COUNTING NEGATIVE EIGENVALUES OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

MARTIN KARUHANGA$^1$ and EUGENE SHARGORODSKY$^2$

Abstract. In this paper, we extend the well known estimates for the number of negative eigenvalues of one-dimensional Schrödinger operators with potentials that are absolutely continuous with respect to the Lebesgue measure to the case of strongly singular potentials.

1. Introduction

Let $N_-(V)$ be the number of negative eigenvalues of a Schrödinger operator

$$H = -\Delta - V, \ V \geq 0$$

on $L^2(\mathbb{R}^d)$. For $d > 2$, the number $N_-(V)$ is estimated above by the well known Cwikel-Lieb-Rozenblum (CLR) inequality [2, 8]. For $d = 2$, the CLR inequality fails and the best known estimates for $N_-(V)$ in this case involve weighted $L^1$ norms and Orlcziz norms of the potential (see, e.g., [9, 10] and [7] in the case where $V$ is supported by a Lipschitz curve). For $d = 1$, an analogue of the CLR inequality holds for potentials that are monotone on $\mathbb{R}_+$ and $\mathbb{R}_-$ (see, e.g., [4]). For general nonnegative potentials that are locally integrable on $\mathbb{R}$ with respect to the standard Lebesgue measure, $N_-(V)$ admits the following estimate

$$N_-(V) \leq 1 + C \sum_{\{j \in \mathbb{Z}, A_j(V) > c\}} \sqrt{A_j(V)},$$

(1.1)

where $C, c$ are positive constants and

$$A_0(V) = \int_{-1}^{1} V(t) \, dt, \quad A_j(V) = 2^j \int_{2^{j-1}}^{2^j} V(t) \, dt, \quad j > 0,$$

$$A_j(V) = 2^{\lvert j \rvert} \int_{-2^{\lvert j \rvert}}^{-2^{\lvert j \rvert-1}} V(t) \, dt, \quad j < 0$$

(see [11] and the references therein). When $V$ is a linear combination of Dirac delta functions, results on $N_-(V)$ can be found for example in [1]. The main purpose of this paper is to extend the estimate (1.1) to the case when $V$ is allowed...
to be a measure that is not necessarily absolutely continuous with respect to the Lebesgue measure. In particular, we study the operator

\[ H_\mu := -\frac{d^2}{dx^2} - \mu \]  

(1.2)
on $L^2(\mathbb{R})$, where $\mu$ is an arbitrary $\sigma$-finite positive Radon measure on $\mathbb{R}$.

2. Main result

We denote by $N_-(\mu, \mathbb{R})$ the number of negative eigenvalues of (1.2) counting multiplicities. Define (1.2) via its quadratic form

\[ q_{\mu, \mathbb{R}}[u] := \int_{\mathbb{R}} |u'(x)|^2 \, dx - \int_{\mathbb{R}} |u(x)|^2 \, d\mu(x), \]

\[ \text{Dom}(q_{\mu, \mathbb{R}}) = W^1_2(\mathbb{R}) \cap L^2(\mathbb{R}, d\mu), \]

where $W^1_2(\mathbb{R})$ denotes the standard Sobolev space of square integrable functions with square integrable weak derivatives. Then $N_-(\mu, \mathbb{R})$ is given by

\[ N_-(\mu, \mathbb{R}) = \sup \{ \dim L : q_{\mu, \mathbb{R}}[u] < 0, \forall u \in L \setminus \{0\} \}, \]  

(2.1)
where $L$ denotes a linear space of $\text{Dom}(q_{\mu, \mathbb{R}})$ (see, e.g., [3, Theorem 10.2.3]).

Let

\[ I_n := [2^{n-1}, 2^n], \ n > 0, \ I_0 := [-1, 1], \ I_n := [-2^n, -2^{n-1}], \ n < 0 \]

and

\[ A_n := \int_{I_n} |x| \, d\mu(x), \ n \neq 0, \ A_0 := \int_{I_0} \, d\mu(x). \]  

(2.2)

Theorem 2.1. Let $\mu$ be a $\sigma$-finite positive Radon measure on $\mathbb{R}$ and let $\{A_n\}$ be the sequence in (2.2). Then there exist constants $c, C > 0$ such that

\[ N_-(\mu, \mathbb{R}) \leq 1 + C \sum_{\{n \in \mathbb{Z}, A_n > c\}} \sqrt{A_n}. \]

