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RICCI SOLITONS ON PARA-KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study Ricci solitons in para-Kenmotsu manifolds when the Weyl-conformal curvature tensor satisfies some geometric properties like flatness, semi-symmetry, pseudo-symmetry, Ricci-pseudosymmetry and Einstein semi-symmetry.

1. Introduction

During 1982, Hamilton [8] made the fundamental observation that Ricci flow is an excellent tool for simplifying the structure of the manifold. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the regularity in the metric. It is given by

$$\frac{\partial g}{\partial t} = -2Ricg$$

where 'g' is a Riemannian metric, '$Ricg$' is the Ricci curvature tensor, 't' is time.

Let $\phi_t : M \to M$, $t \in R$ be a family of diffeomorphisms which is one parameter group of transformations then it gives rise to a vector field known as infinitesimal generator and integral curves. Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are stationary points of the Ricci flow in space of metrics of $\phi_t : M \to M$. Here the metric $g(t)$ is the pull back of the initial metric $g(0)$ of ϕ_t.

A Ricci soliton is a generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field generated by $\phi_t : t \in R$ and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0 \tag{1.1}$$

where S is a Ricci tensor of M, L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding when λ is negative, zero and positive respectively.

In 1976, Sato [16] introduced the notion of almost para-contact manifolds. Before Sato, Takahashi [17], defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated pseudo-Riemannian metric. In

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1985, Kaneyuki et al. [11] defined the notion of almost paracontact structure on pseudo-Riemannian manifold of dimension $n = (2m + 1)$. Later Zamkovoy [20] showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n + 1, n)$. The notion of para-Kenmotsu manifold was introduced by Welyczko [19]. This structure is a analogy of Kenmotsu manifold [10] in para-contact geometry. para-Kenmotsu (briefly p-Kenmotsu) and special para-Kenmotsu (briefly sp-Kenmotsu) manifolds was studied by Sinha et al. [15], Blaga [4] and Sai prasad et al. [14] and others.

The authors R. Sharma and M.M. Tripathi ([13], [18]) initiated the study of Ricci solitons in contact manifolds. But later the study was extended by Calin and Crasmareanu [5], C.S. Bagewadi and G. Ingalahalli ([9], [2]), S. Debnath and A. Bhattacharya [6] for f-Kenmotsu, α-Sasakian, Lorentzian α-Sasakian and Trans-Sasakian manifolds using L.P. Eisenhart problem [7]. In ([1], [12]) S.R. Ashoka, C.S. Bagewadi and Gurupadavva Ingalahalli and H.G. Nagaraja and C.R. Premalatha studied Ricci solitons in $(LCS)_n$, Kenmotsu manifolds using semi-symmetric and pseudo-symmetric conditions.

Recently C.S. Bagewadi and K.R. Vidyavathi [3] studied Ricci solitons of almost $C(\alpha)$ manifolds using semi-symmetric and pseudosymmetric conditions. In this paper we study Ricci solitons in para-Kenmotsu manifolds satisfying some geometric properties like flatness, semi-symmetry, pseudo-symmetry, Ricci-pseudo symmetry and Einstein semi-symmetry using weyl-conformal curvature tensor.

2. Preliminaries

Let (M^n, g) be an n-dimensional smooth manifold equipped with an almost paracontact metric structure (ϕ, ξ, η, g) that is ϕ is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form and g is a pseudo-Riemannian metric such that

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, Y) = \eta(X) \tag{2.3}$$

for all vector fields $X, Y \in TM^n$. Then (M^n, g) is called almost paracontact metric manifold.

