x)) + d_2(y)d_1(g_2(x)) + d_1(y)g_1(d_2(x)) + yd_1(d_2(x)) \\
&= d_1(d_2(y))g_1(g_2(x)) + d_2(y)d_1(x) + d_1(y)d_2(x) + yd_1(d_2(x)). \qquad (2)
\end{aligned}$$

Comparing (1) and (2), we obtain $d_1(x)d_2(y) = d_2(x)d_1(y) = 0$ for all $x, y \in S$. Hence d_1 and d_2 are orthogonal. Similarly, we can prove if d_2d_1 is a reverse semiderivation associated with the function $g_1g_2 : S \to S$, then d_1 and d_2 are orthogonal. □

Theorem 3.14. *Let S be a 2-torsion free semiprime semiring and multiplicatively commutative and let d_1 and d_2 be reverse semiderivations of S into S associated with surjective functions $g_1 : S \to S$ and $g_2 : S \to S$, respectively. Then d_1 and d_2 are orthogonal if and only if there exists $a, b \in S$ such that $d_1d_2(x) = ax + xb$ for all $x \in S$.*

Proof. Suppose that d_1 and d_2 are orthogonal. Then by Theorem 3.11, $d_1d_2 = 0$. That is, $d_1d_2(x) = 0$. Choosing $a = b = 0$, we have $d_1d_2(x) = ax + xb$ for all $x \in S$. Conversely, assume that $d_1d_2(x) = ax + xb$ for all $x \in S$. Replacing x by xy, we get for all $x, y \in S$,

$$\begin{aligned}
&d_1d_2(xy) = axy + xyb\\
&d_1(d_2(y)g_2(x) + yd_2(x)) = axy + xyb\\
&d_1(g_2(x))g_1(d_2(y)) + g_2(x)d_1(d_2(y)) + d_1(d_2(x))g_1(y) + d_2(x)d_1(y) = axy + xyb\\
&d_1(x)d_2(y) + xd_1(d_2(y)) + d_1(d_2(x))y + d_2(x)d_1(y) = axy + xyb\\
&d_1(x)d_2(y) + x(ay + yb) + (ax + xb)y + d_2(x)d_1(y) = axy + xyb\\
&d_1(x)d_2(y) + d_2(x)d_1(y) + xay + xyb + axy + xby = axy + xyb\\
&xby + xay + d_2(x)d_1(y) + d_1(x)d_2(y) = 0. \qquad (1)
\end{aligned}$$

Replacing y by yx, we have for all $x, y \in S$,

$$\begin{aligned}
&xbyx + xayx + d_2(x)d_1(yx) + d_1(x)(d_2(yx)) = 0\\
&xbyx + xayx + d_2(x)(d_1(x)g_1(y) + xd_1(y)) + d_1(x)(d_2(x)g_2(y) + xd_2(y)) = 0\\
&xbyx + xayx + d_2(x)d_1(x)g_1(y) + d_2(x)xd_1(y) + d_1(x)d_2(x)g_2(y) + d_1(x)xd_2(y) = 0\\
&xbyx + xayx + d_2(x)d_1(x)y + d_2(x)xd_1(y) + d_1(x)d_2(x)y + d_1(x)xd_2(y) = 0\\
&xbyx + xayx + d_2(x)yd_1(x) + d_2(x)d_1(y)x + d_1(x)yd_2(x) + d_1(x)d_2(y)x = 0\\
&(xby + xay + d_2(x)d_1(y) + d_1(x)d_2(y))x + 2d_1(x)yd_2(x) = 0
\end{aligned}$$

Hence we obtain $2d_1(x)yd_2(x) = 0$ by (1), for all $x, y \in S$. Then by Lemma 3.9, $d_1(x)sd_2(y) = 0$ for all $x, y \in S$. That is, $d_1(x)sd_2(y) = 0 = d_2(y)sd_1(x)$ for all $x, y \in S$. Hence d_1 and d_2 are orthogonal. □