3. Auxiliary results

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set and let $\mu$ be a positive $\sigma$-finite Radon measure on $\mathbb{R}^n$. Further, let $V$ be a non-negative $\mu$-measurable real valued function and $V \in L^1_{\text{loc}}(\overline{\Omega}, \mu)$. Define the following quadratic form

\[ E_{V, \mu, \Omega}[w] := \int_{\Omega} |\nabla w|^2 \, dx - \int_{\overline{\Omega}} V |w|^2 \, d\mu(x), \]

with the domain $\text{Dom}(E_{V, \mu, \Omega})$, which is a linear subspace of $W^1_2(\Omega) \cap L^2(\overline{\Omega}, V \, d\mu)$. Note that $\mu$ does not have to be the $n$ dimensional Lebesgue measure, and it may well happen that $\mu(\partial \Omega) > 0$.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a (finite or infinite) sequence $\{\Omega_k\}$ of non-empty open subsets $\Omega_k \subset \Omega$ is a $\mu$-partition of $\Omega$ if $\Omega_k \cap \Omega_l = \emptyset$ when $k \neq l$, $\Omega \setminus \bigcup_k \Omega_k$ has zero Lebesgue measure, and $\mu(\overline{\Omega} \setminus \bigcup_k \overline{\Omega_k}) = 0$. 

The following result can be found, e.g., in [5, Ch.6, §2.1, Theorem 4] in the case when \( \mu \) is absolutely continuous with respect to the Lebesgue measure.

**Lemma 3.2.** Let \( \{ \Omega_k \} \) be a \( \mu \)-partition of \( \Omega \) and suppose \( \text{Dom}(\mathcal{E}_{V,\mu},\Omega) \), \( \text{Dom}(\mathcal{E}_{V,\mu,\Omega_k}) \) are such that for every \( k \),

\[
w|_{\Omega_k} \in \text{Dom}(\mathcal{E}_{V,\mu,\Omega_k}), \quad \forall w \in \text{Dom}(\mathcal{E}_{V,\mu,\Omega}).
\]

Then

\[
N_-(\mathcal{E}_{V,\mu,\Omega}) \leq \sum_k N_-(\mathcal{E}_{V,\mu,\Omega_k}).
\] (3.1)

**Proof.** Let

\[
\Sigma := \oplus \{ \text{Dom}(\mathcal{E}_{V,\mu,\Omega_k}), k = 1, 2, \ldots \}.
\]

Here \( \oplus \) denotes the direct sum. We consider \( \sum_k \mathcal{E}_{V,\mu,\Omega_k} \) as a form defined on \( \Sigma \).

Let \( J : \text{Dom}(\mathcal{E}_{V,\mu,\Omega}) \rightarrow \Sigma \) be the embedding defined by

\[
w \mapsto (w|_{\Omega_1}, w|_{\Omega_2}, \ldots).
\]

Let \( \Gamma := J(\text{Dom}(\mathcal{E}_{V,\mu,\Omega})) \). Then \( \forall w \in \text{Dom}(\mathcal{E}_{V,\mu,\Omega}) \), we have

\[
\mathcal{E}_{V,\mu,\Omega}[w] = \int_{\Omega} |\nabla w(x)|^2 dx - \int_{\Omega} V(x)|w(x)|^2 d\mu(x)
\]

\[
\geq \sum_k \left( \int_{\Omega_k} |\nabla w(x)|^2 dx - \int_{\Omega_k} V(x)|w(x)|^2 d\mu(x) \right)
\]

\[
= \sum_k \mathcal{E}_{V,\mu,\Omega_k}[w|_{\Omega_k}] = \left( \sum_k \mathcal{E}_{V,\mu,\Omega_k} \right)[Jw].
\]

Hence

\[
N_-(\mathcal{E}_{V,\mu,\Omega}) \leq N_\left( \left( \sum_k \mathcal{E}_{V,\mu,\Omega_k} \right) |_\Gamma \right) \leq N_\left( \sum_k \mathcal{E}_{V,\mu,\Omega_k} \right) = \sum_k N_-(\mathcal{E}_{V,\mu,\Omega_k}).
\]

\[\square\]

Let \( I \) be a bounded interval in \( \mathbb{R} \) of length \( l \). For simplicity, take \( I = (0, l) \).