If an almost paracontact metric manifold satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.4}$$

for any $X, Y \in TM^n$, then (M^n, g) is called a almost para-Kenmotsu manifold. A normal almost para-Kenmotsu manifold is a para-Kenmotsu manifold. The para-Kenmotsu structure for 3-dimensional normal almost paracontact metric structure was introduced by Welyczko [19].
From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.5}$$

Moreover, In a para-Kenmotsu manifold the following relations hold:

$$\begin{aligned} R(X,Y,Z,W) &= g(X,Z)g(Y,W) - g(Y,Z)g(X,W) && (2.6)\\ R(\xi,X)Y &= \eta(Y)X - g(X,Y)\xi && (2.7)\\ R(\xi,Y)\xi &= Y - \eta(Y)\xi && (2.8)\\ S(X,Y) &= -(n-1)g(X,Y) && (2.9)\\ S(X,\xi) &= -(n-1)\eta(X) && (2.10)\\ S(\xi,\xi) &= -(n-1) && (2.11)\\ QX &= -(n-1)X && (2.12) \end{aligned}$$

Where R is the Riemannian curvature tensor, S is the Ricci tensor and $g(QX,Y) = S(X,Y)$.

For an n-dimensional almost contact metric manifold the Weyl-conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \quad (2.13) \end{aligned}$$

$$C(\xi,X)Y = \left[1 - \frac{2(n-1)}{(n-2)} - \frac{r}{(n-1)(n-2)}\right][\eta(Y)X - g(X,Y)\xi] \quad (2.14)$$

$$C(\xi,X)\xi = \left[1 - \frac{2(n-1)}{(n-2)} - \frac{r}{(n-1)(n-2)}\right][X - \eta(X)\xi] \quad (2.15)$$

$$C(\xi,\xi)X = 0 \quad (2.16)$$

We use the following definitions for our results:

Definition 2.1. An n-dimensional para-Kenmotsu manifold (M^n, g) is called Weyl-conformally flat if $C(X,Y)Z = 0$ for any vector fields X, Y, Z.

Definition 2.2. An n-dimensional para-Kenmotsu manifold (M^n, g) is called Weyl-semisymmetric if $R \cdot C = 0$.

Definition 2.3. An n-dimensional para-Kenmotsu manifold (M^n, g) is called Einstein-semisymmetric if $R \cdot E = 0$. Where E is the Einstein tensor defined by

$$E(X,Y) = S(X,Y) - \frac{r}{n}g(X,Y) \quad (2.17)$$

Where S is the Ricci tensor and r is the scalar curvature.

Definition 2.4. An n-dimensional para-Kenmotsu manifold (M^n, g) is called Weyl-pseudosymmetric if the tensors $R \cdot C$ and $Q(g,C)$ are linearly dependent. This is equivalent to

$$R \cdot C = L_C Q(g,C) \quad (2.18)$$

holding on the set $U_C = \{x \in M : C \neq 0 \ \ at \ \ x\}$. where L_C is some function on U_C.

Definition 2.5. An n-dimensional para-Kenmotsu manifold (M^n, g) is called partially Ricci-pseudosymmetric if and only if the relation

$$R \cdot S = f(p)Q(g, S) \tag{2.19}$$

holds on the set $A = \{x \in M : Q(g, S) \neq 0 \ \ at \ \ x\}$. where $f \in C^\infty(M)$ for $p \in A$. $R \cdot S$, $Q(g, S)$ and $(X \wedge_g Y)$ are respectively defined as

$$\begin{aligned} (R(X, Y) \cdot S)(U, V) &= -S(R(X, Y)U, V) - S(U, R(X, Y)V) && (2.20)\\ Q(g, S) &= ((X \wedge_g Y) \cdot S)(U, V) && (2.21)\\ (X \wedge_g Y) &= g(Y, Z)X - g(X, Z)Y && (2.22) \end{aligned}$$

for all X, Y, U and $V \in TM^n$.

Definition 2.6. An n-dimensional para-Kenmotsu manifold (M^n, g) is called Weyl Ricci-pseudosymmetric if the tensors $C \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$C \cdot S = L_S Q(g, S) \tag{2.23}$$

holding on the set $U_S = \{x \in M : C \neq 0 \ \ at \ \ x\}$. where L_S is some function on U_S.

