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SINGULAR VALUES OF TWO-PARAMETER FAMILIES OF FUNCTIONS ARISING FROM APOSTOL-GENOCCI POLYNOMIALS OF HIGHER ORDER

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Abstract. The singular values of two-parameter families of functions arising from Apostol-Genocchi polynomials of higher order are investigated. For this purpose, we consider the functions $f_{\lambda,\mu}(z) = \lambda(\frac{e^z+1}{2z})^\mu$, $\mu > 0$, $z \in \mathbb{C}\backslash\{0\}$ and $g_{\lambda,\eta}(z) = \lambda(\frac{2z}{e^z+1})^\eta$, $\eta > 0$, $z \in \mathbb{C}$; $\lambda \in \mathbb{R}\backslash\{0\}$ in this paper. It is shown that the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ have infinitely many singular values. The critical values of $f_{\lambda,\mu}(z)$ lie in the exterior of the open disk and interior of the open disk, whereas the critical values of $g_{\lambda,\eta}(z)$ lie in the interior of the open disk and exterior of the open disk according to two different regions.

1. Introduction

The singular values are very important for the description of Julia sets, Fatou sets and other results in the complex dynamics of transcendental functions. The dynamics of entire and meromorphic functions with infinitely many bounded or unbounded singular values [2, 3, 4, 5, 11] are crucial to determine in comparison to that of function with finitely many singular values. The singular values of one parameter family of functions are found in [6, 7, 8, 9] and the singular values of two-parameter family of functions are investigated in [10]. The singular values of transcendental meromorphic functions were also discussed by Zheng [12].

A point z^* is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0, \infty) \to \hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. A function f is called critically bounded or functions of bounded type if the set of all singular values of f is bounded, otherwise it is called unbounded type.

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Our families of functions are associated to the Apostol-Genocchi polynomials of higher order α (real or complex parameter) in variable x are given in [1] as:

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}$$

For $x = 0$ and choosing $\lambda = 1$, we have

$$\left(\frac{2z}{e^z+1}\right)^{\alpha} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(0;1)\frac{z^n}{n!} \tag{1.1}$$

Now we consider the following two-parameter families by taking $\mu = -\alpha > 0$ for $\alpha < 0$ and $\eta = \alpha > 0$, $\alpha > 0$ in Equation 1.1:

$$\mathcal{F} = \left\{f_{\lambda,\mu}(z) = \lambda\left(\frac{e^z+1}{2z}\right)^{\mu} : \mu > 0, \lambda \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C}\backslash\{0\}\right\}$$

$$\mathcal{G} = \left\{g_{\lambda,\eta}(z) = \lambda\left(\frac{2z}{e^z+1}\right)^{\eta} : \eta > 0, \lambda \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C}\right\}$$

The functions $f_{\lambda,\mu} \in \mathcal{F}$ and $g_{\lambda,\eta} \in \mathcal{G}$ are transcendental meromorphic, neither even nor odd and not periodic. The function $f_{\lambda,\mu} \in \mathcal{F}$ has only one pole at $z = 0$ and the function $g_{\lambda,\eta} \in \mathcal{G}$ has infinitely many poles at $z = (2j+1)\pi i$, where j is an integer.

This paper is organized as follows: In Theorem 2.1, It is shown that the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ have infinitely many singular values. It is seen that, in Theorem 2.3, the functions $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in the right half plane except one real positive zero. In Theorem 2.4 and Theorem 2.5, it is found that the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ map two different regions exterior of the open disk and interior of the open disk centered at origin and vise versa respectively. The critical values of $f_{\lambda,\mu}(z)$ lie in the exterior of the open disk and interior of the open disk and the critical values of $g_{\lambda,\eta}(z)$ lie in the interior of the open disk, whereas exterior of the open disk according to two different regions, are described in Theorem 2.6 and Theorem 2.7 respectively.

2. SINGULAR VALUES OF $f_{\lambda,\mu} \in \mathcal{F}$ AND $g_{\lambda,\eta} \in \mathcal{G}$

In the following theorem, it is shown that the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ have infinitely many singular values.:

Theorem 2.1. *Let $f_{\lambda,\mu} \in \mathcal{F}$ and $g_{\lambda,\eta} \in \mathcal{G}$. Then, the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ have infinitely many singular values.*

Proof. For critical points of functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$, we have $f'_{\lambda,\mu}(z) = \lambda\mu(\frac{e^z+1}{2z})^{\mu-1}\frac{(z-1)e^z-1}{2z^2} = 0$ and $g'_{\lambda,\eta}(z) = \lambda\eta(\frac{2z}{e^z+1})^{\eta-1}\frac{2[(1-z)e^z+1]}{(e^z+1)^2} = 0$. Then, the critical points of $f_{\lambda,\mu}(z)$ are $z = (2j+1)\pi i$, where j is an integer, and solutions of the equation $(z-1)e^z - 1 = 0$; and the critical points of $g_{\lambda,\eta}(z)$ are $z = 0$ and solutions of the equation $(z-1)e^z - 1 = 0$. Now, we solve $(z-1)e^z - 1 = 0$ for

critical points of both $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$. Using real and imaginary parts of this equation, then

$$\frac{y}{\sin y} + e^{y\cot y - 1} = 0 \tag{2.1}$$

$$x = 1 - y\cot y \tag{2.2}$$

From Figure 1, it is observed that Equation (2.1) has infinitely many solutions, say $\{y_k\}_{k=-\infty,k\neq 0}^{k=\infty}$, since number of intersections increases when the size of interval increases on the horizontal axis.