Let \( 0 = t_0 < t_1 < \ldots < t_n = l \) be a partition of the interval \( I \) into \( n \) subintervals \( I_k = (t_{k-1}, t_k) \). Let \( P \) stand for any such partition and \( |P| \) denote the number of subintervals, i.e. \( |P| = n \). Let \( \nu \) be a positive Radon measure on \( \mathbb{R} \) and for any real number \( a > 0 \), consider the following function of partitions:

\[
\Theta_a(P) := \max_k (t_k - t_{k-1})^a \nu(I_k).
\] (3.2)

**Lemma 3.3.** Suppose \( \nu(\{x\}) = 0 \) for all \( x \in I \). Then for any \( n \in \mathbb{N} \), there exists a partition \( P \) of the interval \( I \) such that \( |P| = n \) and

\[
\Theta_a(P) \leq l^a n^{-1-a} \nu(I).
\] (3.3)

**Proof.** The proof is similar to that of [11, Lemma 7.1] where measures absolutely continuous with respect to the Lesbegue measure were considered. By scaling, it is enough to prove (3.3) for \( l = 1 \) and \( \nu(I) = 1 \). For \( n = 1 \), there is nothing to prove. Now suppose (3.3) is true for some \( n \). We need to show that then this is
true for \( n + 1 \). Since \( x \mapsto \nu([x, 1]) \) is continuous, there exists a point \( x \in (0, 1) \) such that

\[
(1 - x)^a \nu([x, 1]) = (n + 1)^{1-a}.
\] \hspace{1cm} (3.4)

Then one has

\[
\nu([x, 1]) = (n + 1)^{1-a}(1 - x)^{-a}.
\]

By the induction assumption, there exists a partition \( P_0 \) of the interval \( (0, x) \) into \( n \) subintervals \( 0 = t_0 < t_1 < \ldots < t_n = x \) such that

\[
\Theta_a(P_0) \leq x^a n^{1-a} \nu((0, x)) = x^a n^{1-a} (1 - (n + 1)^{1-a}(1 - x)^{-a}).
\]

Let \( P \) be the partition \( 0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = 1 \). Since (3.4) holds, (3.3) with \( n + 1 \) in place of \( n \) will follow if one proves that \( \Theta_a(P_0) \leq (n + 1)^{1-a} \). The latter is achieved this by showing that

\[
n^{-1-a} \leq (n + 1)^{1-a} x^{-a} + n^{-1-a}(n + 1)^{1-a}(1 - x)^{-a}.
\]

Let \( h(x) = (n + 1)^{1-a} x^{-a} + n^{-1-a}(n + 1)^{1-a}(1 - x)^{-a} \). Then \( h \) is convex on \( (0, 1) \), and solving \( h'(x) = 0 \) we see that \( h \) attains its minimum on \( (0, 1) \) at the point \( x = n(n + 1)^{-1} \) and that this minimum value is \( n^{-1-a} \).

**Lemma 3.4.** Suppose \( \nu(\{t\}) = 0 \) for all \( t \in \mathcal{T} \). For any \( n \in \mathbb{N} \), there exists a partition \( P \) of the interval \( I \) such that \( |P| = n \) and

\[
\int_I |u(t)|^2 d\nu(t) \leq \frac{l}{n^2} \nu(I) \int_I |u'(t)|^2 dt
\]

for all \( u \in \mathcal{L}_n \), where \( \mathcal{L}_n \) is the subspace of \( W^1_2(I) \) of co-dimension \( n \) formed by the functions satisfying \( u(t_1) = \ldots = u(t_n) = 0 \).

**Proof.** For any \( t \in I_k \), the Cauchy-Schwartz inequality implies

\[
|u(t)|^2 = |u(t) - u(t_k)|^2 = \left| \int_t^{t_k} u'(s) \, ds \right|^2 \leq |t - t_k| \int_t^{t_k} |u'(s)|^2 \, ds
\]

\[
\leq |t_k - t_{k-1}| \int_{t_{k-1}}^{t_k} |u'(s)|^2 \, ds.
\]

Hence

\[
\int_{I_k} |u(t)|^2 d\nu(t) \leq \sup_{t \in I_k} |u(t)|^2 \nu(I_k)
\]

\[
\leq |t_k - t_{k-1}| \nu(I_k) \int_{t_{k-1}}^{t_k} |u'(s)|^2 \, ds.
\]
With $a = 1$, (3.2) and Lemma 3.3 imply
\[
\int_I |u(t)|^2 \, d\nu(t) = \sum_{k=1}^n \int_{I_k} |u(t)|^2 \, d\nu(t)
\]
\[
\leq \sum_{k=1}^n |t_k - t_{k-1}| \nu(I_k) \int_{t_{k-1}}^{t_k} |u'(s)|^2 \, ds
\]
\[
\leq \Theta_a(P) \sum_{k=1}^n \int_{I_k} |u'(s)|^2 \, ds \leq \frac{l}{n^2} \nu(I) \int_I |u'(s)|^2 \, ds.
\]

The above Lemma excludes measures with atoms. However, one can show that the lemma still holds true even when $\nu$ has atoms by approximating $\nu$ by measures that are absolutely continuous with respect to the Lebesgue measure.