3. Ricci solitons in Weyl-conformally flat para-Kenmotsu manifold

We consider a para-Kenmotsu manifold (M^n, g) which is Weyl-conformally flat. Then from definition (2.1) and (2.13) we have

$$\begin{aligned} R(X, Y)Z &= \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]\\ &- \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] \end{aligned} \tag{3.1}$$

Taking inner product with respect to U on both sides of equation (3.1), we get

$$\begin{aligned} R(X, Y, Z, U) &= \frac{1}{(n-2)}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U)\\ &+ g(Y, Z)S(X, U) - g(X, Z)S(Y, U)]\\ &- \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)] \end{aligned} \tag{3.2}$$

Where $R(X, Y, Z, U) = g(R(X, Y)Z, U)$ and $g(QX, Y) = S(X, Y)$.
Taking $X = U = e_i$ and summing over $i = 1, 2, .., n$ in (3.2), we obtain

$$S(Y, Z) = \frac{n(n-1)}{(n-2)}g(Y, Z) \tag{3.3}$$

Hence we state the following lemma:

Lemma 3.1. *A Weyl-conformally flat para-Kenmotsu manifold (M^n, g) is an Einstein manifold.*

Theorem 3.2. *Let (g, V, λ) be a Ricci soliton in para-Kenmotsu manifold (M^n, g). Then (M^n, g) is Weyl-conformally flat iff V is conformal killing. Further deduce that (g, ξ, λ) is shrinking.*

Proof. Let (M^n, g) be an n-dimensional para-Kenmotsu manifold and Let (g, V, λ) be a Ricci soliton on (M^n, g). If V is conformal killing vector field. then by definition

$$L_V g = \rho g \tag{3.4}$$

for some scalar function ρ and from (1.1), we have

$$S = -\left(\lambda g + \frac{1}{2} L_V g\right) \tag{3.5}$$

From (3.4) and (3.5), we get

$$S(X, Y) = -\left(\lambda + \frac{\rho}{2}\right) g(X, Y) \tag{3.6}$$

Taking inner product (2.13) with respect to W and by virtue of (2.6) and (3.6), we obtain

$$C(X, Y, Z, W) = 0 \tag{3.7}$$

This shows that para-Kenmotsu manifold (M^n, g) is Weyl-conformally flat.

Conversely, Let (M^n, g) be a Weyl-conformally flat para-Kenmotsu manifold and (g, V, λ) be a Ricci soliton on (M^n, g). then from (3.3), we have

$$S(Y, Z) = \frac{n(n-1)}{(n-2)} g(Y, Z) \tag{3.8}$$

Substituting this in (1.1), we get

$$(L_V g)(Y, Z) = \rho g(Y, Z) \tag{3.9}$$

Where $\rho = -2\left[\lambda + \frac{n(n-1)}{(n-2)}\right]$ i.e V is conformal killing.
Further, If a Weyl-conformally flat para-Kenmotsu manifold (M^n, g) admits Ricci soliton (g, ξ, λ). then from the relation (1.1) implies

$$2S(X, Y) = -(L_\xi g)(X, Y) - 2\lambda g(X, Y) \tag{3.10}$$

On a para-Kenmotsu manifold (M^n, g), from equation (2.5) we have

$$(L_\xi g)(X, Y) = 2\{g(X, Y) - \eta(X)\eta(Y)\} \tag{3.11}$$

Using (3.11) in (3.10), we obtain

$$S(X, Y) = -(\lambda + 1) g(X, Y) + \eta(X)\eta(Y) \tag{3.12}$$

Then from equations (3.3) and (3.12), we get

$$\left[\frac{n^2 + \lambda(n-2) - 2}{(n-2)}\right] g(X, Y) - \eta(X)\eta(Y) = 0 \tag{3.13}$$

Substitution of $X = \xi$ in (3.13), we get the relation:

$$\lambda = \frac{-n(n+1)}{(n-2)} \tag{3.14}$$

Therefore, λ is negative. Hence the proof. □

4. Ricci solitons in Weyl semi-symmetric para-Kenmotsu manifold

Let us consider Weyl semi-symmetric para-Kenmotsu manifold. Then from definition (2.2), we have

$$(R(X,Y)\cdot C)(U,V)W = 0 \tag{4.1}$$

Equation (4.1) can be written as

$$\begin{aligned} & R(X,Y)C(U,V)W - C(R(X,Y)U,V)W - C(U,R(X,Y)V)W \\ - \; & C(U,V)R(X,Y)W = 0 \end{aligned} \tag{4.2}$$

