Gulf Journal of Mathematics
Vol 14, Issue 1 (2022) 25-32

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v14i1.883

THE BIGRADED HILBERT FUNCTION OF UNIONS OF LINES IN THE 3-DIMENSIONAL FLAG VARIETY

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ABSTRACT. Let $\mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ be the 3-dimensional flag. Let $\pi_1 : \mathbb{F} \to \mathbb{P}^2$ and $\pi_2 : \mathbb{F} \to \mathbb{P}^{2\vee}$ be the projections. For all $u, v \in \mathbb{N} \setminus \{(0,0)\}$ let $M(u,v)$ denote the set of all curves $\pi_1^{-1}(F) \cup \pi_2^{-1}(E)$ such that $\pi_1^{-1}(F) \cap \pi_2^{-1}(E) = \emptyset$, $\#F = v$ and $\#E = u$. Any $A \in M(u,v)$ has $u+v$ connected components, all of them smooth and rational and embedded as lines by the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$. In this paper we study the bigraded Hilbert function $H^0(\mathcal{I}_A(a,b))$, $(a,b) \in \mathbb{N}^2$, for a general $A \in M(u,v)$. We also give geometric properties of $\mathcal{I}_A(a,b)$ (spannedness and a uniqueness result for non-general $A \in M(u,v)$).

1. INTRODUCTION

Let $\mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ be the 3-dimensional flag variety, i.e. the set of all pairs (p, ℓ), where $p \in \mathbb{P}^2$, $\ell \subset \mathbb{P}^2$ is a line, and $p \in \ell$. Since we do not need to consider the twistor structure ([3]) we identify $\mathbb{P}^{2\vee}$ with $\mathbb{P}^2$ and see $\mathbb{F}$ as a submanifold of $\mathbb{P}^2 \times \mathbb{P}^2$ (up to an element of $(\mathrm{Aut}(\mathbb{P}^2))^2$ we may identify $\mathbb{F}$ with the diagonal of $\mathbb{P}^2 \times \mathbb{P}^2$).

Let $\Pi_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$, $i = 1, 2$, denote the projection of $\mathbb{P}^2 \times \mathbb{P}^2$ onto its i-th factor. Set $\pi_i := \Pi_{i|\mathbb{F}}$. For any $(x,y) \in \mathbb{Z}^2$ set $\mathcal{O}_{\mathbb{F}}(x,y) := \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(x)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^2}(y))$. We have $\mathrm{Pic}(\mathbb{F}) = \cup_{(x,y)\in\mathbb{Z}^2} \mathcal{O}_{\mathbb{F}}(x,y)$ and the cohomology groups of all $\mathcal{O}_{\mathbb{F}}(x,y)$ are well-known. In particular $h^1(\mathcal{O}_{\mathbb{F}}(a,b)) = 0$ and $h^0(\mathcal{O}_{\mathbb{F}}(a,b)) = \binom{a+2}{2}\binom{b+2}{2} - \binom{a+1}{2}\binom{b+1}{2}$ for all $(a,b) \in \mathbb{N}^2$ (section 2).

In this paper we study the bigraded Hilbert function of the following class of curves $A \subset \mathbb{F}$ which generalise the case of general unions of lines in a projective space done by R. Hartshorne and A. Hirschowitz ([7]). Fix $(u,v) \in \mathbb{N}^2 \setminus \{(0,0)\}$. For all finite sets $E, F \subset \mathbb{P}^2$ such that $\#F = v$, $\#E = u$ the curve $\pi_1^{-1}(F) \cup \pi_2^{-1}(E)$ has $u+v$ irreducible components, all of them isomorphic to a smooth rational curve. Let $M(u,v)$ be the set of all such curves $\pi_1^{-1}(F) \cup \pi_2^{-1}(E)$ for some E, F with the additional assumption that $\pi_1^{-1}(F) \cap \pi_2^{-1}(E) = \emptyset$, i.e., that each irreducible component is a connected component. The set $M(u,v)$ is an irreducible quasi-projective variety of dimension $2u+2v$. For any $(a,b) \in \mathbb{N}^2$ and any $A \in M(u,v)$ we have $h^1(\mathcal{O}_A(a,b)) = 0$ and $h^1(\mathcal{O}_A(a,b)) = u(b+1) + v(b+1)$. Thus $h^0(\mathcal{I}_A(a,b)) \ge \max\{\binom{a+2}{2}\binom{b+2}{2} - \binom{a+1}{2}\binom{b+1}{2} - u(b+1) - v(b+$

Date: Received: Jun 3, 2022; Accepted: Oct 17, 2022.
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2010 *Mathematics Subject Classification.* Primary 14JN05; Secondary 14M15; 14H50.
Key words and phrases. flag variety; flag 3-fold; lines; bigraded Hilbert function.