**Lemma 3.5.** Let $\nu$ be an arbitrary positive Radon measure on $\mathbb{R}$. For any $c > 1$ and any $n \in \mathbb{N}$ there exists a partition $P$ of $I$ such that $|P| = n$ and
\[
\int_I |u(t)|^2 \, d\nu(t) \leq c \frac{1}{n^2} \nu(T) \int_I |u'(t)|^2 \, dt,
\]
for all $u \in W^1_2(I)$ such that $u(t_1) = u(t_2) = \ldots = u(t_n) = 0$.

**Proof.** Let $\varphi \in C^\infty_c(\mathbb{R})$ such that $\varphi(t) = 0$ if $|t| \geq 1$, $\varphi \geq 0$, and $\int_\mathbb{R} \varphi(t) \, dt = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(t) = \frac{1}{\varepsilon} \varphi(\frac{t}{\varepsilon})$. Then $\varphi_\varepsilon(t) = 0$ if $|t| \geq \varepsilon$ and $\int_\mathbb{R} \varphi_\varepsilon(t) \, dt = 1$. Extend $\nu$ to $\mathbb{R}$ by $\nu(J) = 0$ for $J = \mathbb{R} \setminus T$. Let $\nu_\varepsilon := \nu \ast \varphi_\varepsilon$, i.e.,
\[
d\nu_\varepsilon(t) = \left( \int_\mathbb{R} \varphi_\varepsilon(t - y) \, d\nu(y) \right) dt.
\]
Then $\text{supp} \, \nu_\varepsilon \subseteq I_\varepsilon$, where $I_\varepsilon := [-\varepsilon, l + \varepsilon]$. By Lemma 3.4, for any $n \in \mathbb{N}$ there exists a partition $P_\varepsilon = \{t^\varepsilon_0, \ldots, t^\varepsilon_n\}$ of $I_\varepsilon$ such that $|P_\varepsilon| = n$ and
\[
\int_{I_\varepsilon} |u_\varepsilon(t)|^2 \, d\nu_\varepsilon(t) \leq \frac{l}{n^2} \nu_\varepsilon(I_\varepsilon) \int_{I_\varepsilon} |u'_\varepsilon(t)|^2 \, dt, \tag{3.5}
\]
for all $u_\varepsilon \in W^1_2(I_\varepsilon)$ such that $u(t^\varepsilon_1) = \ldots = u(t^\varepsilon_n) = 0$.

Let
\[
\xi(x) := \frac{l + 2\varepsilon}{l} x - \varepsilon.
\]
Then
\[
\xi^{-1}(y) = \frac{l}{l + 2\varepsilon} (y + \varepsilon)
\]
and
\[

\xi : I \to I_\varepsilon, \quad \xi^{-1} : I_\varepsilon \to I.
\]
Let
\[
t_k = \xi^{-1}(t^\varepsilon_k), \quad k = 0, ..., n.
\]
Take any $u \in W^1_2(I)$ such that $u(t_1) = \ldots = u(t_n)$. Consider
\[
u_\varepsilon(y) := u(\xi^{-1}(y)).
\]
Then \( u_\varepsilon \in W^1_2(I_\varepsilon) \) and \( u_\varepsilon(t_1^\varepsilon) = \ldots = u_\varepsilon(t_n^\varepsilon) = 0 \), so (3.5) holds.