Put $X = U = \xi$ in (4.2) and using (2.7), we get

$$\begin{aligned} & \eta(C(\xi,V)W)Y - g(Y,C(\xi,V)W)\xi - C(Y,V)W \\ + \; & \eta(Y)C(\xi,V)W - \eta(V)C(\xi,Y)W + g(Y,V)C(\xi,\xi)W \\ - \; & \eta(W)C(\xi,V)Y + g(Y,W)C(\xi,V)\xi = 0 \end{aligned} \tag{4.3}$$

Using (2.13), (2.14), (2.15) and (2.16) in (4.3), we obtain

$$R(Y,V)W = g(Y,W)V - g(V,W)Y \tag{4.4}$$

Taking inner product with Z of (4.4), we have

$$R(Y,V,W,Z) = g(Y,W)g(V,Z) - g(V,W)g(Y,Z) \tag{4.5}$$

Taking $V = W = e_i$ and summing over $i = 1, 2, ..., n$ in (4.5), we get

$$S(Y,Z) = -(n-1)g(Y,Z) \tag{4.6}$$

Hence we state the following lemma:

Lemma 4.1. *A Weyl semi-symmetric para-Kenmotsu manifold is an Einstein manifold.*

Theorem 4.2. *Let (g,V,λ) be a Ricci soliton in para-Kenmotsu manifold (M^n,g). Then (M^n,g) is Weyl semi-symmetric iff V is conformal killing. Further deduce that (g,ξ,λ) is expanding.*

Proof. Let (M^n,g) be an n-dimensional para-Kenmotsu manifold and Let (g,V,λ) be a Ricci soliton on (M^n,g). If V is conformal killing vector field. then by definition

$$L_V g = \rho g \tag{4.7}$$

for some scalar function ρ and from (1.1), we have

$$S = -\left(\lambda g + \frac{1}{2}L_V g\right) \tag{4.8}$$

From (4.7) and (4.8), we get

$$S(X,Y) = -\left(\lambda + \frac{\rho}{2}\right) g(X,Y) \tag{4.9}$$

Then equation (4.2), we have

$$\begin{aligned} R \cdot C &= R(X,Y)C(U,V)W - C(R(X,Y)U,V)W \\ &- C(U,R(X,Y)V)W - C(U,V)R(X,Y)W \end{aligned} \tag{4.10}$$

Using (2.6) and (4.9) in (4.10), we get

$$R \cdot C = 0 \tag{4.11}$$

This shows that para-Kenmotsu manifold (M^n, g) is Weyl semi-symmetric.

Conversely, Let (M^n, g) be a Weyl semi-symmetric para-Kenmotsu manifold and (g, V, λ) be a Ricci soliton on (M^n, g). then from (4.6), we have

$$S(Y,Z) = -(n-1)g(Y,Z) \tag{4.12}$$

Substituting this in (1.1), we get

$$(L_V g)(Y,Z) = \rho g(Y,Z) \tag{4.13}$$

Where $\rho = (2(n-1) - 2\lambda)$ i.e V is conformal killing.
Further, If a Weyl semi-symmetric para-Kenmotsu manifold (M^n, g) admits Ricci soliton (g, ξ, λ). then from the relation (3.13), we have

$$S(X,Y) = -(\lambda+1)g(X,Y) + \eta(X)\eta(Y) \tag{4.14}$$

Then from equations (4.12) and (4.14), we get

$$(-\lambda + n - 2)g(X,Y) + \eta(X)\eta(Y) = 0 \tag{4.15}$$

Substitution of $X = \xi$ in (4.15), we get the relation:

$$\lambda = n - 1 \tag{4.16}$$

Therefore, λ is positive. Hence the proof. □

5. Ricci solitons in Einstein semi-symmetric para-Kenmotsu manifold

Let us consider Einstein semi-symmetric para-Kenmotsu manifold (M^n, g). Then from definition (2.3), we have

$$(R(X,Y) \cdot E)(Z,U) = 0 \tag{5.1}$$

Equation (5.1) can be written as

$$E(R(X,Y)Z,U) + E(Z,R(X,Y)U) = 0 \tag{5.2}$$

Using (2.17) in (5.2), we get

$$\begin{aligned} &S(R(X,Y)Z,U) + S(Z,R(X,Y)U) \\ &- \frac{r}{n}[g(R(X,Y)Z,U) + g(Z,R(X,Y)U)] = 0 \end{aligned} \tag{5.3}$$