$1),0\}$, $h^1(\mathcal{I}_A(a,b)) \ge \max\{0, u(b+1)+v(b+1)-\binom{a+2}{2}\binom{b+2}{2}+\binom{a+1}{2}\binom{b+1}{2}\}$ and $h^0(\mathcal{I}_A(a,b))-h^1(\mathcal{I}_A(a,b)) = \binom{a+2}{2}\binom{b+2}{2}-\binom{a+1}{2}\binom{b+1}{2}-u(b+1)-v(b+1)$. Since $h^1(\mathcal{O}_{\mathbb{F}}(a,b)) = h^1(\mathcal{O}_A(a,b)) = 0$ for all $(a,b,u,v)\in\mathbb{N}^4$ and all $A\in M(u,v)$, we have $h^0(\mathcal{I}_A(a,b))=0$ (resp. $h^1(\mathcal{I}_A(a,b))=0$) if an only if the restriction map $H^0(\mathcal{O}_{\mathbb{F}}(a,b))\to H^0(\mathcal{O}A(a,b))$ is injective (resp. surjective). Easy examples show that for many (a,b,u,v) the integer $h^0(\mathcal{I}_A(a,b))$ is not the same for all $A\in M(u,v)$ (see Proposition 3.7 for a general result).

Since $M(u,v)$ is an irreducible variety, the semicontinuity theorem for cohomology says that $h^0(\mathcal{I}_A(a,b)$ is constant in a non-empty open subset of $M(u,v)$ and outside it it is bigger. Since $h^1(\mathcal{I}_A(a,b))=0$ for all $A\in M(u,v)$ if $a\ge v$ and $b\ge u$ (Remark 3.3) there is a non-empty open subset U of $M(u,v)$ such that $h^0(\mathcal{I}_A(a,b))$ has its minimal value for all $(a,b)\in\mathbb{N}^2$ and all $A\in U$. These values $h^0(\mathcal{I}_A(a,b))$ are called the generic values. We say that $A\in M(u,v)$ has *maximal rank in bidegree* (a,b) if either $h^0(\mathcal{I}_A(a,b))=0$ or $h^1(\mathcal{I}_A(a,b))=0$ and that A has *maximal rank* if it has maximal rank in all bidegrees. For many (u,v) a general $A\in M(u,v)$ has not maximal rank, contrary to the case of general lines in projective spaces ([7]). We describe it for all (a,b,u,v) and in particular we describe if a general $A\in M(u,v)$ has maximal rank: only if $u\le 1$ and $v\le 1$ (Proposition 1.2). We prove the following results.

Theorem 1.1. *Fix $(a,b,u,v)\in\mathbb{N}^4$ such that $(u,v)\ne(0,0)$ and take a general $A\in M(u,v)$.*

(i) *If either $v\le\binom{a+1}{2}$ and $u\le\binom{b+2}{2}$ or $v\le\binom{a+2}{2}$ and $u\le\binom{b+1}{2}$, then $h^1(\mathcal{I}_A(a,b))=0$.*
(ii) *If $\binom{a+1}{2}\le v\le\binom{a+2}{2}$ and $\binom{b+1}{2}\le u\le\binom{b+2}{2}$, then $h^0(\mathcal{I}_A(a,b))=(\binom{b+2}{2}-u)(\binom{a+2}{2}-v)$.*
(iii) *If $u>\binom{b+2}{2}$ (resp. $v>\binom{a+2}{2}$), then $h^1(\mathcal{I}_A(a,b))\ge u-\binom{b+2}{2}$ (resp. $h^1(\mathcal{I}_A(a,b))\ge v-\binom{a+2}{2}$).*

Proposition 1.2. *Fix $(u,v)\in\mathbb{N}^2\setminus\{(0,0)\}$ and take a general $A\in M(u,v)$. A has maximal rank with respect to all $\mathcal{O}_{\mathbb{F}}(a,b)$, $(a,b)\in\mathbb{N}^2$, if and only if $u\le 1$ and $v\le 1$.*

In section 4 we describe some geometric properties of the bigraded ideals $H^0(\mathcal{I}_A(a,b))$, $(a,b)\in\mathbb{N}^2$. We give a condition for the integers a, b, u and v which give $h^1(\mathcal{I}_A(a,b))=0$ and $\mathcal{I}_A(a,b)$ is globally generated (Theorem 4.3).

In the last section we consider multiple structures on a general $A\in M(u,v)$. We give a cohomological vanishing result in terms of the corresponding vanishing results for general fat points on $\mathbb{P}^2$ for which several vanishing results are kinown.

There are several good reasons to study the multidegrees of closed subvarieties contained in a multiprojective space $Y := \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$, i.e. their Hilbert function with respect to all line bundles $\mathcal{O}_Y(a_1,\dots,a_k)$, $(a_1,\dots,a_k)\in\mathbb{N}^k$. ([6, 10]).