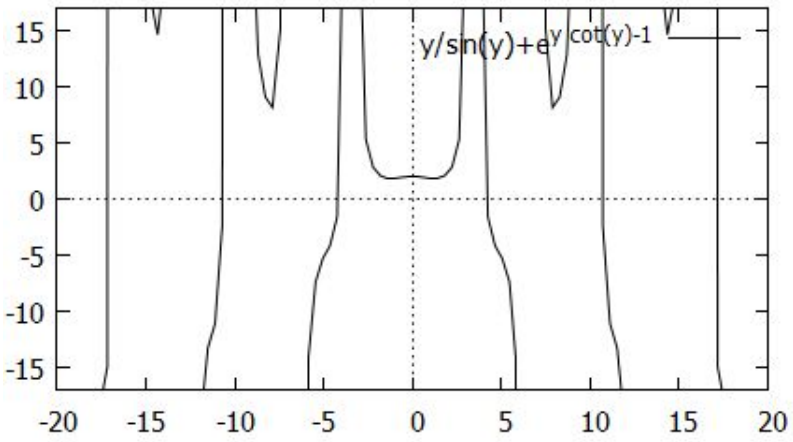


FIGURE 1. Graph of $\frac{y}{\sin y} + e^{y\cot y - 1}$

From Equation (2.2), $x_k = 1 - y_k \cot y_k$ for k nonzero integer. Then, $z_k = x_k + iy_k$ are critical points of $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$. The critical values of both functions $f_{\lambda,\mu}(z_k) = \lambda(\frac{e^{z_k}+1}{2z_k})^{\mu}$ and $g_{\lambda,\eta}(z_k) = \lambda(\frac{2z_k}{e^{z_k}+1})^{\eta}$ are distinct for k nonzero integer. It shows that the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ have infinitely many critical values.

Since $f_{\lambda,\mu}(z) \to 0$ as $z \to \infty$ along negative real axis and $g_{\lambda,\eta}(z) \to 0$ as $z \to \infty$ along positive real axis, it follows that the finite asymptotic value of $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ is 0.

This proves that the functions $f_{\lambda,\mu} \in \mathcal{F}$ and $g_{\lambda,\eta} \in \mathcal{G}$ have infinitely many singular values. □

Remark 2.2. It is easily seen that $f'_{\lambda,\mu}(z)$ has purely imaginary zeros at $(2j+1)\pi i$, where j is a nonzero integer, while $g'_{\lambda,\eta}(z)$ has no purely imaginary zeros.

Let $H^+ = \{z \in \hat{\mathbb{C}} : Re(z) > 0\}$ and $H^- = \{z \in \hat{\mathbb{C}} : Re(z) < 0\}$ be the right half plane and the left half plane respectively. The following theorem shows that the functions $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in the right half plane:

Theorem 2.3. *Let $f_{\lambda,\mu} \in \mathcal{F}$ and $g_{\lambda,\eta} \in \mathcal{G}$. Then, the functions $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in the right half plane H^+ except one real positive zero.*

Proof. Since $f'_{\lambda,\mu}(z) = \lambda\mu(\frac{e^z+1}{2z})^{\mu-1}\frac{(z-1)e^z-1}{2z^2} = 0$, the zeros of $f'_{\lambda,\mu}(z)$ are $z = (2j+1)\pi i$, where j is an integer, and solutions of the equation $e^{-z} = z - 1$.

Similarly, since $g'_{\lambda,\eta}(z) = \lambda\eta(\frac{2z}{e^z+1})^{\eta-1}\frac{2[(1-z)e^z+1]}{(e^z+1)^2} = 0$, the zeros of $g'_{\lambda,\eta}(z)$ are $z = 0$ and solutions of the equation $e^{-z} = z - 1$. Writing real and imaginary parts of $e^{-z} = z - 1$,

$$e^{-x}\cos y = x - 1 \tag{2.3}$$

$$e^{-x}\sin y = -y \tag{2.4}$$

When $y = 0$, then $z = x > 0$ and, by Equation (2.3), $e^{-x} = x - 1$. This equation has only one real positive solution.

When $y \neq 0$, then, by Equation (2.4), $\frac{\sin y}{y} = -e^x < -1$ for $x > 0$. This is not true for $y > 0$ since $|\frac{\sin y}{y}| < 1$. Moreover, since $\frac{\sin y}{y}$ is an even function, it is also false for $y < 0$.