In view of equation (2.9) and by replacing $Y = U = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i = 1, 2, ..., n$ we get

$$S(X, Z) = -\frac{r}{n}g(X, Z) \tag{5.4}$$

Hence, we state the following lemma:

Lemma 5.1. *A Einstein semi-symmetric para-Kenmotsu manifold (M^n, g) is an Einstein manifold.*

Theorem 5.2. *Let (g, V, λ) be a Ricci soliton in para-Kenmotsu manifold (M^n, g). Then (M^n, g) is Einstein semi-symmetric iff V is conformal killing. Further deduce that (g, ξ, λ) is shrinking.*

Proof. Let (M^n, g) be an n-dimensional para-Kenmotsu manifold and Let (g, V, λ) be a Ricci soliton on (M^n, g). If V is conformal killing. then by definition

$$L_V g = \rho g \tag{5.5}$$

for some scalar function ρ and from (1.1), we have

$$S = -\left(\lambda g + \frac{1}{2}L_V g\right) \tag{5.6}$$

From (5.5) and (5.6), we get

$$S(X, Y) = -\left(\lambda + \frac{\rho}{2}\right) g(X, Y) \tag{5.7}$$

Then from equation (5.2), we have

$$R \cdot E = E(R(X, Y)Z, U) + E(Z, R(X, Y)U) \tag{5.8}$$

$$R \cdot E = S(R(X, Y)Z, U) + S(Z, R(X, Y)U) - \frac{r}{n}[g(R(X, Y)Z, U) + g(Z, R(X, Y)U)] \tag{5.9}$$

Using (2.6) and (5.7) in (5.9), we get

$$R \cdot E = 0 \tag{5.10}$$

This shows that para-Kenmotsu manifold (M^n, g) is Einstein semi-symmetric.

Conversely, Let (M^n, g) is an Einstein semi-symmetric para-Kenmotsu manifold and (g, V, λ) be a Ricci soliton on (M^n, g). then from (5.4), we have

$$S(X, Y) = -\frac{r}{n}g(X, Y) \tag{5.11}$$

Substituting this in (1.1), we get

$$(L_V g)(X, Y) = \rho g(X, Y) \tag{5.12}$$

Where $\rho = 2(\frac{r}{n} - \lambda)$ i.e V is conformal killing.
Further, If an Einstein semi-symmetric para-Kenmotsu manifold (M^n, g) admits Ricci soliton (g, ξ, λ). then from the relation (3.13), we have

$$S(X, Y) = -(\lambda + 1)g(X, Y) + \eta(X)\eta(Y) \tag{5.13}$$

Then from equations (5.11) and (5.13), we get

$$\left[\frac{r}{n} - (\lambda + 1)\right] g(X, Y) + \eta(X)\eta(Y) = 0 \tag{5.14}$$

Substitution of $X = \xi$ in (5.14), we get the relation:

$$\lambda = -(n - 1) \tag{5.15}$$

Therefore, λ is negative. Hence the proof. □

6. Ricci solitons in Weyl pseudo-symmetric para-Kenmotsu manifold

Let us consider weyl pseudo-symmetric para-Kenmotsu manifold (M^n, g). Then from definition (2.4), we have

$$(R(X, Y) \cdot C)(U, V)W = L_C[Q(g, C)(U, V, W; X, Y)] \tag{6.1}$$

Equation (6.1) can be written as

$$\begin{aligned} & R(X, Y)C(U, V)W - C(R(X, Y)U, V)W - C(U, R(X, Y)V)W \\ - \; & C(U, V)R(X, Y)W = L_C[(X \wedge Y)C(U, V)W - C((X \wedge Y)U, V)W \\ - \; & C(U, (X \wedge Y)V)W - C(U, V)(X \wedge Y)W] \end{aligned} \tag{6.2}$$

