CURVILINEAR SUBSCHEMES OF SEGRE VARIETIES AND THE CACTUS RANK

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ABSTRACT. We study zero-dimensional linearly dependent subschemes $W$ of the Segre variety with $\deg(W) \leq 5$. If $W$ is connected and curvilinear with arbitrary degree we give a strong restriction on the number of factors of the concise Segre containing $W$.

1. Introduction

Our main aim is to study linearly dependent zero-dimensional subschemes $W$ with low degree contained in a Segre variety and linearly dependent. As a first crucial step one can classify the subschemes $W$ such that all proper subschemes of $W$ are linearly independent. With this restriction the case $\deg(W) \leq 3$ is obvious (Remark 5.2) and we prove the case $\deg(W) = 4$ (Theorem 5.1) and $\deg(W) = 5$, $W$ connected and curvilinear (Theorem 6.1). In [1, 2] we did the case in which $W$ is reduced, i.e. a finite set of points, up to the case $\#W = 6$ using [6]. As in [1, 2] we also introduce concepts which allow to measure the complexity of linearly dependent zero-dimensional subschemes of a Segre variety. We also prove several related results.

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k > 0$, $n_i > 0$ for all $i$, be a multiprojective space. Let $\nu : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^{k}(n_i + 1)$, be the Segre embedding. Set $X := \nu(Y)$. For any $q \in \mathbb{P}^r$ the tensor rank of $q$ or the $X$-rank $r_X(q)$ of $q$ is the minimal cardinality of a set $A \subset X$ such that $q \in \langle A \rangle$, where $\langle \cdot \rangle$ denote the linear span. The cactus rank $cr_X(q)$ of $q$ is the minimal degree of a zero-dimensional scheme $W \subset X$ such that $q \in \langle A \rangle$. Obviously $cr_X(q) \leq r_X(q)$. Let $S(Y, q)$ denote the set of all finite sets $S \subset Y$ such that $\#S = r_X(q)$ and $q \in \langle \nu(S) \rangle$. Let $Z(Y, q)$ denote the set of all zero-dimensional schemes $W \subset Y$ such that $\deg(W) = cr_X(q)$ and $q \in \langle \nu(W) \rangle$. A very useful result on the tensor rank is that if $Y' \subset Y$ is a multiprojective subspace and $q \in \langle \nu(Y') \rangle$, then $r_X(q) = r_{\nu(Y')}(q)$ and every $S \in S(Y, q)$ is contained in $Y'$ ([11, Proposition 3.1.3.1]).

Question 1.1. Does concision hold for the cactus rank of tensors and/or the cactus rank of homogeneous polynomials? Are all zero-dimensional schemes evincing the cactus rank of a tensor or of a homogeneous polynomial linearly independent?
As a justification for the cactus rank, see [7, 8, 9, 10].

As in [2] we consider the width of a multiprojective space and of a zero-dimensional subscheme of a multiprojective space. For any multiprojective subspace \( Y' \subseteq Y \) let \( w(Y') \) denote the number of positive-dimensional factors of \( Y' \). For instance, \( w(Y) = k \), because we assumed \( n_i > 0 \) for all \( i \). For any zero-dimensional scheme \( W \subset Y \) set \( w(W) = w(Y') \), where \( Y' \) is the minimal multiprojective subspace of \( Y \) containing \( W \). It is straightforward to compute \( Y' \) knowing \( W \) (Remark 3.1). We recall that a curvilinear zero-dimensional scheme of a smooth quasi-projective variety is a zero-dimensional scheme whose connected components are either reduced or with one-dimensional Zariski tangent space. We prove the following result in which \( \pi_i : Y \rightarrow \mathbb{P}^{n_i} \) is the projection of \( Y \) onto its \( i \)-th factor.

**Theorem 1.2.** Let \( W \) be a connected curvilinear scheme. Set \( o = (o_1, \ldots, o_k) := W_{\text{red}} \). Assume \( \deg(\pi_i^{-1}(o_i) \cap W) = 1 \) for all \( i \) and that \( \nu(W) \) is linearly dependent. Then \( w(W) \leq \deg(W) - 2 \).

The assumptions in Theorem 1.2 are very strong, but also its thesis is very strong (compare it to the thesis of [2, Theorem 1.1]).

## 2. Notation

Let \( M \) be an integral projective variety. For all integers \( t > 0 \) and \( a_1 \geq \cdots \geq a_t > 0 \) let \( A(M, a_1, \ldots, a_t) \) denote the set of all zero-dimensional and curvilinear schemes \( Z \subset M_{\text{reg}} \) with \( t \) connected components \( Z_1, \ldots, Z_t \) with \( \deg(Z_i) = a_i \) for all \( i \).

Let \( Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) be a multiprojective space. Fix \( i \in \{1, \ldots, k\} \). Set \( Y_i := \prod_{i \neq h} \mathbb{P}^{n_h} \). Let \( \nu_i \) denote the Segre embedding of the multiprojective space \( Y_i \). Let \( \pi_i: Y \rightarrow \mathbb{P}^{n_i} \), \( i = 1, \ldots, k \), denote the projection onto its \( i \)-th factor. Let \( \eta_i : Y \rightarrow Y_i \) be the surjection which forget the \( i \)-th coordinate of any \( p = (p_1, \ldots, p_k) \in Y \). For any \( i = 1, \ldots, k \) let \( \mathcal{O}_Y(\epsilon_i) \) (resp. \( \mathcal{O}_Y(\hat{\epsilon}_i) \)) denote the line bundle on \( Y \) with multidegree \((a_1, \ldots, a_k)\) with \( a_i = 1 \) and \( a_h = 0 \) for all \( h \neq i \) (resp. \( a_i = 0 \) and \( a_h = 1 \) for all \( h \neq i \)).

## 3. Preliminary observations and lemmas

**Remark 3.1.** Let \( W \subset Y \) be a zero-dimensional subscheme. The minimal multiprojective space containing \( W \) is the multiprojective subspace \( \prod_{i=1}^{k} \langle \pi_i(W) \rangle \), where \( \pi_i(W) \) denote the scheme-theoretic image.