Therefore, the functions $f'_{\lambda,\mu}(z)$ and $g'_{\lambda,\eta}(z)$ have no zeros in H^+ except one real positive zero. □

Suppose that the left half plane is divided in three regions $|z| < 1$, $1 \leq |z| < 2$ and $|z| \geq 2$. The following theorems prove that the functions $f_{\lambda,\mu}(z)$ and $g_{\lambda,\eta}(z)$ map two different regions exterior of the open disk and interior of the open disk centered at origin and vise versa respectively:

Theorem 2.4. *Let $f_{\lambda,\mu} \in \mathcal{F}$. Then, the function $f_{\lambda,\mu}(z)$ maps the left half plane H^-*

(i) *into the exterior of the open disk centered at origin and having radius $|\lambda|$ for $|z| < 1$.*
(ii) *into the interior of the open disk centered at origin and having radius $|\lambda|$ for $|z| \geq 2$.*

Proof. Suppose the function $h(z) = e^z$ for an arbitrary fixed $z \in H^-$ and the line segment γ is defined by $\gamma(t) = tz$, $t \in [0, 1]$. Then,

$$\int_\gamma h(z)dz = \int_0^1 h(\gamma(t))\gamma'(t)dt = z\int_0^1 e^{tz}dt = e^z - 1$$

$$|e^z + 1| = \left|\int_\gamma h(z)dz + 2\right| \tag{2.5}$$

Since $m \equiv \min_{t\in[0,1]}|h(\gamma(t))| = \min_{t\in[0,1]}|e^{tz}| > 0$ for $z \in H^-$, by Equation (2.5),

$$|e^z + 1| \geq m|z| + 2 > 2 > 2|z|$$

$$\left|\frac{e^z + 1}{2z}\right| > 1 \text{ for } |z| < 1.$$

It gives that, for $\mu > 0$,

$$|f_{\lambda,\mu}(z)| = \left|\lambda\left(\frac{e^z + 1}{2z}\right)^\mu\right| > |\lambda| \text{ for } |z| < 1.$$

It shows that $f_{\lambda,\mu}(z)$ maps H^- into the exterior of the open disk centered at origin and having radius $|\lambda|$ for $|z| < 1$.

Since $M \equiv \max_{t\in[0,1]} |h(\gamma(t))| = \max_{t\in[0,1]} |e^{tz}| < 1$ for $z \in H^-$, by Equation (2.5),

$$|e^z + 1| \leq M|z| + 2 < |z| + 2 < 2|z|$$
$$\left|\frac{e^z + 1}{2z}\right| < 1 \text{ for } |z| \geq 2.$$

It follows that, for $\mu > 0$,

$$|f_{\lambda,\mu}(z)| = \left|\lambda\left(\frac{e^z + 1}{2z}\right)^{\mu}\right| < |\lambda| \text{ for } |z| \geq 2.$$

The function $f_{\lambda,\mu}(z)$ maps H^- into the interior of the open disk centered at origin and having radius $|\lambda|$ for $|z| \geq 2$. □

Theorem 2.5. *Let $g_{\lambda,\eta} \in \mathcal{G}$. Then, the function $g_{\lambda,\eta}(z)$ maps the left half plane H^-*

(i) *into the interior of the open disk centered at origin and having radius $|\lambda|$ for $|z| < 1$.*
(ii) *into the exterior of the open disk centered at origin and having radius $|\lambda|$ for $|z| \geq 2$.*

The proof of this theorem can be obtained using similar arguments as Theorem 2.4.

The following theorems show that the critical values of $f_{\lambda,\mu} \in \mathcal{F}$ and $g_{\lambda,\eta} \in \mathcal{G}$ lie in the exterior of the open disk and interior of the open disk according to mapping of two regions and vise versa respectively:

Theorem 2.6. *Let $f_{\lambda,\mu} \in \mathcal{F}$. Then, the critical values of $f_{\lambda,\mu}(z)$ lie*

(i) *in the exterior of the open disk centered at origin and having radius $|\lambda|$ for $|z| < 1$.*
(ii) *in the interior of the open disk centered at origin and having radius $|\lambda|$ for $|z| \geq 2$.*

Proof. Using Theorem 2.3 and Remark 2.2, all the critical points of $f_{\lambda,\mu} \in \mathcal{F}$ lie in H^- except one real positive point and on imaginary axis at $(2j + 1)\pi i$. But $f_{\lambda,\mu}((2j + 1)\pi i) = 0$. Hence, by Theorem 2.4, $f_{\lambda,\mu} \in \mathcal{F}$ maps in H^- into the exterior of the open disk and interior of the open disk for different regions. The proof of theorem is completed. □

Theorem 2.7. *Let $g_{\lambda,\eta} \in \mathcal{G}$. Then, the critical values of $g_{\lambda,\eta}(z)$ lie in the*

(i) *in the interior of the open disk centered at origin and having radius $|\lambda|$ for $|z| < 1$.*
(ii) *in the exterior of the open disk centered at origin and having radius $|\lambda|$ for $|z| \geq 2$.*

The proof of this theorem can be obtained using similar arguments as Theorem 2.6.

3. CONCLUSION

In this paper, we have investigated the singular values of the two-parameter families of functions arising from Apostol-Genocchi polynomials of higher order. It was shown that both the families of functions have infinitely many singular values. Further, we have proved that the critical values of one family of functions lie in the exterior of the open disk and interior of the open disk, and other family of functions lie vise versa.

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