Put $X = U = \xi$ in (6.2) and using (2.7), (2.8) and (2.22), we get

$$\begin{aligned} & [L_C + 1][g(Y, C(\xi, V)W)\xi - \eta(C(\xi, V)W)Y + C(Y, V)W \\ - \; & \eta(Y)C(\xi, V)W + \eta(V)C(\xi, Y)W - g(Y, V)C(\xi, \xi)W \\ + \; & \eta(W)C(\xi, V)Y - g(Y, W)C(\xi, V)\xi] = 0 \end{aligned} \tag{6.3}$$

Using (2.13), (2.14), (2.15) and (2.16) in (6.3), we get

$$[L_C + 1][R(Y, V)W + g(V, W)Y - g(Y, W)V] = 0 \tag{6.4}$$

Therefore, either

$$L_C = -1 \quad or \quad R(Y, V)W = g(Y, W)V - g(V, W)Y \tag{6.5}$$

Taking inner product with Z of (6.5), we get

$$R(Y, V, W, Z) = g(Y, W)g(V, Z) - g(V, W)g(Y, Z) \tag{6.6}$$

Taking $V = W = e_i$ and summing over $i = 1, 2, ..., n$ in (6.6), we obtain

$$S(Y, Z) = -(n - 1)g(Y, Z) \tag{6.7}$$

Hence we state the following lemma:

Lemma 6.1. *A Weyl pseudo-symmetric para-Kenmotsu manifold is an Einstein manifold with $L_C \neq -1$.*

Let a Weyl pseudo-symmetric para-Kenmotsu manifold (M^n, g) admits Ricci soliton on (g, ξ, λ). Then from equations (3.13) and (6.7), we have

$$(-\lambda + n - 2)g(X, Y) + \eta(X)\eta(Y) = 0 \tag{6.8}$$

Substitution of $X = \xi$ in (6.8), we get the relation:

$$\lambda = n - 1$$

Therefore, λ is positive. Hence we state the following theorem:

Theorem 6.2. *A Ricci soliton in (g, ξ, λ) Weyl semi-symmetric para-Kenmotsu manifold (M^n, g) is expanding provided $L_C \neq -1$.*

7. Ricci solitons in partially Ricci-pseudosymmetric para-Kenmotsu manifold

Let us consider partially Ricci-pseudo symmetric para-Kenmotsu manifold. Then from definition (2.5), we have

$$(R(X,Y)\cdot S)(Z,U) = f(p)[((X \wedge_g Y).S)(Z,U)] \tag{7.1}$$

From equations (2.20) and (2.21), it follows that

$$\begin{aligned} & S(R(X,Y)Z,U) + S(Z,R(X,Y)U) \\ = \; & f(p)[S((X \wedge_g Y)Z,U) + S(Z,(X \wedge_g Y)U)] \end{aligned} \tag{7.2}$$

Taking $Y = U = \xi$ in (7.2), we have

$$\begin{aligned} & S(R(X,\xi)Z,\xi) + S(Z,R(X,\xi)\xi) \\ = \; & f(p)[S((X \wedge_g \xi)Z,\xi) + S(Z,(X \wedge_g \xi)\xi)] \end{aligned} \tag{7.3}$$

Applying (2.11), (2.20) in (7.3), we obtain

$$\begin{aligned} & -(n-1)\eta(R(X,\xi)Z) + \eta(X)S(Z,\xi) - S(X,Z) \\ = \; & f(p)[\eta(Z)S(X,\xi) - g(X,Z)S(\xi,\xi) + S(X,Z) - \eta(X)S(Z,\xi)] \end{aligned} \tag{7.4}$$

Using (2.6)and (2.11) in (7.4), we get

$$\begin{aligned} & (n-1)\{\eta(Z)\eta(X) - g(X,Z)\} - S(X,Z) \\ - \; & (n-1)\eta(X)\eta(Z) = f(p)[-(n-1)\eta(X)\eta(Z) \\ + \; & (n-1)g(X,Z) + S(X,Z) + (n-1)\eta(X)\eta(Z)] \end{aligned} \tag{7.5}$$