**Remark 3.2.** Let \( E \subset Y \) be a zero-dimensional scheme such that \( \deg(E) \leq 3 \) and \( \dim \nu(E) \leq \deg(E) - 2 \). Since \( \nu \) is an embedding, we have \( \deg(E) = 3 \) and \( (\nu(E)) \) is a line. Since \( X \) is scheme-theoretical cut out by quadrics, we have \( (\nu(E)) \subset \nu(Y) \), i.e. there are \( i \in \{1, \ldots, k\} \) and \( o \in \mathbb{P}^{n_i} \) such that \( E \subset \eta_i^{-1}(o) \).

In particular \( \nu(A) \) is linearly independent for any degree 2 scheme \( A \subset Y \).

**Remark 3.3.** Assume \( \deg(A) = 3 \) and that \( A \in \mathcal{Z}(Y, q) \) for some tensor \( q \). Since \( A \) evinces the cactus rank of \( q \), it is not a so-called fat point or 2-point of a plane. Thus \( A \) is curvilinear. Either \( A \) is connected or \( A \) has two connected component,
one of degree 2 and one of degree 1, or it is reduced, i.e. the union of 3 different points. Since \( q \in \sigma_3(X) \setminus \sigma_2(X) \), in the latter case we have \( r_X(q) = 3 \). In all cases the linear space \( \langle \nu(A) \rangle \) is a plane not contained in \( X \), because \( q \notin X \) and any line \( L \subset \mathbb{P}^r \) with \( \deg(L \cap X) \geq 3 \) is contained in \( X \).

**Lemma 3.4.** Fix \( u \in \mathbb{P}^r \) and any \( E \in \mathcal{Z}(Y,u) \) with \( \deg(E) \leq 3 \). Then \( \dim(\nu(E)) = \deg(E) - 1 \) and \( \eta_i|E \) is an embedding for \( i = 1, \ldots, k \).

**Proof.** The first assertion is true for any degree 2 scheme (since \( \nu \) is an embedding) and it is true if \( \deg(E) = 3 \), because any line \( L \subset \mathbb{P}^r \) containing a degree 3 subscheme of \( X \) is contained in \( X \). Assume that \( \eta_i|E \) is not an embedding for some \( i \), say for \( i = k \). Since the scheme-theoretic fibers of \( \eta_k \) are projective schemes embedded as linear subspaces by \( \nu \), there is a degree 2 scheme \( v \subset E \) such that \( \eta_k(v) \) is a point and \( \langle \nu(v) \rangle \) is a line contained in \( X \). Since \( \deg(v) = 2 \) and \( E \) evinces a cactus rank, we have \( E \neq v \) and hence \( \deg(E) = 3 \). Thus \( \langle \nu(E) \rangle \) is a plane containing \( \nu(E) \) and the line \( \langle \nu(v) \rangle \subset X \). Since \( X \) is cut out by quadrics, either \( \langle \nu(E) \rangle \subset X \) (excluded because \( cr_X(u) = 3 \)) or \( X \cap \langle \nu(E) \rangle \) is the union of two different lines. Thus all \( u' \in \langle \nu(E) \rangle \) have \( r_X(u') \leq 2 \), contradicting the assumption \( cr_X(u) = 3 \). \( \square \)

**Lemma 3.5.** Let \( E \subset Y \) be a zero-dimensional scheme such that \( \deg(E) \leq 3 \). There is \( i \in \{1, \ldots, k\} \) such that \( h^1(I_E(\xi_i)) > 0 \) if and only if either there is \( v \subset E \) such that \( \deg(v) = 2 \) and \( \deg(\eta_i(v)) = 1 \) or \( \deg(E) = 3 \) and there is \( j \in \{1, \ldots, k\} \setminus \{i\} \) such that \( \deg(\pi_h(E)) = 1 \) for all \( h \in \{1, \ldots, k\} \setminus \{i,j\} \).

**Proof.** We have \( h^1(I_E(\xi_i)) > 0 \) if and only if either \( \eta_i|E \) is not an embedding or \( \deg(E) = \deg(\eta_i(E)) \) and \( h^1(Y_i, I_{\eta_i(E)}(1, \ldots, 1)) > 0 \). \( \eta_i|E \) is not an embedding if and only if there is \( v \subset E \) such that \( \deg(v) = 2 \) and \( \deg(\eta_i(v)) = 1 \). Now assume that \( \eta_i|E \) is an embedding, i.e. assume \( \deg(\eta_i(E)) = \deg(E) \). Since \( \eta_i|E \) is an embedding, we have \( h^1(I_E(\xi_i)) = h^1(Y_i, I_{\eta_i(E)}(1, \ldots, 1)) \). Since \( \mathcal{O}_{Y_i}(1, \ldots, 1) \) is very ample, the structure of lines on the Segre variety \( \nu_i(Y_i) \) gives that \( h^1(Y_i, I_{\eta_i(E)}(1, \ldots, 1)) > 0 \) if and only if there is \( j \in \{1, \ldots, k\} \setminus \{i\} \) such that \( \deg(\pi_h(E)) = 1 \) for all \( h \in \{1, \ldots, k\} \setminus \{i,j\} \). \( \square \)

4. Cactus rank for matrices

In this section we consider the case \( k = 2 \), i.e. \( Y = \mathbb{P}^m \times \mathbb{P}^n \), \( m > 0 \), \( n > 0 \).

**Proposition 4.1.** Assume \( k = 2 \), i.e. assume \( Y = \mathbb{P}^m \times \mathbb{P}^n \) for some \( m > 0 \), \( n > 0 \). Fix \( q \in \mathbb{P}^r \) and set \( a := r_X(q) \). Let \( Y' \subset Y \) be the minimal multiprojective subspace of \( Y \) such that \( q \in \langle \nu(Y') \rangle \) Then:

(i) \( cr_{\nu(Y)}(q) = cr_{\nu(Y')}\rangle(q) = a \) and \( \mathcal{Z}(Y,q) = \mathcal{Z}(Y',q) \).

(ii) For all integers \( t > 0 \) and \( a_1 \geq \cdots \geq a_t > 0 \) such that \( a_1 + \cdots + a_t = a \) we have \( \mathcal{Z}(Y',a_1, \ldots, a_t) \cap \mathcal{Z}(Y',q) \neq \emptyset \).