Now,

\[
\nu_\varepsilon(I_\varepsilon) = \int_{I_\varepsilon} \int_R \varphi(\tau - y) dy \nu(y) dt = \int_{I_\varepsilon} \int_R \varphi_\varepsilon(t - y) dt \nu(y)
\]

\[
= \int_{I_\varepsilon} \int_R \varphi_\varepsilon(t - y) dt \nu(y) = \int_{I_\varepsilon} \varphi_\varepsilon(t - y) dt \nu(y)
\]

\[
= \int_{I_\varepsilon} \nu(y) = \nu(T),
\]

(3.6)

\[
\int_{I_\varepsilon} |u'_\varepsilon(t)|^2 dt = \int_{I_\varepsilon} \left| \frac{d}{dt} u(\xi^{-1}(t)) \right|^2 dt = \frac{l}{l + 2\varepsilon} \int_I |u'(x)|^2 dx
\]

\[
\leq \int_I |u'(x)|^2 dx,
\]

(3.7)

\[
\left| \int_T |u(y)|^2 d\nu(y) - \int_{I_\varepsilon} |u_\varepsilon(t)|^2 d\nu_\varepsilon(t) \right|
\]

\[
= \left| \int_R |u(y)|^2 d\nu(y) - \int_R |u_\varepsilon(t)|^2 d\nu_\varepsilon(t) \right|
\]

\[
= \left| \int_R |u(y)|^2 d\nu(y) - \int_R |u_\varepsilon(t)|^2 \int_{I_\varepsilon} \varphi_\varepsilon(t - y) dy \nu(y) dt \right|
\]

\[
= \left| \int_R |u(y)|^2 d\nu(y) - \int_R \int_{I_\varepsilon} |u_\varepsilon(\tau + y)|^2 \varphi_\varepsilon(\tau) d\tau d\nu(y) \right|
\]

\[
\leq \int_R \int_{I_\varepsilon} |u(y)|^2 - |u_\varepsilon(\tau + y)|^2 |\varphi_\varepsilon(\tau)| d\tau d\nu(y)
\]

\[
\leq \max_{y \in I} \left| |u(y)|^2 - |u_\varepsilon(\tau + y)|^2 \right| \nu(T),
\]

\[
|u(y)|^2 - |u_\varepsilon(\tau + y)|^2 = |u(y)|^2 - \left| u \left( \frac{l}{l + 2\varepsilon} (y + \tau + \varepsilon) \right) \right|^2
\]

\[
\leq \left( |u(y)| - \left| u \left( \frac{l(y + \tau + \varepsilon)}{l + 2\varepsilon} \right) \right| \right) \left( |u(y)| + \left| u \left( \frac{l(y + \tau + \varepsilon)}{l + 2\varepsilon} \right) \right| \right)
\]

\[
\leq 2 \sqrt{|I|} \left| y - \frac{l}{l + 2\varepsilon} (y + \tau + \varepsilon) \right|^\frac{1}{2} \|u'\|_{L^2}^2
\]

\[
= 2 \sqrt{\frac{l}{l + 2\varepsilon}} \left| 2\varepsilon y - l\varepsilon \right|^\frac{1}{2} \|u'\|_{L^2}^2
\]

\[
\leq 4 \sqrt{\frac{l}{l + 2\varepsilon}} \|u'\|_{L^2}^2 \leq 4 \sqrt{\varepsilon} \|u'\|_{L^2}^2.
\]
Hence
\[ \left| \int_{\mathcal{I}} |u(y)|^2 \, d\nu(y) - \int_{I_\varepsilon} |u_\varepsilon(t)|^2 \, d\nu_\varepsilon(t) \right| \leq 4\sqrt{l}\sqrt{\varepsilon} \|u'\|_{L^2}\nu(\mathcal{I}). \]

This combined with (3.5), (3.6), and (3.7) implies
\[ \int_{I} |u(y)|^2 \, d\nu(y) \leq \int_{I_\varepsilon} |u_\varepsilon(t)|^2 \, d\nu_\varepsilon(t) + 4\sqrt{l}\sqrt{\varepsilon}\nu'(\mathcal{I}) \int_{\mathcal{I}} |u'(x)|^2 \, dx. \]

It is now left to take \( \varepsilon > 0 \) such that
\[ \frac{l}{n^2} + 4\sqrt{l}\sqrt{\varepsilon} \leq \frac{l}{n^2}, \]
i.e.
\[ \varepsilon \leq \left( \frac{c - 1}{4n^2} \right)^2 l. \]

\[ \square \]

**Lemma 3.6.** For every \( y \in \mathbb{R}_+ \), there exists \( c > 1 \) such that
\[ \lceil cy \rceil - 1 \leq y, \]
where \( \lceil x \rceil \) is the smallest integer not less than \( x \).