Many results on curves in a multiprojective space use induction on the integer k or on the integer $n_1+\cdots+n_k$. Thus the case $k=2$ and $n_1=n_2$ must be done. Now consider the case of curves C of multidegree $(b_1,\dots,b_k)$ in a multiprojective space $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. All cases with $n_1 + \cdots + n_k = 2$ are very special and in this case all curves are described and their multigraded ring easily understood. The case $\mathbb{P}^2 \times \mathbb{P}^2$ need to be proved and it seems that often using $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^2$, i.e. the divisors of bidegree $(0,1)$ and $(1,0)$ of $\mathbb{P}^2 \times \mathbb{P}^2$, are not enough to get the case of $\mathbb{P}^2 \times \mathbb{P}^2$. The flag $\mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth divisor of bidegree $(1,1)$ of $\mathbb{P}^2 \times \mathbb{P}^2$, the only one up to an element of $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$.

Our main motivation for study curves in $\mathbb{F}$ cam from twistor theory and a theorem of N. J. Hitchin in [8] (see [4, 9] for motivations of Hitchin's work). Each oriented 4-dimensional riemannian manifold has canonically associated a 6-dimensional manifold Z, together with an almost complex structure J on Z with a distinguished family of curves, called twistor fibers. In [8] he proved that there are exactly 2 compact complex Kähler 3-folds of the form (Z, J): $\mathbb{CP}^3$ and $\mathbb{F}$. See [1, 11] for the study of the twistor geometry of $\mathbb{CP}^3$. See [2, 3] for the twistor geometry of $\mathbb{F}$.

2. Preliminaries

For all $(a,b) \in \mathbb{N}^2$ we have $h^1(\mathcal{O}_{\mathbb{F}}(a,b)) = 0$ and

$$h^0(\mathcal{O}_{\mathbb{F}}(a,b)) = \binom{a+2}{2}\binom{b+2}{2} - \binom{a+1}{2}\binom{b+1}{2} = (a+1)(b+1)(a+b+2)/2. \tag{2.1}$$

For all $(a,b,u,v) \in \mathbb{N}^4$ such that $(u,v) \neq (0,0)$ and all $A \in M(u,v)$ we have

$$h^0(\mathcal{O}_A(a,b)) = v(b+1) + u(a+1). \tag{2.2}$$

By (2.2) for all $A \in M(\binom{b+1}{2}, \binom{a+2}{2})$ we have $h^0(\mathcal{O}_A(a,b)) = \binom{a+2}{2}(b+1) + \binom{b+1}{2}(a+1) = (a+1)(b+1)(a+b+2)/2$ and hence $h^0(\mathcal{O}_A(a,b)) = h^0(\mathcal{O}_{\mathbb{F}}(a,b))$.

Let M be a projective variety, $D \subset M$ an effective Cartier divisor of M and $\mathcal{L}$ a line bundle on M. For any closed subset T of M let $\mathrm{Res}_D(T)$ denote the closure in W of the locally closed algebraic set $T \setminus T \cap D$, i.e., let $\mathrm{Res}_D(T)$ be the union of all irreducible components of T not contained in D. The following sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(T)} \otimes \mathcal{L}(-D) \to \mathcal{I}_T \otimes \mathcal{L} \to \mathcal{I}_{T\cap D,D} \otimes \mathcal{L}_{|D} \to 0 \tag{2.3}$$

is exact and it will be called the residual exact sequence of D without mentioning $\mathcal{L}$ and T.

Remark 2.1. We recall the description of all elements of $|\mathcal{O}_{\mathbb{F}}(0,1)|$ and of all elements of $|\mathcal{O}_{\mathbb{F}}(1,0)|$ ([3, §4]). Each element of $|\mathcal{O}_{\mathbb{F}}(0,1)|$ (resp. $|\mathcal{O}_{\mathbb{F}}(1,0)|$) is equal to $\pi_2^{-1}(L)$ (resp. $\pi_1^{-1}(L)$) for some line $L \subset \mathbb{P}^2$. Each $D \in |\mathcal{O}_{\mathbb{F}}(0,1)|$ and each $W \in |\mathcal{O}_{\mathbb{F}}(1,0)|$ is isomorphic to the Hirzebruch surface F_1. We take as a basis of $\mathrm{Pic}(F_1)$ a fiber of the ruling $|f|$ of F_1 and the only integral curve $h \subset F_1$ such that $h^2 = -1$. We have $h \cdot f = 1$, $h^2 = -1$ and $f^2 = 0$. Up to this identification we have $\mathcal{O}_D(1,0) \cong \mathcal{O}_{F_1}(ah + (a+b)f)$. The divisor D contains a unique element of $M(1,0)$ (it is h) and all other elements of $M(1,0)$ intersect transversally D at one point (a general point of D for a general element of $M(1,0)$). All elements of $|f|$ are curves of bidegree $(0,1)$, while $T \cap D = \emptyset$ for all other curves T of bidegree $(0,1)$. We have $h^1(\mathcal{O}_{F_1}(xh + yf)) = 0$ for all $x \geq 0$ and all $y \geq x - 1$. If $x \geq 0$ and