**Proof.** By the classification of matrices we have \( Y' \cong \mathbb{P}^{a_1-1} \times \mathbb{P}^{a_2-1} \), all matrices with rank \( a \) are equivalent and \( \mathcal{S}(Y,q) = \mathcal{S}(Y',q) \). Thus \( cr_{\nu(Y)}(q) \leq cr_{\nu(Y')}\rangle(q) \leq a \). Take a zero-dimensional scheme \( Z \subset Y \) such that \( q \in \langle \nu(Z) \rangle \). Since \( \nu(Y') \) is the minimum (not just minimal, by concision for the tensor decomposition)
multiprojective space with \( q \in \langle \nu(Y') \rangle \), we have \( \langle \nu(Z) \rangle \supseteq \langle \pi_i(Y') \rangle \) for all \( i = 1, \ldots, k \). Thus \( c_{r_\nu(Y)}(q) \geq a \), concluding the proof of all assertions of part (i).

Fix integers \( t > 0 \) and \( a_1 \geq \cdots \geq a_t > 0 \). Fix \( t \) distinct points \( p_1, \ldots, p_t \in \mathbb{P}^1 \). Fix a basis of \( |O_{\mathbb{P}^1}(a - 1)| \) and use it to define an embedding \( f_1 : \mathbb{P}^1 \to \mathbb{P}^{a - 1} \) with a rational normal curve as its image. Since \( f_1(\mathbb{P}^1) \) is a rational normal curve and \( a_1 + \cdots + a_t = a \), \( f_1(a_1p_1 + \cdots + a_tp_t) \) is a linearly independent degree \( a \) zero-dimensional scheme spanning \( \mathbb{P}^{a - 1} \). Let \( f : \mathbb{P}^1 \to Y' = \mathbb{P}^{a - 1} \times \mathbb{P}^{a - 1} \) be the morphism \( f = (f_1, f_1) \). The scheme \( Z := f(a_1p_1 + \cdots + a_tp_t) \) is an element of \( A(Y', a_1, \ldots, a_t) \subseteq A(Y, a_1, \ldots, a_t) \) and its images by the projections onto the factors of \( Y' \) span the factors. Thus \( Y' \) is the minimal multiprojective space containing \( Z \). Take a general \( q' \in \langle \nu(Z) \rangle \). Since \( Z \subset Y' \), \( r_X(q') \leq a \). Since all \( m \times n \) matrices with rank \( a \) are projectively equivalent, to prove part (ii) it is sufficient to prove \( r_{\nu(Y')}(q') \geq a \). Assume \( r_{\nu(Y')}(q') \leq a - 1 \) and take \( A \subset Y \) such that \( \#A \leq a - 1 \) and \( q' \in \langle \nu(A) \rangle \). We identify \( Y' \) with \( \mathbb{P}^{a - 1} \times \mathbb{P}^{a - 1} \). Set \( B := A \cup Z \). Since \( Z \) is curvilinear, it has only finitely many proper subschemes. Since \( q' \) is general in \( \langle \nu(Z) \rangle \) and \( \nu(Z) \) is linearly independent, there is no \( Z' \subseteq Z \) such that \( q' \in \langle \nu(Z') \rangle \). In particular \( A \not\subseteq Z \). Since \( q' \in \langle \nu(A) \rangle \cap \langle \nu(Z) \rangle \), we have \( h^1(I_B(1, 1)) > 0 \). Since \( \#A \leq a - 1 \), there is \( H \in |O_{|Y'}(1, 0)| \) containing \( A \). Since \( \deg(H \cap \mathbb{P}^{a - 1} \cap f(\mathbb{P}^1)) = a - 1 < \deg(Z) \), we have \( \text{Res}_H(B) \neq 0 \). Thus \( h^1(I_{\text{Res}_H(B)}(0, 1)) > 0 \) ([5, Lemma 5.1]). This is absurd, because \( \text{Res}_H(B) \subseteq Z \), \( \pi_{2|Z} \) is an embedding and \( \pi_2(Z) \) is linearly independent. \( \square \)

5. \( \deg(W) \leq 4 \)

The aim of this section is the proof of the following result.

**Theorem 5.1.** Let \( W \subset Y \) be a zero-dimensional scheme such that \( \deg(W) = 4 \), \( Y \) is the minimal multiprojective space containing \( W \), \( \nu(W) \) is linearly dependent and \( \nu(W') \) is linearly independent for all \( W' \subsetneq W \). Then either \( k = 1 \) and \( n_1 = 2 \) or \( k = 2 \), \( n_1 = n_2 = 1 \) and \( W \) is as in one of the Examples 5.3 and 5.4.

**Remark 5.2.** Let \( W \subset B \) be a zero-dimensional scheme such that \( \deg(W) \leq 3 \) and \( \nu(W) \) is linearly dependent. Since \( \nu \) is an embedding, \( \deg(W) = 3 \) and \( \langle \nu(W) \rangle \) is a line. Since \( \nu(Y) \) is scheme-theoretically cut out by quadrics and \( W \subseteq \langle \nu(E) \rangle \cap \nu(Y) \), then \( \langle \nu(E) \rangle \subset Y \). The structure of linear subspaces of a Segre variety shows the existence of \( i \in \{1, \ldots, k\} \) such that \( \deg(\pi_h(W)) = 1 \) for all \( h \neq i \), \( \pi_{i|W} \) is an embedding and \( \pi_i(W) \subseteq \mathbb{P}^{n_i} \) is a line.

**Example 5.3.** Let \( W \subset \mathbb{P}^n \) be a zero-dimensional scheme such that \( \deg(W) = 4 \), \( W \) spans \( \mathbb{P}^a \), \( W \) is linearly dependent, but all proper subschemes are linearly independent. Obviously \( n = 2 \). Since \( h^0(O_{\mathbb{P}^2}(2)) = 6 \) and \( \deg(L \cap W) \leq 3 \) for all lines \( L \subset \mathbb{P}^2 \), \( \dim |I_{W}(2)| = 2 \) and either \( W \) is the complete intersection of 2 conics or \( \deg(L \cap W) = 3 \) for exactly one line \( L \) and there is \( o \in \mathbb{P}^2 \) such that all \( E \in |I_{W}(2)| \) is of the form \( E = L \cup R \) with \( R \in |I_o(1)| \). In both cases there are non-curvilinear \( W \)'s. In the latter case the latter case \( o \in L \) and \( W \) is the union of the fat point \( 2o \) and some \( p \in L \setminus \{o\} \).

**Example 5.4.** Let \( W \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a degree 4 scheme. Since \( h^0(O_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)) = 4 \), \( h^1(I_{W}(1, 1)) > 0 \) if and only if there is \( D \in |O_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)| \) containing \( W \). If \( \nu(W') \)
is linearly independent for all \( W' \subsetneq W \), then \( D \) is unique, because we have \( h^1(\mathcal{I}_W(1, 1)) = 1 \) in this case.