**Proof.** Case 1: Suppose \( y \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \). Then there exists \( l \in \mathbb{Z}_+ \) such that
\[ l < y < l + 1. \]
Take \( c > 1 \) such that
\[ l < cy < l + 1. \]
Then
\[ \lceil cy \rceil - 1 = l + 1 - 1 = l < y. \]

Case 2: Suppose \( y \in \mathbb{Z}_+ \). Take \( c > 1 \) such that
\[ cy < y + 1. \]
Then
\[ \lceil cy \rceil - 1 = y + 1 - 1 = y. \]

\[ \square \]

We will need the following estimate. For any \( 0 \leq a < b \) and \( u \in W^1_2([a, b]) \),
\[ \frac{|u(x)|^2}{|x|} \leq C(\kappa) \left( \int_a^b |u'(t)|^2 \, dt + \kappa \int_a^b \frac{|u(t)|^2}{|t|^2} \, dt \right), \quad \forall x \in [a, b], \tag{3.8} \]
where
\[ C(\kappa) = \frac{1}{2\kappa} \left( 1 + \sqrt{1 + 4\kappa} \frac{b^{\sqrt{1 + 4\kappa}} + a^{\sqrt{1 + 4\kappa}}}{b^{\sqrt{1 + 4\kappa}} - a^{\sqrt{1 + 4\kappa}}} \right) \tag{3.9} \]
(see [9, Appendix A]). In the case \( a = 0 \), one should take \( x > 0 \) and assume that \( u(0) = 0 \), since otherwise the right-hand side of the above inequality is infinite.
4. Proof of Theorem 2.1

Let

\[ X := W^1_2(\mathbb{R}), \quad X_0 := \{ u \in X : u(0) = 0 \}, \]
\[ X_1 := \left\{ u \in W^1_{2,loc}(\mathbb{R}) : u(0) = 0, \int_{\mathbb{R}} |u'(x)|^2 \, dx < \infty \right\}. \]

Then, \( \dim(X/X_0) = 1 \) and \( X_0 \subset X_1 \). Let \( \mathcal{E}_{X,\mu} \), \( \mathcal{E}_{X_0,\mu} \), and \( \mathcal{E}_{X_1,\mu} \) denote the forms on the domains \( X \cap L^2(\mathbb{R}, d\mu) \), \( X_0 \cap L^2(\mathbb{R}, d\mu) \) and \( X \cap L^2(\mathbb{R}, d\mu) \) respectively. Then

\[ N_-(\mathcal{E}_{\mathbb{R},\mu}) = N_-(\mathcal{E}_{X,\mu}) \leq N_-(\mathcal{E}_{X_0,\mu}) + 1 \leq N_-(\mathcal{E}_{X_1,\mu}) + 1 \quad (4.1) \]

(see (2.1)). An estimate for the right hand of (4.1) is presented in [9] (see also [11]) for the case when \( \mu \) is absolutely continuous with respect to the Lebesgue measure. We follow a similar argument. It follows from Hardy’s inequality (see, e.g., [6, Theorem 327]) that

\[
\int_{\mathbb{R}} |u'(x)|^2 \, dx + \kappa \int_{\mathbb{R}} \frac{|u(x)|^2}{\lambda} \, dx \leq \int_{\mathbb{R}} |u'(x)|^2 \, dx + 4\kappa \int_{\mathbb{R}} |u'(x)|^2 \, dx
\]
\[
= (4\kappa + 1) \int_{\mathbb{R}} |u'(x)|^2 \, dx, \quad \forall u \in X_1, \quad \forall \kappa \geq 0.
\]

Hence

\[ N_-(\mathcal{E}_{X_1,\mu}) \leq N_-(\mathcal{E}_{\kappa,\mu}), \quad (4.2) \]

where

\[ \mathcal{E}_{\kappa,\mu}[u] := \int_{\mathbb{R}} |u'(x)|^2 \, dx + \kappa \int_{\mathbb{R}} \frac{|u(x)|^2}{\lambda} \, dx - (4\kappa + 1) \int_{\mathbb{R}} |u(x)|^2 \, d\mu(x), \]

\[ \text{Dom} (\mathcal{E}_{\kappa,\mu}) = X_1 \cap L^2(\mathbb{R}, d\mu). \]

It follows from (4.1) and (4.2) that

\[ N_-(\mathcal{E}_{\mathbb{R},\mu}) \leq N_-(\mathcal{E}_{\kappa,\mu}) + 1. \quad (4.3) \]

Let

\[ I_n := [2^{n-1}, 2^n], \quad n > 0, \quad I_0 := [-1, 1], \quad I_n := [-2^n, -2^{n-1}], \quad n < 0. \]