$y \ge x-1$ we have $h^0(\mathcal{O}_{F_1}(xh+yf)) = \sum_{i=0}^{x}(y+1-i) = (x+1)(2y+2-x)/2$. Thus $h^0(\mathcal{O}_{F_1}(xh+xf)) = \binom{x+2}{2}$. The exact sequences

$$0 \to \mathcal{O}_{\mathbb{F}}(a-1,b) \to \mathcal{O}_{\mathbb{F}}(a,b) \to \mathcal{O}_Y(a,b) \to 0 \tag{2.4}$$

$$0 \to \mathcal{O}_{\mathbb{F}}(a,b-1) \to \mathcal{O}_{\mathbb{F}}(a,b) \to \mathcal{O}_W(a,b) \to 0 \tag{2.5}$$

give that $\mathcal{O}_Y(1,0)$, $\mathcal{O}_Y(0,1)$, $\mathcal{O}_W(1,0)$ and $\mathcal{O}_W(0,1)$ are globally generated, that $h^0(\mathcal{O}_Y(1,0) = h^0(\mathcal{O}_W(0,1)) = 2$ and that (up to the identification with F_1) they induce the ruling u, i.e. $|\mathcal{O}_Y(0,1)| = |\mathcal{O}_W(0,1)| = |\mathcal{O}_{F_1}(f)|$. We also get that $h^0(\mathcal{O}_Y(0,1) = h^0(\mathcal{O}_W(1,0)) = 3$ and (up to the identification of Y and F_1) they induce the morphism $u_1 : Y \to \mathbb{P}^2$ and $u_2 : W \to PP^2$ associated to $|\mathcal{O}_{F_1}(h+f)|$ which contracts h. Take a union $T \subset Y$ of c fibers of the rulings. We have $\mathcal{O}_Y(a,b)(-T) \cong \mathcal{O}_Y(ah+(a+b-c)f)$ and hence $h^1(\mathcal{O}_Y(a,b)(-T)$ if an only if $c \le b+1$. We have $h^1(\mathcal{O}_{\mathbb{F}}(a,b)(-Y)) = h^1(\mathcal{O}_{\mathbb{F}}(a-1,b))$ and hence $h^1(\mathcal{O}_{\mathbb{F}}(a,b)(-Y)) = 0$ if $b \ge 0$ and $a > 0$.

3. Proof of Theorem 1.1 and related results

Lemma 3.1. *Fix $(x,y) \in \mathbb{N}^2$ and finite subsets E, F of $\mathbb{P}^2$ such that $h^0(\mathbb{P}^2, \mathcal{I}_E(y)) = 0$ and $h^0(\mathbb{P}^2, \mathcal{I}_F(x)) = 0$. Set $Z := \Pi_1^{-1}(E) \cup \Pi_2^{-1}(F)$. Then*

$$h^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{I}_Z(x,y)) = h^1(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{I}_Z(x,y)) = 0.$$

Proof. Note that $\mathcal{I}_Z(x,y) \cong \Pi_1^{-1}(\mathcal{I}_E)(y) \otimes \Pi_2^{-1}(\mathcal{I}_F(x))$ and apply the Künneth formula. □

Proposition 3.2. *Fix $(a,b,u,v) \in \mathbb{N}^4$ such that $\binom{a+1}{2} \le v \le \binom{a+2}{2}$ and $\binom{b+1}{2} \le u \le \binom{b+2}{2}$. Then $h^0(\mathcal{I}_A(a,b)) = (\binom{b+2}{2}-u)(\binom{a+2}{2}-v)$ for a general $A \in M(u,v)$.*

Proof. Write $A = U \cup V$ with U general in $M(u,0)$, V general in $M(0,v)$ and $U \cap V = \emptyset$. Set $E := \pi_1(V)$, $F := \pi_2(U)$, and $Z := \Pi_1^{-1}(E) \cup \Pi_2^{-1}(F)$. Note that $A = Z \cap \mathbb{F}$. Consider the exact sequence

$$0 \to \mathcal{I}_{Z,\mathbb{P}^2\times\mathbb{P}^2}(a-1,b-1) \to \mathcal{I}_{Z,\mathbb{P}^2\times\mathbb{P}^2}(a,b) \to \mathcal{I}_{A,\mathbb{F}}(a,b) \to 0 \tag{3.1}$$

The Künneth formula and the generality of E and F give $h^0(\mathbb{P}^2\times\mathbb{P}^2, \mathcal{I}_{Z,\mathbb{P}^2\times\mathbb{P}^2}(a,b)) = (\binom{b+2}{2}-u)(\binom{a+2}{2}-v)$. Lemma 3.1 gives $h^0(\mathbb{P}^2\times\mathbb{P}^2, \mathcal{I}_Z(x,y)) = h^1(\mathbb{P}^2\times\mathbb{P}^2, \mathcal{I}_Z(x,y)) = 0$. Use the cohomology exact sequence of (3.1). □

Remark 3.3. Fix $(a,b,u,v) \in \mathbb{N}^4$ such that $(u,v) \ne (0,0)$, $a \ge v$ and $b \ge u$. As in Lemma 3.1 we see that $h^1(\mathcal{I}_A(a,b)) = 0$ for all $A \in M(u,v)$.