Theorem 3.15. *Let S be a 2-torsion free semiprime semiring and multiplicatively commutative and let d_1 and d_2 be reverse semiderivations of S into S associated with surjective functions $g_1 : S \to S$ and $g_2. : S \to S$, respectively. such that d_1d_2 is also a semiderivation associated with the function $g_1g_2 : S \to S$. Then d_1 is zero or d_2 is zero.*

Proof. Since d_1d_2 is a semiderivation, we have for all $x, y \in S$,

$$d_1d_2(xy) = d_1d_2(y)g_1g_2(x) + yd_1d_2(x). \qquad (1)$$

Also, we get for all $x, y \in S$,

$$\begin{aligned}
d_1d_2(xy) &= d_1(d_2(y)g_2(x) + yd_2(x))\\
&= d_1(g_2(x)d_2(y) + d_2(x)y)\\
&= d_1(d_2(x))g_1(g_2(y)) + d_2(x)d_1(g_2(x)) + d_1(y)g_1(d_2(x)) + yd_1(x)d_2(x)\\
&= d_1d_2(x)g_1g_2(y) + d_2(x)d_1(x) + d_1(y)d_2(x) + yd_1(x)d_2(x) \qquad (2)
\end{aligned}$$

Comparing (1) and (2), we have $x, y \in S$,

$$d_1(x)d_2(y) + d_2(x)d_1(y) = 0 \qquad (3)$$

Replacing x by $xd_1(z)$ in the above equation (3), we obtain

$$0 = d_1(xd_1(z))d_2(y) + d_2(xd_1(z))d_1(y) = d_1(d_1(z)x)d_2(y) + d_2(d_1(z)x)d_1(y)$$

for all $x, y \in S$. Hence we have by (3) and since g is surjective.

$$\begin{aligned} 0 &= d_1(d_1(z)x)d_2(y) + d_2(d_1(z)x)d_1(y) \\ &= d_1(x)g_1(d_1(z))d_2(y) + xd_1(d_1(z))d_2(y) + d_2(x)g_2(d_1(z))d_1(y) + xd_2(d_1(z))d_1(y) \\ &= d_1(x)g_1(d_1(z))d_2(y) + d_2(x)g_2(d_1(z))d_1(y) \\ &= d_1(x)d_1(z)d_2(y) + d_2(x)d_1(z)d_1(y). \qquad (4) \end{aligned}$$

Replacing x by z in the above equation (3), we have for all $y, z \in S$,

$$d_1(z)d_2(y) + d_2(z)d_1(y) = 0.$$

This implies $d_1(z)d_2(y) = -d_2(z)d_1(y)$ for all $y, z \in S$. Hence (4) becomes $-d_1(x)d_2(z)d_1(y)+d_2(x)d_1(z)d_1(y) = 0$ for all $x, y, z \in S$. Hence we have $(d_2(x)d_1(z)-d_1(x)d_2(z))d_1(y) = 0$ for all x, y, z. Now, by lemma 3.6, we have $d_2(x)d_1(z) - d_1(x)d_2(z) = 0$ or $d_1(y) = 0$. Let

$$d_2(x)d_1(z) - d_1(x)d_2(z) = 0. \qquad (5)$$

In the above equation (3), replacing y by z, we have

$$d_1(x)d_2(z) + d_2(x)d_1(z) = 0 \qquad (6)$$

for all $x, z \in S$. Adding (5) and (6), $2d_2(x)d_1(z) = 0$ for all $x, z \in S$. Since S is 2-torsion free, we have $d_2(x)d_2(z) = 0$ for all $x, z \in S$. Again, by Lemma 3.6, $d_2(x) = 0$ or $d_1(z) = 0$ for all $x, z \in S$. That is, $d_1 = 0$ or $d_2 = 0$. □

Corollary 3.16. *Let S be a 2-torsion free semiprime semiring and multiplicatively commutative and let d_1 and d_2 be reverse semiderivations of S into S associated with surjective functions $g_1 : S \to S$ and $g_2 : S \to S$, respectively. such that d_1^2 is also a semiderivation associated with the function $g_1^2 : S \to S$. Then d_1 is zero.*

Proof. In Theorem 3.15, replacing d_2 by d_1, we obtain the result $d_1 = 0$. □

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