**Lemma 5.5.** Let \( W \subset Y \) be a zero-dimensional scheme such that \( \deg(W) = 4 \), \( Y \) is the minimal multiprojective space containing \( W \), \( \nu(W) \) is linearly dependent, \( \nu(W') \) is linearly independent for all \( W' \subsetneq W \) and \( \#W_{\text{red}} = 3 \). Then either \( k = 1 \) and \( n_1 = 2 \) or \( k = 2 \) and \( n_1 = n_2 = 1 \).

*Proof.* Since the case \( k = 1 \) is obvious, we may assume \( k \geq 2 \). Write \( W = v \cup \{a, b\} \) with \( v \) connected and \( \deg(v) = 2 \).

(a) Assume \( n_i \geq 2 \) for some \( i \), say \( n_1 \geq 2 \). Thus there is \( H \in |\mathcal{I}_O(\epsilon_1)| \). Since \( Y \) is the minimal multiprojective space containing \( W \), \( W \not\subset H \) and hence \( h^1(\mathcal{I}_{H \cap W}(1, \ldots, 1)) = 0 \). The residual exact sequence of \( H \) gives the inequality \( h^1(\mathcal{I}_{\text{Res}_H(W)}(\epsilon_1)) > 0 \). Thus \( H \cap W = v \) and \( \eta_1(a) = \eta_1(b) \). Thus there is \( D \in |\mathcal{O}_Y(\epsilon_2)| \) containing \( \{a, b\} \). The residual exact sequence of \( D \) gives \( D \cap W = \{a, b\} \) and \( \deg(\eta_2(v)) = 1 \). Thus there is \( M \in |\mathcal{O}_Y(\epsilon_1)| \) with \( M \supseteq \{v, a\} \). The residual exact sequence of \( M \) gives a contradiction.

(b) Now assume \( k \geq 3 \) and \( n_i = 1 \) for all \( i \). Take \( H \in |\mathcal{O}_Y(\epsilon_1)| \) with \( H \cap W \neq \emptyset \). Easy modifications of the proof of step (a) gives \( \deg(H \cap W) = 1 \). Take \( M \in |\mathcal{O}_Y(\epsilon_2)| \) with \( M \cap \text{Res}_H(W) \neq \emptyset \). As in step (a) we get \( \deg(\text{Res}_{H \cap M}(W)) = 2 \) and \( h^1(\mathcal{I}_{\text{Res}_{H \cap M}(W)}(0, 0, 1, \ldots, 1)) > 0 \). Thus there is \( H' \in |\mathcal{O}_Y(\epsilon_3)| \) such that \( \deg(H' \cap W) \geq 2 \). Using \( H' \) instead of \( H \) we get a contradiction. \( \square \)

**Lemma 5.6.** Let \( W \subset Y \) be a zero-dimensional scheme such that \( \deg(W) = 4 \), \( Y \) is the minimal multiprojective space containing \( W \), \( \nu(W) \) is linearly dependent and \( W \) has a connected component of degree 3 which is not curvilinear. Then there is \( W' \subsetneq W \) such that \( \nu(W') \) is linearly dependent.

*Proof.* Call \( A \) the degree 3 connected component of \( W \) and write \( W = A \cup \{a\} \). Set \( \{o\} : = A_{\text{red}} \). Since \( A \) is not curvilinear there is a 2-dimensional linear subspace \( U \) of the tangent space \( T_Y \) (a vector space of dimension \( n_1 + \cdots + n_k \) such that \( A = (2o, U) \) is the 2-fat point of \( U \) with \( o \) as its reduction. Since \( \nu \) is an embedding, \( \dim(\nu(A)) = 2 \). Since \( A = (2o, U) \), the line \( \langle \nu(o), \nu(a) \rangle \) has degree 3 intersection with \( \nu(W) \). Thus \( \nu(W) \) has a degree 3 linearly dependent subscheme. \( \square \)

**Lemma 5.7.** Let \( E \subset Y \) be a degree 3 curvilinear scheme and \( o \in Y \setminus E_{\text{red}} \). Assume \( h^1(\mathcal{I}_E(1, \ldots, 1)) = 0 \), \( h^1(\mathcal{I}_{E \cup \{o\}}(1, \ldots, 1)) > 0 \) and that \( Y \) is the minimal multiprojective space containing \( E \cup \{o\} \). Then either \( k = 1 \) or \( k = 2 \) and \( n_1 = n_2 = 1 \).

*Proof.* Since \( h^1(\mathcal{I}_E(1, \ldots, 1)) = 0 \), \( \langle \nu(E) \rangle \) is a plane containing \( \nu(o) \). Take \( H \in |\mathcal{O}_Y(\epsilon_1)| \) containing \( o \). By Lemma 3.5 either there is \( v \subset \text{Res}_H(E) \) with \( \deg(v) = 2 \) and \( \deg(\eta_i(v)) = 1 \) or there is \( i > 1 \) such that \( \deg(\pi_h(E)) = 1 \) for all \( h \notin \{1, i\} \). In the second case the minimal multiprojective space \( Y' \) containing \( E \) is contained in a multiprojective space isomorphic to \( \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \). Since \( \nu(o) \in \langle \nu(Y') \rangle \), concision for the tensor decomposition ([11, Proposition 3.1.3.1]) applied to the tensor decomposition of \( \nu(o) \) gives \( Y' = Y \). Hence \( k = 2 \) and \( n_1 \leq 2 \) and \( n_2 \leq 2 \). Assume \( (n_1, n_2) \neq (1, 1) \), say \( n_1 = 2 \). Take \( H \in |\mathcal{I}_o(\epsilon_1)| \). Since \( E \not\subset H \), [5,
5.1] gives \( h^1(\mathcal{I}_{\text{Res}_H}(1,0)) > 0 \) and hence \( h^1(\mathcal{I}_E(1,0)) \). The definition of \( Y' \) gives \( \langle \pi_1(E) \rangle = \mathbb{P}^2 \). Thus \( h^1(\mathcal{I}_E(1,0)) = 0 \), a contradiction. \( \square \)

As in Lemma 5.5 we get the following lemma.