Theorem 3.4. *Fix $(a,b) \in \mathbb{N}^2$. Then $h^i(\mathcal{I}_A(a,b)) = 0$, $i = 0,1$, for a general $A \in M(\binom{b+1}{2}, \binom{a+2}{2})$.*

Proof. By the semicontinuity theorem for cohomology it is sufficient to prove the existence of one $A \in M(\binom{b+1}{2}, \binom{a+2}{2})$ such that $h^0(\mathcal{I}_A(a,b)) = 0$. Write $A = U \cup V$ with $U \in M(\binom{a+2}{2}, 0)$ and $V \in M(0, \binom{b+1}{2})$. Fix $D \in |\mathcal{O}_{\mathbb{F}}(0,1)|$.

(a) Assume $b = 0$. Thus $U = \emptyset$. For a general $A = U \cup V$ the scheme $\pi_1(A)$ is a general union of $\binom{a+2}{2}$ general points. Thus $h^i(\mathcal{I}_A(a,0)) = h^i(\mathbb{P}^2, \mathcal{I}_{\pi_1(A)}(a)) = 0$, $i = 0,1$.

(b) Assume $b > 0$. By step (a) and induction on the integer b we may assume that the proposition is true for the triple $(a, b-1, u)$. Take $A' = U' \cup V \in M(\binom{b+1}{2}, \binom{a+2}{2})$. Since A' is general, $A' \cap D$ is a general union of $\binom{a+2}{2}$ points of D. Up to the identification of D and F_1 we have $\mathcal{O}_D(a,b) \cong \mathcal{O}_{F_1}(ah + (a+b)f)$ and the elements of $|f|$ are elements of $M(0,1)$. Take a general union V of b elements of $|f|$ and set $V := V' \cup V''$ and $A = U \cup V$. We have $h^i(D, \mathcal{I}_{D\cap A,D}(a,b)) = h^i(F_1, \mathcal{I}_{U\cap D,D}(ah + af)) = 0$, $i = 0,1$ (Remark 2.1). Use the residual exact sequence of D. □

Remark 3.5. Exchanging π_1 and π_2 in the statement of Theorem 3.4 we get $h^i(\mathcal{I}_B(a,b)) = 0$, $i = 0,1$, for a general $B \in M(\binom{b+2}{2}, \binom{a+1}{2})$. Thus $h^1(\mathcal{I}_E(a,b)) = 0$ for a general $E \in M(u,v)$ if either $u \le \binom{b+1}{2}$ and $v \le \binom{a+2}{2}$ or $u \le \binom{b+2}{2}$ and $v \le \binom{a+1}{2}$ and $h^0(\mathcal{I}_F(a,b)) = 0$ for a general $F \in M(u,v)$ if either $u \ge \binom{b+1}{2}$ and $v \ge \binom{a+2}{2}$ or $u \ge \binom{b+2}{2}$ and $v \ge \binom{a+1}{2}$

Theorem 3.6. *Fix $(a,b,u,v) \in \mathbb{N}^4$ such that $(a,b) \ne (0,0)$ and $(u,v) \ne (0,0)$. Take any $A \in M(u,v)$. If $u > \binom{b+2}{2}$ (resp. $v > \binom{a+2}{2}$), then $h^1(\mathcal{I}_A(a,b)) \ge u - \binom{b+2}{2}$ (resp. $h^1(\mathcal{I}_A(a,b)) \ge v - \binom{a+2}{2}$).*

Proof. Write $A = U \cup V$ with $U \in M(u,0)$, $V \in M(0,v)$ and $U \cap V = \emptyset$. Since $U \cap V = \emptyset$, $h^1(\mathcal{I}_A(a,b)) \ge \max\{h^1(\mathcal{I}_U(a,b)), h^1(\mathcal{I}_V(a,b)\}$. Call $\rho_{a,b,V} : H^0(\mathcal{O}_{\mathbb{F}}(a,b)) \to H^0(\mathcal{O}_V(a,b))$ and $\rho_{a,b,U} : H^0(\mathcal{O}_{\mathbb{F}}(a,b)) \to H^0(\mathcal{O}_U(a,b))$ the restriction maps. To conclude the proof note that $\mathrm{Im}(\rho_{a,b,V}) = \mathrm{Im}(\rho_{a,0,V})$ and $\mathrm{Im}(\rho_{a,b,U}) = \mathrm{Im}(\rho_{0,b,U})$. □

Proof of Theorem 1.1: Part (i) is Theorem 2.1 and Remark 3.5 . Part (ii) is Proposition 3.2. Part (iii) is Theorem 2.1. □

Proposition 3.7. *Fix $(u,v) \in \mathbb{N}^2 \setminus \{(0,0)\}$. We have $h^0(\mathcal{I}_A(a,b)) = h^0(\mathcal{I}_B(a,b))$ for all $A, B \in M(u,v)$ and all $(a,b) \in \mathbb{N}^2$ if and only if $u \le 2$ and $v \le 2$.*

Proof. Since the natural action of $\mathrm{Aut}(\mathbb{P}^2)$ on $\mathbb{P}^2$ is 2-transitive, $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^2)$ is transitive on the set $M(0,x) \times M(y,0)$ if $1 \le x \le 2$ and $1 \le y \le 2$. Thus if $u \le 2$ and $v \le 2$ all elements of $M(u,v)$ have the same bigraded Hilbert function.