This can be written as

$$-[S(X,Z) + (n-1)g(X,Z)] = f(p)[S(X,Z) + (n-1)g(X,Z)] \tag{7.6}$$

Thus, we have

$$[f(p)+1][S(X,Z) + (n-1)g(X,Z)] = 0 \tag{7.7}$$

This can be hold only if either

$$f(p) = -1 \quad or \quad S(X,Z) = -(n-1)g(X,Z) \tag{7.8}$$

Hence, we state the following lemma:

Lemma 7.1. *A partially Ricci-pseudosymmetric para-Kenmotsu manifold (M^n, g) is an Einstein manifold with $f(p) \neq -1$*

Let a partially Ricci-pseudosymmetric para-Kenmotsu manifold admits Ricci soliton on (M^n, g). Then from equations (3.13) and (7.8), we get

$$[-\lambda + n - 2]g(X,Y) + \eta(X)\eta(Y) = 0 \tag{7.9}$$

Substitution of $X = \xi$ in (7.9), we obtain

$$\lambda = n - 1 \tag{7.10}$$

Therefore, λ is positive. Hence we state the following theorem:

Theorem 7.2. *A Ricci soliton (g, ξ, λ) in paratially Ricci-pseudosymmetric para-Kenmotsu manifold (M^n, g) is expanding provided $f(p) \neq -1$.*

8. Ricci solitons in Weyl Ricci-pseudo symmetric para-Kenmotsu manifold

Let us consider Weyl Ricci-pseudo symmetric para-Kenmotsu manifold (M^n, g). Then from definition (2.6), we have

$$(C(X,Y)\cdot S)(U,V) = L_S Q(g,S) \tag{8.1}$$

Equation (8.1) can be written as

$$\begin{aligned} & S(C(X,Y)U,V) + S(U,C(X,Y)V) \\ = \; & L_S[S((X\wedge Y)U,V) + S(U,(X\wedge Y)V)] \end{aligned} \tag{8.2}$$

Putting $X = V = \xi$ in (8.2), we get

$$S(C(\xi,Y)U,\xi) + S(U,C(\xi,Y)\xi) = L_S[S((\xi\wedge Y)U,\xi) + S(U,(\xi\wedge Y)\xi)] \tag{8.3}$$

Using (2.9), (2.10), (2.14), (2.15) and (2.22) in (8.3), we obtain

$$\left[L_S + 1 - \frac{2(n-1)}{(n-2)} - \frac{r}{(n-1)(n-2)}\right][S(Y,U) + (n-1)g(Y,U)] = 0 \tag{8.4}$$

This can be hold only if either

$$L_S = \left[\frac{2(n-1)}{(n-2)} + \frac{r}{(n-1)(n-2)} - 1\right] \quad or \quad S(Y,U) = -(n-1)g(Y,U) \tag{8.5}$$

Hence, we state the following lemma:

Lemma 8.1. *A Weyl Ricci-pseudosymmetric para-Kenmotsu manifold (M^n, g) is an Einstein manifold with $L_S \neq \left[\frac{2(n-1)}{(n-2)} + \frac{r}{(n-1)(n-2)} - 1\right]$*

Let a Weyl Ricci-pseudosymmetric para-Kenmotsu manifold (M^n, g) admits Ricci soliton. Then from equations (3.13) and (8.5), we get

$$[-\lambda + n - 2]g(X,Y) + \eta(X)\eta(Y) = 0 \tag{8.6}$$

Substitution of $X = \xi$ in (8.6), we obtain

$$\lambda = n - 1 \tag{8.7}$$

Therefore, λ is positive. Hence we state the following theorem:

Theorem 8.2. *A Ricci soliton (g, ξ, λ) in Weyl Ricci-pseudosymmetric para-Kenmotsu manifold (M^n, g) is expanding provided $L_S \neq \left[\frac{2(n-1)}{(n-2)} + \frac{r}{(n-1)(n-2)} - 1\right]$*

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