**Lemma 5.8.** Take \( v, w \subset Y \) degree 2 connected zero-dimensional scheme such that \( v \cap w = \emptyset \), \( \nu(v \cup w) \) is linearly dependent and \( \nu(W') \) is linearly independent for all \( W' \subsetneq v \cup w \). Then the minimal multiprojective space \( Y' \) containing \( v \cup w \) is either \( \mathbb{P}^m \), \( m \leq 2 \), or \( \mathbb{P}^1 \times \mathbb{P}^1 \).

**Example 5.9.** Take \( Y = \mathbb{P}^1 \times \mathbb{P}^1 \). Fix \( o \in Y \) and set \( \{D_i\} := [\mathcal{I}_o(\epsilon_i)], i = 1, 2 \), and \( T := D_1 \cup D_2 \). In the plane \( M := \langle \nu(T) \rangle \) take any conic \( E \subset M \) such that \( \nu(o) \in \text{Sing}(E) \), \( \nu(D_1) \not\subset E \) and \( \nu(D_2) \not\subset E \). Thus \( B := E \cap \nu(T) \) degree 4 zero-dimensional scheme, which is the complete intersection of two conics. Thus \( h^0(M, \mathcal{I}_W(2)) = 2 \), i.e. the scheme \( B \) is the scheme-theoretic base locus of the pencils of all conics of \( M \) singular at \( \nu(o) \). Thus \( B \) is independent of the choice of \( T \) and \( E \). Since \( B \subset \nu(T) \), there is a unique scheme \( W \subset T \subset Y \) such that \( B = \nu(W) \). Since \( \deg(W) = 4 \) and \( W \subset T \), \( \nu(W) \) is linearly dependent. Since \( B \) is the complete intersection of 2 conics singular at \( 4\nu(o) \) the connected degree 4 schemes \( B \) and \( W \) are Gorenstein, i.e. there are unique \( W' \subset W \) and \( B' \subset B \) such that \( \deg(B') = \deg(W') = 3 \). The uniqueness of \( B' \) gives \( \nu(W') = B' \).

Since \( B \) is the complete intersection of 2 conics of \( M \), \( \deg(B) \cap D_i \geq 2 \) for \( i = 1, 2 \), i.e. \( B \) contains the first infinitesimal neighborhood \( 2\nu(o) \) of \( \nu(o) \) in \( M \). Since \( \deg(\nu(o)) = 3 \), \( 2n(o) = B' \). Since \( 2\nu(o) \) spans \( M \), \( B' \) is linearly independent. Thus each \( \nu(W'') \), \( W'' \subset W \), is linearly independent. The construction shows that \( W \) is the only connected and not curvilinear degree 4 zero-dimensional scheme \( Z \) such that \( Z_{\text{red}} = \{o\} \), \( \nu(Z) \) is linearly dependent and each \( \nu(Z') \), \( Z' \subset Z \), is linearly independent.

**Lemma 5.10.** Let \( W \subset Y \) be a degree 4 connected and not curvilinear subscheme such that \( \nu(W) \) is linearly dependent, while all \( \nu(W') \), \( W' \subset W \) are linearly independent. Assume that \( Y \) is the minimal multiprojective subspace containing \( W \). Then \( k \leq 2 \) and \( k = 2 \) if and only if \( n_1 = n_2 = 1 \) and \( W \) is as in Example 5.9.

**Proof.** Set \( \{o\} := W_{\text{red}} \). Write \( o = (o_1, \ldots, o_k) \). For each \( i = 1, \ldots, k \) set \( L_i := \eta_i^{-1}(o_i) \). Note that \( \nu(L_i) \) is an \( n_i \)-dimensional projective space and that \( \nu(L_1), \ldots, \nu(L_k) \) are the \( k \) maximal linear subspaces of the Segre variety \( \nu(Y) \).

By assumption \( \langle \nu(W) \rangle \) is a plane and hence the Zariski tangent space \( T_oW \subset T_oY \) has dimension \( \leq 2 \). Since \( W \) is not curvilinear, its Zariski tangent space has dimension \( > 1 \). Thus \( \dim T_oW = 2 \). Thus \( W \) contains the 2-fat point of \( o \) in \( T_oW \), i.e. the degree 3 zero-dimensional scheme \( (2o, T_oW) \subset Y \cap T_oY \). This implies \( \deg(D \cap W) \geq 2 \) for every effective divisor \( D \) of \( Y \) containing \( o \).

Assume \( k \geq 2 \) and take \( H \in [\mathcal{I}_o(\epsilon_k)] \). Thus \( \deg(H \cap W) \geq 2 \). Since \( Y \) is the minimal multiprojective space containing \( W \), \( W \not\subset H \). Thus \( 2 \leq \deg(H \cap W) \leq 3 \).

(a) Assume \( \deg(H \cap W) = 3 \). If \( \langle \nu(W) \rangle \subset \nu(Y) \), then \( k = 1 \), a contradiction. Thus \( \langle \nu(W) \rangle \not\subset \nu(Y) \). Since \( \nu(Y) \) is scheme-theoretically cut out by quadric, either \( \langle \nu(W) \rangle \cap \nu(Y) \) is a conic or it is a complete intersection of of two conics.
If the intersection is a conic, then \( cr_X(q) = 2 \) and \( Z(Y, q) \) is infinite for a general \( q \in \nu(W) \). Lemma 5.8 gives \( k = 2 \) and \( n_1 = n_2 = 1 \). The last sentence of Example 5.9 gives that \( W \) is as in Example 5.9.

(b) Assume \( \deg(H \cap W) = 2 \). We have \( h^1(\mathcal{I}_{H \cap W}(1, \ldots, 1)) = 0 \), because \( \nu \) is an embedding. Using the residual exact sequence of \( H \) we obtain the inequality \( h^1(\mathcal{I}_{Res_H(W)}(\hat{e}_k)) > 0 \). Since \( \deg(Res_H(W)) = 2 \), Lemma 3.5 gives \( \deg(\eta_i(Res_H(W))) = 1 \). Since \( Res_H(W) \subset W \), the see that \( \deg(W \cap L_k) \geq 2 \). Using \( i \in \{1, \ldots, k\} \) instead of \( K \) we get \( \deg(W \cap L_i) \geq 2 \) for all \( i \). Since the linear spaces \( \nu(L_i) \) are as general as possible in the \( (n_1 + \cdots + n_k) \)-dimensional Zariski tangent space \( T_{\nu(o)}\nu(Y) \) with the only restriction that they contain \( \nu(o) \) and have prescribed dimensions whose sum is \( \dim T_{\nu(o)}\nu(Y) \), \( W \) has embedding dimensions at least \( k \). Thus \( k = 2 \). Since \( \deg(\eta_i(W)) = 2 \) for all \( i \), we have \( \deg(\pi_h(W)) \leq 2 \) for all \( h \). Since \( Y \) is the minimal multiprojective space containing \( W \), \( n_1 = 1 \) for all \( i \). The last sentence of Example 5.9 gives that \( W \) is as in Example 5.9. □