Assume $v \ge 3$. Take $V, W \in M(0,v)$ such that V is general (i.e. $\pi_1(V)$ is general in $\mathbb{P}^2$) and $\pi_1(W)$ is formed by v collinear points. Take $E, F \in M(u,0)$ such that E is general and $\pi_1(F)$ is formed by collinear points (the assumption on F is always satisfied if $u \in \{0,1,2\}$). Set $A := V \cup E$ and $B := W \cup F$. Assume for the moment $u \ge 2$. We get $h^0(\mathcal{I}_A(1,1)) = 0$ and $h^0(\mathcal{I}_B(1,1)) \ne 0$. Now assume $u = 0$. We get $h^0(\mathcal{I}_A(0,1)) = 0$ and $h^0(\mathcal{I}_B(0,1)) = 1$. Now assume $u = 1$. We see that $h^0(\mathcal{I}_B(1,1)) = 2$. Since $h^0(\mathcal{I}_V(1,0) = 0$, a residual exact sequence shows that $h^0(\mathcal{I}_A(1,1)) \le 1$.

In the same way we conclude if $v \le 2$ and $u \ge 3$. □

Proof of Proposition 1.2: Write $A = U \cup V$ with U general in $M(u,0)$, V general in $M(0,v)$ and $U \cap V = \emptyset$. First assume $v \ge 2$. Let a be the maximal non-negative integer such that $\binom{a+2}{a} < v$. For $b \gg 0$ we have $(a+1)(b+1)(a+b+2)/2 > u(b+1) + v(a+1)$. Thus $h^0(\mathcal{I}_A(a,b)) > 0$. Part (iii) of Theorem 1.1 gives

$h^1(\mathcal{I}_A(a,b)) > 0$. In the same way we analyse the case $u \ge 2$. If $u \le 1$ and $v \le 1$, then $h^1(\mathcal{I}_A(a,b)) = 0$ for all $(a,b) \in \mathbb{N}^2 \setminus \{(0,0)\}$. Since $u+v > 0$, $h^0(\mathcal{I}_A) = 0$. □

4. Geometric properties

The case $(a,b) = (c,d)$ of the following result gives a sufficient condition for the uniqueness of an integral $X \in |\mathcal{O}_{\mathbb{F}}(a,b)|$ containing at least one element of $M(u,v)$. Obviously the assumption that X is integral is essential.

Proposition 4.1. *Fix $(a,b,c,d,u,v) \in \mathbb{N}^6$ such that $(u,v) \ne (0,0)$, $(a,b) \ne (0,0)$, $c > 0$, $d > 0$ and $(a,b) \ne (c,d)$. Take $A \in M(u,v)$. Assume the existence of an integral $X \in |\mathcal{I}_A(a,b)|$. Assume one of the following conditions:*

(1) *$ac + ad + bc < v$;*
(2) *$bd + ad + bc < u$;*
(3) *$ac + ad + bc = v$ and $bd + ad + bc = u$.*

Then each element of $|\mathcal{I}_A(c,d)|$ contains X and in particular $c \ge a$ and $d \ge b$.

Proof. Assume the existence of $X' \in |\mathcal{I}_A(c,d)|$ such that $X' \not\supseteq X$. We have $\mathcal{O}_{\mathbb{F}}(a,b) \cdot \mathcal{O}_{\mathbb{F}}(c,d) = ac\mathcal{O}_{\mathbb{F}}(1,0) \cdot \mathcal{O}_{\mathbb{F}}(1,0) + (ad+bc)\mathcal{O}_{\mathbb{F}}(1,0) \cdot \mathcal{O}_{\mathbb{F}}(0,1) + bd\mathcal{O}_{\mathbb{F}}(0,1) \cdot \mathcal{O}_{\mathbb{F}}(0,1)$. Note that $\mathcal{O}_{\mathbb{F}}(1,0) \cdot \mathcal{O}_{\mathbb{F}}(1,0)$ (resp. $\mathcal{O}_{\mathbb{F}}(1,0) \cdot \mathcal{O}_{\mathbb{F}}(0,1)$, resp. $\mathcal{O}_{\mathbb{F}}(0,1) \cdot \mathcal{O}_{\mathbb{F}}(0,1)$) is a one-dimensional cycle of bidegree $(0,1)$ (resp. bidegree $(1,1)$, resp. bidegree $(1,0)$). Thus the one-dimensional cycle $\mathcal{O}_{\mathbb{F}}(a,b) \cdot \mathcal{O}_{\mathbb{F}}(c,d)$ has bidegree $(bd + ad + bc, ad + ac + bd)$. Since $c > 0$ and $d > 0$, $\mathcal{O}_{\mathbb{F}}(c,d)$ is ample. Since $X' \not\supseteq X$, X is irreducible, A has bidegree (u,v) and A is not connected, either $ac+ad+bc > x$ and $bd+ad+bc \ge x$ or $ac+ad+bc \ge x$ and $bd+ad+bc > x$. □