The variational principle (see (3.1)) implies

\[ N_-(\mathcal{E}_{\kappa,\mu}) \leq \sum_{n \in \mathbb{Z}} N_-(\mathcal{E}_{\kappa,\mu,n}), \quad (4.4) \]

where

\[ \mathcal{E}_{\kappa,\mu,n}[u] := \int_{I_n} |u'(x)|^2 \, dx + \kappa \int_{I_n} \frac{|u(x)|^2}{\lambda} \, dx - (4\kappa + 1) \int_{I_n} |u(x)|^2 \, d\mu(x), \]

\[ \text{Dom} (\mathcal{E}_{\kappa,\mu,n}) = W^1_2(I_n) \cap L^2(I_n, d\mu), \quad n \in \mathbb{Z} \setminus \{0\}, \]

\[ \text{Dom} (\mathcal{E}_{\kappa,\mu,0}) = \{ u \in W^1_2(I_0) : u(0) = 0 \} \cap L^2(I_0, d\mu). \]
Let \( n > 0 \). For any \( c > 1 \) and \( N \in \mathbb{N} \), by Lemma 3.5 there exists a subspace \( \mathcal{L}_N \in \text{Dom} (\mathcal{E}_{\kappa, \mu, n}) \) of co-dimension \( N \) such that
\[
\int_{I_n} |u(x)|^2 \, d\mu(x) \leq c \left( \frac{|I_n|}{N^2} \mu(I_n) \right) \int_{I_n} |u'(x)|^2 \, dx, \quad \forall u \in \mathcal{L}_N.
\]
If
\[
c(4\kappa + 1) \frac{|I_n|}{N^2} \mu(I_n) \leq 1,
\]
then \( \mathcal{E}_{\kappa, \mu, n}[u] \geq 0, \quad \forall u \in \mathcal{L}_N \), and \( N_- (\mathcal{E}_{\kappa, \mu, n}) \leq N \). Let
\[
\mathcal{A}_n := \int_{I_n} |x| \, d\mu(x), \quad n \neq 0, \quad \mathcal{A}_0 := \int_{I_0} d\mu(x).
\]
Since \( |I_n| \int_{I_n} d\mu(x) \leq \mathcal{A}_n, \quad n \neq 0 \), it follows from the above that
\[
c(4\kappa + 1) \mathcal{A}_n \leq N^2 \quad \Rightarrow \quad N_- (\mathcal{E}_{\kappa, \mu, n}) \leq N.
\]
Hence
\[
N_- (\mathcal{E}_{\kappa, \mu, n}) \leq \left\lceil \sqrt{c(4\kappa + 1) \mathcal{A}_n} \right\rceil, \quad \text{(4.5)}
\]
where \( \lceil \cdot \rceil \) denotes the ceiling function, i.e. \( \lceil a \rceil \) is the smallest integer not less than \( a \). Suppose \( \text{supp} \mu \cap I_n \neq \{2^{n-1}\} \), i.e., \( \mu|_{I_n} \neq \text{const} \delta_{2^{n-1}} \). Then
\[
|I_n| \int_{I_n} d\mu(x) < \mathcal{A}_n.
\]
Take \( c > 1 \) such that
\[
c|I_n| \int_{I_n} d\mu(x) \leq \mathcal{A}_n.
\]
Then applying Lemma 3.5 with this \( c \) implies
\[
N_- (\mathcal{E}_{\kappa, \mu, n}) \leq \left\lceil \sqrt{(4\kappa + 1) \mathcal{A}_n} \right\rceil. \quad \text{(4.6)}
\]
If \( \mu|_{I_n} = \text{const} \delta_{2^{n-1}} \neq 0 \), then
\[
\int_{I_n} |u(x)|^2 \, d\mu(x) = 0
\]
on the subspace of co-dimension one consisting of functions \( u \in W^1_2(I_n) \) such that \( u(2^{n-1}) = 0 \), and clearly (4.6) holds. Finally, if \( \mu|_{I_n} = 0 \), then (4.6) takes the form \( 0 \leq 0 \).
If \( \mu|_{I_n} \neq 0 \), the right-hand side of (4.6) is at least 1, so one cannot feed it straight into (4.4). One needs to find conditions under which \( N_- (\mathcal{E}_{\kappa, \mu, n}) = 0 \). By (3.8), we have that
\[
\int_{I_n} |u(x)|^2 \, d\mu(x) \leq C_0(\kappa) \int_{I_n} |x| \, d\mu(x) \left( \int_{I_n} |u'(x)|^2 \, dx + \kappa \int_{I_n} \frac{|u(x)|^2}{|x|^2} \, dx \right)
\]= \mathcal{A}_n C_0(\kappa) \left( \int_{I_n} |u'(x)|^2 \, dx + \kappa \int_{I_n} \frac{|u(x)|^2}{|x|^2} \, dx \right)
for all $u \in W^1_2(I_n)$, where