**Lemma 5.11.** Let \( W \subset Y \) be a connected and curvilinear scheme such that \( \deg(W) = 4 \), \( Y \) is the minimal multiprojective space containing \( W \) and all \( \nu(W') \), \( W' \subsetneq W \) are linearly dependent. The scheme \( \nu(W) \) is linearly dependent if and only if either \( k = 1 \) and \( n_1 = 2 \) or \( k = 2 \) and \( n_1 = n_2 = 1 \).

**Proof.** Since the case \( k = 1 \) is obvious, we assume \( k \geq 2 \). Call \( i_1 \) the integer \( j \in \{1, \ldots, k\} \) such that there is \( D_1 \in \{\mathcal{O}_Y(\epsilon_{i_1})\} \) with \( w_1 := \deg(D \cap W) \) maximal. Up to a permutation of the factors of \( W \) we may assume \( i_1 = 1 \). Note that \( D_1 \cap W = W_{e_1} \) and \( Res_{D_1}(W) = W_{4-e_1} \). If \( e_1 < 4 \) by the minimality \( f Y \). Let \( i_2 \) be the integer \( j \in \{2, \ldots, k\} \) such that there is \( D_2 \in \{\mathcal{O}_Y(\epsilon_{i_2})\} \) with \( e_2 : \deg(D_2 \cap W_{4-e_1}) \) maximal. Permuting the last \( k - 1 \) factors of \( Y \) we may assume \( i_2 = 2 \). Since \( W_{4-e_1} \subset W \), the maximality property of \( i_1 \) implies \( e_1 \geq e_2 \).

(a) Assume \( e_1 + e_2 = 4 \). Thus either \( e_1 = 3 \) and \( e_2 = 1 \) or \( e_1 = e_2 = 2 \).

(a1) Assume \( e_1 = 3 \) and \( e_2 = 1 \). Since \( e_1 < 4 \), \( h^1(\mathcal{I}_{W \cap D_1}(1, \ldots, 1)) = 0 \). The residual exact sequence of \( D_1 \) gives \( h^1(\mathcal{I}_{W_{e_1}}(\hat{e}_1)) > 0 \). Since \( W(1) = \{\emptyset\} \) and \( \mathcal{O}_Y(\hat{e}_1) \) is globally generated, \( h^1(\mathcal{I}_{W_{e_1}}(\hat{e}_1)) = 0 \), a contradiction. (a) Assume \( e_1 = e_2 = 2. e_1 < 4 \), \( h^1(\mathcal{I}_{W \cap D_1}(1, \ldots, 1)) = 0 \). The residual exact sequence of \( D_1 \) gives \( h^1(\mathcal{I}_{W_{e_1}}(\hat{e}_1)) > 0 \). Lemma 3.5 gives \( \deg(\eta_i(W_2) = 0 \), i.e. \( \pi_{ij}W_2 \) is constant for all \( i > 1 \). Since \( e_2 = e_1 \), we may first use \( D_2 \) and get \( h^1(\mathcal{I}_{W_{e_1}}(\hat{e}_1)) > 0 \). Thus \( \deg(\mathcal{I}_{W_2}(\hat{e}_2)) = 1 \) for all \( h \neq 2 \). Thus the inclusion \( W_2 \subset Y \) is not an embedding by the universal property of the product, a contradiction.

(b) Assume \( e_1 + e_2 < 4 \). Let \( i_3 \) be the integer \( j \in \{3, \ldots, k\} \) such that there is \( D_3 \in \{\mathcal{O}_Y(\epsilon_{i_3})\} \) with \( e_3 : \deg(D_3 \cap W(4 - e_1 - e_2) \) maximal. Permuting the last \( k - 1 \) factors of \( Y \) we may assume \( i_3 = 3 \). Either \( e_1 = 2 \) and \( e_2 = e_3 = 1 \) or \( e_1 = e_2 = e_3 = 1 \). First assume \( e_1 = 2 \). Since \( (D_1 \cup D_2) \cap W = W(3) \), we have \( h^1(\mathcal{I}_{W \cap (D_1 \cup D_2)}(1, \ldots, 1)) = 0 \). The residual exact sequence of \( D_1 \cup D_2 \) gives \( h^1(\mathcal{I}_{W_3}(0, 0, 1, \ldots, 1)) > 0 \), a contradiction. If \( e_1 = e_2 = e_3 = 1 \) we use the residual exact sequence of \( D_1 \cup D_2 \cup D_3 \).

(c) Thus \( k = 2 \). Now we check that \( n_1 = n_2 = 1 \). Assume \( n_1 + n_2 > 2 \), say \( n_1 \geq 2 \). In this case there is \( H \in \{\mathcal{O}_Y(1, 0)\} \) such that \( \deg(H \cap W) \geq 2 \). Since \( W \) is not contained in a proper multiprojective subspace of \( Y \), \( 2 \leq \deg(W \cap H) \leq 3 \).
Since \(\nu(W \cap H)\) is linearly independent, the residual exact sequence of \(H\) gives a contradiction if \(\deg(H \cap W) = 3\). Thus \(\deg(H \cap W) = 2\) and hence we have \(n_1 = 2\). We get \(\deg(\eta_1(W(2))) = 1\). Thus there is \(D \in |O_Y(0, 2)|\) (even if \(n_2 = 1\) containing \(W_2\). Since \(h^1(I_{W(2)}(1, 1)) = 0\), the residual exact sequence of \(D\) gives \(h^1(I_{W_2}(1, 0)) > 0\). This is absurd, because the assumption \(\deg(H \cap W) < 3\) gives that \(\pi_1(W_2)\) spans a line.

\[\square\]

**Remark 5.12.** Take a degree 4 zero-dimensional scheme \(W \subset \mathbb{P}^1 \times \mathbb{P}^1\) with \(\deg(W) = 4\). \(\nu(W)\) is linearly dependent if and only if there is \(D \in |O_Y(1, 1)|\) containing \(W\) and if \(D \supset W\) exists it is unique because \(O_Y(1, 1) \cdot O_Y(1, 1) = 2\) (intersection number). If \(W\) is not curvilinear, then \(D\) must be reducible with \(W_{\text{red}}\) as its singular point.