Lemma 4.2. *Fix integers $d > 0$ and x such that $0 \le x \le \binom{d+2}{2} - 3$. Let $S \subset \mathbb{P}^2$ be a general subset of $\mathbb{P}^2$ such that $\#S = x$. Then $h^1(\mathcal{I}_S(d)) = 0$ and S is the scheme-theoretic base locus of $|\mathcal{I}_S(d)|$.*

Proof. The equality $h^1(\mathcal{I}_S(d)) = 0$ holds, because $x \le \binom{d+2}{2}$ and S is general. Call $\mathcal{B}$ the scheme-theoretic base locus of $|\mathcal{I}_S(d)|$. Take a general $S' \subset \mathbb{P}^2$ such that $\#S' = \binom{d+2}{2} - 2 - x$ (more precisely we take $S \cap S' = \emptyset$ and we assume that $S \cup S'$ is a general subset of $\mathbb{P}^2$ of cardinality $\binom{d+2}{2} - 2$. Thus $h^0(\mathcal{I}_{S' \cup S'}(d)) = 2$ and the pencil $|\mathcal{I}_{S \cup S'}(d)|$ is the general pencil of degree d plane curves. Call C, D be two distinct elements of $|\mathcal{I}_{S \cup S'}(d)|$. Thus the scheme $Z := C \cap D$ is a general complete intersection of 2 degree d plane curves. By Bertini's theorem Z is smooth. Thus Z is a union of d^2 distinct points. Obviously $S \subset Z$. Since $|\mathcal{I}_{S \cup S'}(d)| \subset |\mathcal{I}_S(d)|$, $\mathcal{B} \subseteq Z$. Since Z is formed buy distinct points, to prove the lemma it is sufficient to prove that for each $p \in Z \setminus S$ there is $H \in |\mathcal{I}_S(d)|$ such that $p \notin S$, i.e. it is sufficient to prove that $h^0(\mathcal{I}_{S \cup \{p\}}(d)) = h^0(\mathcal{I}_S(d)) - 1$. This is true, because by a monodromy argument we may assume that p is one of the points of S'. □

Theorem 4.3. *Fix $(a,b,u,v) \in \mathbb{N}^4$ such that $a > 0$ $b > 0$, $v \le \binom{a+2}{2} - 3$ and $u \le \binom{a+2}{2} - 3$. Let A be a general element of $M(u,v)$. Then A is the scheme-theoretic base locus of the linear system $|\mathcal{I}_A(a,b)|$.*

Proof. Write $A = U \cup V$ with V a general element of $M(0,v)$ and U a general element of $M(u,0)$. Thus $\pi_1(V)$ is a general subset of $\mathbb{P}^2$ with cardinality v and

$\pi_2(U)$ is a general subset of $\mathbb{P}^2$ with cardinality 1. Set $W_1 := H^0(\mathcal{I}_U(a,b)) \cong \pi_1^*(H^0(\mathbb{P}^2,\mathcal{I}_{\pi_1(V)}(a)))$ and $W_2 := H^0(\mathcal{I}_U(a,b)) \cong \pi_2^*(H^0(\mathbb{P}^2,\mathcal{I}_{\pi_2(U)}(b)))$The projective space $|\mathcal{I}_A(a,b)|$ contains the set W_1+W_2 of all unions $G\cup F$ with $G\in W_1$ and $F\in W_2$. Lemma 4.2 gives that V is the scheme-theoretic base locus of W_1 and that U is the scheme-theoretic base locus of W_2. A is the scheme-theoretic base locus of the linear system $|\mathcal{I}_A(a,b)|$. □

5. Multiple components

For any $p\in\mathbb{P}^2$ and any positive integer m let mp or $(mp,\mathbb{P}^2)$ denote the closed subscheme of $\mathbb{P}^2$ with $(\mathcal{I}_p)^m$ as its ideal sheaves. For all positive integers s and m_i, $1\le s$, and any s distinct points $p_1,\dots,p_s\in\mathbb{P}^2$ the scheme $m_1p_1\cup\cdots\cup m_sp_s$ is called a fat subscheme of $\mathbb{P}^2$ with multiplicities $m_1,\dots,m_s$. The Hilbert function of such general fat points (the so-called Segre-Harbourne-Hirschowitz conjecture and the many papers proving particular cases of it) is a vast topic. Let C be a smooth curve embedded in a smooth projective variety T such that $\dim T\ge 3$. For any positive integer m let (mC,T) denote the closed subscheme of T with $(\mathcal{I}_{C,T})^m$ as its ideal sheaf. We may take general unions of finitely many schemes $m_iC_i\subset T$ and consider their Hilbert function with respect to the line bundles on T. General unions of multiple lines in projective spaces are quite difficult to handle. However, if we take $T=\mathbb{F}$ and give $u+v$ prescribed multiplicities to the connected component of a general $A\in M(u,v)$, then we land in a far easier problem because each connected component C of A has trivial normal bundle and its m-fattening is the pull-back by π_i of the scheme $\pi_i(p)$ with $p=\pi_i(C)$ (here $i=1$ if $C\in M(0,1)$ and $i=2$ if $C\in M(1,0)$). Thus the proofs of Theorem 3.4 and Remark 3.5 give the following result.