$$C_0(\kappa) = \frac{1}{2\kappa} \left( 1 + \sqrt{1 + 4\kappa} \frac{2\sqrt{1 + 4\kappa} + 1}{2\sqrt{1 + 4\kappa} - 1} \right)$$

(cf. (3.9)).

Hence $N_-(\mathcal{E}_{\kappa,\mu,n}) = 0$, i.e. $\mathcal{E}_{\kappa,\mu,n}[u] \geq 0$, provided $A_n \leq \Phi(\kappa)$, where

$$\Phi(\kappa) := \frac{1}{(4\kappa + 1)C_0(\kappa)} = \frac{2\kappa}{4\kappa + 1} \left( 1 + \sqrt{4\kappa + 1} \frac{2\sqrt{4\kappa + 1} + 1}{2\sqrt{4\kappa + 1} - 1} \right)^{-1}.$$

The above estimates for $N_-(\mathcal{E}_{\kappa,\mu,n})$ clearly hold for $n < 0$ as well, but the case $n = 0$ requires some changes. Since $u(0) = 0$ for any $u \in \text{Dom}(\mathcal{E}_{\kappa,\mu,0})$, one can use the same argument as the one leading to (4.5), but with two differences: a) $\mathcal{L}_N$ can be chosen to be of co-dimension $N - 1$, and b) $|I_0| \int_{I_0} d\mu(x) = 2A_0$. This gives the following analogue of (4.5)

$$N_-(\mathcal{E}_{\kappa,\mu,0}) \leq \left\lceil \sqrt{2c(4\kappa + 1)A_0} \right\rceil - 1.$$

for any $c > 1$. We can choose $c > 1$ such that

$$N_-(\mathcal{E}_{\kappa,\mu,0}) \leq \sqrt{2(4\kappa + 1)A_0}$$

(see Lemma 3.6). It is also easy to see that the implication $A_n \leq \Phi(\kappa) \implies N_-(\mathcal{E}_{\kappa,\mu,n}) = 0$ remains true for $n = 0$. Now it follows from (4.3) and (4.4) that

$$N_-(\mathcal{E}_{R,2\mu}) \leq 1 + \sum_{\{n \in \mathbb{Z} \setminus \{0\} : A_n > \Phi(\kappa)\}} \sqrt{(4\kappa + 1)A_n} + \sqrt{2(4\kappa + 1)A_0}, \quad (4.7)$$

and one can drop the last term if $A_0 \leq \Phi(\kappa)$. The presence of the parameter $\kappa$ in this estimate allows a degree of flexibility. In order to decrease the number of terms in the sum in the right-hand side, one should choose $\kappa$ in such a way that $\Phi(\kappa)$ is close to its maximum. A Mathematica calculation shows that the maximum is approximately 0.092 and is achieved at $\kappa \approx 1.559$. For values of $\kappa$ close to 1.559, one has

$$A_n > \Phi(\kappa) \implies \sqrt{(4\kappa + 1)A_n} > \sqrt{(4\kappa + 1)\Phi(\kappa)} \approx 0.816.$$

Since $[a] \leq 2a$ for $a \geq 1/2$, (4.7) implies

$$N_-(\mathcal{E}_{R,\mu}) \leq 1 + 2\sqrt{(4\kappa + 1)} \sum_{A_n > \Phi(\kappa)} \sqrt{A_n}$$

with $\kappa \approx 1.559$. Hence

$$N_-(\mathcal{E}_{R,\mu}) \leq 1 + 5.38 \sum_{\{n \in \mathbb{Z}, A_n > 0.092\}} \sqrt{A_n}.$$
References


1 Department of Mathematics, Mbarara University of Science and Technology, P.O BOX 1410, Mbarara, Uganda.
   Email address: mkaruhanga@must.ac.ug

2 Department of Mathematics, King’s College London, Strand, London, WC2R 2LS, UK.
   Email address: eugene.shargorodsky@kcl.ac.uk