**Proof of Theorem 5.1:** Since the case \(k = 1\) is obvious, we may assume \(k \geq 2\). The case \(W\) reduced is true by [2, Proposition 5.2]. The case \(W\) connected and not curvilinear is true by Lemma 5.10. The case \(W\) connected and curvilinear is true by Lemma 5.11. Lemma 5.8 proves the case in which \(W\) has 2 connected components of degree 2. Lemma 5.7 proves the case in which \(W\) has a curvilinear connected component of degree 3. Lemma 5.5 proves the case in which \(W\) has a non-curvilinear connected component of degree 3. Lemma 5.5 gives the case in which \(W\) has 3 connected components. 

\(\square\)

6. \(\deg(W) = 5\)

We expect that the next result holds for non-connected curvilinear schemes and that the cases with \(k = 2\) and \(n_1 + n_2 = 3\) and \(k = 3, n_1 = n_2 = n_3 = 1\) are only the ones described in [1, Example 5.7 and Lemma 5.8].

**Theorem 6.1.** Let \(W \subset Y\) be a connected and curvilinear scheme such that \(\deg(W) = 5\) and \(Y\) is the minimal multiprojective space containing \(W\) and that all \(\nu(W')\), \(W' \subsetneq W\) are linearly dependent. The scheme \(\nu(W)\) is linearly dependent if and only if \(Y\) is in one of the following cases:

1. \(k = 1, n_1 = 3\);
2. \(k = 2, n_1 = n_2 = 1\);
3. \(k = 2\) and \(n_1 + n_2 = 3\);
4. \(k = 3, n_1 = n_2 = n_3 = 1\).

**Proof.** Call \(i_1\) the integer \(j \in \{1, \ldots, k\}\) such that there is \(D_1 \in |O_Y(\epsilon_{i_1})|\) with \(e_1 := \deg(D \cap W)\) maximal. Up to a permutation of the factors of of \(Y\) we may assume \(i_1 = 1\). Note that \(D_1 \cap W = W_{\epsilon_1}\) and \(\text{Res}_{D_1}(W) = W_{5-\epsilon_1}\). By the minimality of \(Y\) we have \(e_1 < 5\). Let \(i_2\) be the integer \(j \in \{2, \ldots, k\}\) such that there is \(D_2 \in |O_Y(\epsilon_{i_2})|\) with \(e_2 := \deg(D_2 \cap W_{4-\epsilon_1})\) maximal. Permuting the last \(k - 1\) factors of \(Y\) we may assume \(i_2 = 2\). Since \(W_{4-\epsilon_1} \subsetneq W\), the maximality property of \(i_1\) implies \(e_1 \geq e_2\). If \(e_1 + e_2 = 5\), then write \(g := 2\). Now assume \(e_1 + e_2 < 5\). We continue until we find an integer \(g \geq 3\) such that \(e_1 + \cdots + e_g = 5\). Note that \(g \leq 5\) and that \(g = 5\) if and only \(e_1 = 1\), because \(e_i \geq e_j\) for all \(i < j\). We assume for the moment that this construction is possible, i.e. that \(g \leq k\). Note that \(n_i = 1\) if \(e_i = 1\) and \(e_1 + \cdots + e_i < 5\) and that \(n_i \leq e_i\) if \(e_1 + \cdots + e_i < 5\).
SEGRE VARIETY

(a) Assume \( g = 2 \). Either \( e_1 = 4 \) and \( e_2 = 1 \) or \( e_1 = 3 \) and \( e_2 = 2 \).

If \( e_1 = 4 \), the residual exact sequence of \( D_1 \) gives a contradiction, because
\( h^1(I_{D_1 \cap W}(1, \ldots, 1)) = 0 \) and \( h^1(I_{W_1}(\hat{\epsilon}_1)) = 0 \), because the line bundle \( \mathcal{O}_Y(\hat{\epsilon}_1) \) is globally generated.

Assume \( e_1 = 3 \) and \( e_2 = 2 \). The residual exact sequence of \( D_1 \) and Lemma 3.5
give \( \deg(\eta_1(W_2)) = 1 \). Since \( \deg(W \cap D_2) \geq 3 \), the residual exact sequence of \( D_2 \)
and Lemma 3.5 give that either \( \deg(\eta_2(W_2)) = 1 \) or \( D_2 \cap W = W_2 \) and there is
\( i \in \{1, \ldots, k\} \setminus \{2\} \) such that \( \deg(\pi_h(W_3)) = 1 \) for all \( h \notin \{1, \ldots, k\} \setminus \{2, i\} \). First
assume \( \deg(\eta_2(W_2)) = 1 \). Since \( \deg(\eta_1(W_2)) = 1 \), the inclusion \( W_2 \subset Y \) is not an
embedding, absurd. Now assume \( k \geq 3 \) and the existence of \( i \in \{1, \ldots, k\} \setminus \{2\} \)
such that \( \deg(\pi_h(W_3)) = 1 \) for all \( h \notin \{1, \ldots, k\} \setminus \{2, i\} \). Since \( e_3 = 1 \), we get
\( n_1 = 1 \). Taking \( H \in |\mathcal{O}_Y(\epsilon_h)| \) containing \( W_3 \) we obtain \( \deg(\eta_1(W_2)) = 1 \).
If we may take \( h \neq 1 \), then \( \deg(\pi(W_3)) = 1 \) and hence \( \deg(\pi_1(W_2)) = 1 \). Since
\( \deg(\pi_j(W_2)) = 1 \) for all \( j > 1 \), \( W_2 \subset Y \) is not an embedding, absurd. Thus
\( i = 1 \). Since \( e_1 = 3 \) and \( \deg(\pi_1(W_3)) = 1 \), \( n_1 = 1 \). Since \( \dim |\mathcal{O}_Y(\epsilon_1)| \geq 3 \),
there is \( G \in |I_{W_1}(1, \ldots, 1)| \). If \( \deg(G \cap W) = 4 \), the residual exact sequence of \( G \)
gives a contradiction. If \( \deg(G \cap W) = 3 \), then \( h^1(I_{W_2}(\hat{\epsilon}_3)) = 1 \) and hence
\( \deg(\pi_j(W_2)) = 1 \) for all \( j \neq 3 \). Since \( W_2 \subset Y \) is an embedding, \( \deg(\pi_3(W_2)) = 2 \),
contradicting the equality \( \deg(\eta_1(W_2)) = 1 \).