Proposition 5.1. *Fix $(a,b,u,v)\in\mathbb{N}^4$ such that $(u,v)\ne(0,0)$ and $u+v$ positive integers $m_1,\dots,m_u,\mu_1,\dots,m_v$. Fix a general $A\in M(u,v)$ and write $A=L_1\cup\cdots\cup L_u\cup R_1\cup\cdots\cup R_v$ with $L_i\in M(1,0)$ and $R_i\in M(0,1)$. Set $Z:=m_1L_1\cup\cdots\cup m_uL_u\cup\mu_1R_1\cup\cdots\cup\mu_vR_v$. Take $u+v$ general points $p_1,\dots,p_u,q_1,\dots,q_v$ of $\mathbb{P}^2$ and set $Z':=\mu_1q_1\cup\cdots\cup\mu_vq_v$ and $Z'':=m_1p_1\cup\cdots\cup m_up_u$. If $h^1(\mathbb{P}^2,\mathcal{I}_{Z',\mathbb{P}^2}(b))=0$ (resp. $h^0(\mathbb{P}^2,\mathcal{I}_{Z',\mathbb{P}^2}(b))=0$) and $h^1(\mathbb{P}^2,\mathcal{I}_{Z'',\mathbb{P}^2}(a-1))=0$ (resp. $h^0(\mathbb{P}^2,\mathcal{I}_{Z'',\mathbb{P}^2}(a-1))=0$), then $h^1(\mathcal{I}_Z(a,b))=0$ (resp. $h^0(\mathcal{I}_Z(a,b))=0$).*

Note that the conditions $h^1(\mathbb{P}^2,\mathcal{I}_{Z',\mathbb{P}^2}(b)) = h^1(\mathbb{P}^2,\mathcal{I}_{Z'',\mathbb{P}^2}(a-1)) = 0$ and $h^10(\mathbb{P}^2,\mathcal{I}_{Z',\mathbb{P}^2}(b)) = h^0(\mathbb{P}^2,\mathcal{I}_{Z'',\mathbb{P}^2}(a-1)) = 0$ are known to be true in many cases by the works on the Segre-Harbourne-Hirschowitz conjecture.

Acknowledgement. The author is a member of GNSAGA of INdAM (Italy).

References

1. A. Altavilla and E. Ballico, *Twistor lines on algebraic surfaces*, Ann. Global Anal. Geom. **55** (2019), no. 3, 555–573.
2. A. Altavilla, E. Ballico and M. C. Brambilla, *Surfaces in the flag threefolds containing smooth conics and twistor fibers*, e-print arXiv:2204.10544.
3. A. Altavilla, E. Ballico, M.C. Brambilla and S. Salamon, *The twistor geometry of the flag manifold*, e-print arXiv:2112.11100; Math. Z. (to appear).

4. M.F. Atiyah, N.J. Hitchin and I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461. https://doi.org/10.1098/rspa.1978.0143
5. G. Gentili, S. Salamon and C. Stoppato, *Twistor transforms of quaternionic functions and orthogonal complex structures,* J. Eur. Math. Soc. (JEMS) **16** (2014), no. 11, 2323–2353. https://doi.org/10.4171/JEMS/488
6. E. Guardo and A. Van Tuyl, Arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, Springerbriefs in Math., p. 1–134, Springer, Cham, 2015.
7. R. Hartshorne and A. Hirschowitz, *Droites en position générale dans l'espaces proiectifs*, in: Algebraic Geometry, Proceedings La Rabida 1981,pp. 169–188, Lect. Notes in Math. 961, Springer-Verlag, Berlin, 1982.
8. N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. (3) **43** (1981), no. 1, 133–150. https://doi.org/10.1112/plms/s3-43.1.133
9. C. LeBrun, *Anti-self-dual metrics and Kähler geometry*, In: Proceedings of the International Congress of Mathematicians, vol. 1, 2 (Zürich, 1994), pp. 498–507. Birkhäuser, Basel, 1995.
10. I. Peeva, Graded syzygies, Algebra and Applications, Vol. 14. Springer-Verlag London, Ltd., London, 2011.
11. S. Salamon and J. Viaclovsky, *Orthogonal complex structures on domains in* $\mathbb{R}^4$, Math. Ann. **343** (2009), no. 4, 853–899. https://doi.org/10.1007/s00208-008-0293-5

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