(b) Assume \( g = 3 \). Either \( e_1 = 3 \) and \( e_2 = e_3 = 1 \) or \( e_1 = e_2 = 2 \) and \( e_3 = 1 \).
Thus \( e_3 = 1 \). We get a contradiction using the residual exact sequence of \( D_1 \cup D_2 \).

(c) Assume \( g = 4 \). Since \( e + 1 + e_2 + e_3 \leq 4 \) and \( e_i \geq e_j \) for all \( i < j \), we get
\( e_1 = 2 \) and \( e_2 = e_3 = e_4 = 1 \). Since \( \deg(\text{Res}_{D_1 \cup D_2 \cup D_3}(W)) = 1 \), the residual exact
sequence of \( D_1 \cup D_2 \cup D_3 \) gives \( h^1(I_{W \cap (D_1 \cup D_2 \cup D_3)(1, \ldots, 1)}) > 0 \), contradicting the
inequality \( e_1 + e_2 + e_3 < 5 \).

(d) Assume \( g = 5 \). Thus \( e_1 = e_2 = e_3 = e_4 = e_5 = 1 \). Therefore we have
\( \deg(\text{Res}_{D_1 \cup D_2 \cup D_3 \cup D_4}(W)) = 1 \). Thus the residual exact sequence of \( D_1 \cup D_2 \cup D_3 \cup D_4 \) gives
\( h^1(I_{W \cap (D_1 \cup D_2 \cup D_3 \cup D_4)(1, \ldots, 1)}) > 0 \), contradicting the inequality
\( e_1 + e_2 + e_3 < 5 \).

(e) Now we discuss all cases in which \( g \) is not defined, i.e. \( e_1 + \cdots + e_k < 5 \).
Thus \( k \leq 4 \). Assume \( k = 4 \). We get \( e_1 = e_2 = e_3 = e_4 = 1 \) and hence
\( 1 = 5 - e_1 - e_2 - e_3 - e_4 \). The residual exact sequence of \( D_1 \cup D_2 \cup D_3 \) gives
a contradiction. Now assume \( k = 3 \). Either \( e_1 = e_2 = e_3 = 1 \) or \( e_1 = 2 \) and
\( e_2 = e_3 = 1 \). In the latter case the residual exact sequence of \( D_1 \cup D_2 \cup D_3 \).
Assume \( k = 3 \) and \( e_1 = e_2 = e_3 = 1 \). Thus \( Y = (\mathbb{P}^1)^3 \) and each \( \pi_{i|W} \) is an
embedding.

(f) Assume \( k = 2 \). Thus \( e_1 \) and \( e_2 \) are well-defined. We excluded the case
with \( g = 2 \) and \( e_1 = 4 \), the case \( e_1 = e_2 = 2 \) and the case \( e_1 = 3 \) and \( e_2 = 1 \).
Thus it is sufficient to handle the cases \( e_1 = 1 \) and \( (e_1, e_2) = (3, 2) \).

(f1) Assume \( e_1 = 1 \). By th definition of \( e_1 \) each \( \pi_{i|W} \) is an embedding.

(f2) Assume \( e_1 = 3 \) and \( e_2 = 2 \). Since \( e_1 \geq n_1 \), we have \( n_1 \leq 3 \).
Now assume \( (n_1, n_2) = (3, 2) \).

(g) Assume \( k = 3 \). By step (e) it is sufficient to consider the case \( e_1 = e_2 = e_3 = 1 \).
Since \( e_1(W) \geq \max\{n_1, n_2, n_3\} \), \( n_1 = n_2 = n_3 = 1 \). Since \( \deg(W) = 5 \)
and \( h^0(\mathcal{O}_Y(1, 1, 1)) = 8 \), the scheme \( \nu(W) \) is linearly dependent if and only if
h^0(\mathcal{I}_W(1,1,1)) \geq 4. Since h^1(\mathcal{I}_W(1,1,1)) = 0 for all \ W' \subset W, h^1(\mathcal{I}_W(1,1,1)) = 1 and hence h^0(\mathcal{I}_W(1,1,1)) = 4. \qed

7. Refined invariants

Let \ W \subset Y \ be a connected and curvilinear zero-dimensional scheme.

Set \ k := \deg(W) \ and \ \ \{o\} := \ W_{\text{red}}. \ Since \ W \ is connected and curvilinear, for each integer \ t \ such that \ 0 \leq t \leq z \ there is a unique degree \ t \ scheme \ W_t \subset W \ with \ \deg(W_t) = t. \ Note \ that \ W_z = W, \ W_1 = \{o\} \ and \ W_0 = \emptyset. \ For \ any \ integer \ t > z \ (resp. \ t < 0) \ set \ W_t := W \ (resp. \ W_t = \emptyset). \ For \ any \ integer \ a \in \{1,\ldots,k\} \ and \ any \ a\text{-tuple} \ (i_1,\ldots,i_a) \ of \ distinct \ integers \ between \ 1 \ and \ k \ define \ the \ scheme \ W(i_1,\ldots,i_a) \subset W \ in \ the \ following \ way. \ Take \ D \in |\mathcal{O}_Y(\epsilon_{i_1})| \ with \ maximal \ \deg(D \cap W). \ The \ divisor \ D \ may \ be \ not \ unique, \ unless \ n_{i_1} = 1, \ but \ all \ of \ them \ have \ the \ same \ \Res_D(W), \ because \ \Res_D(W) = W_{-\deg(W \cap D)} \ and \ the \ integer \ \deg(D \cap W) \ only \ depends \ on \ W \ and \ i_1. \ Set \ W(i_1) := \Res_D(W). \ If \ a \geq 2 \ define \ recursively \ W(i_1,\ldots,i_a) \ by \ the \ formula \ W(i_1,\ldots,i_a) := \Res_W(i_2,\ldots,i_a). \ We \ also \ get \ integers \ e_1(W),\ldots,e_a(W) \ with \ e_1(W) := \deg(D \cap W) \ and, \ if \ a \geq 2, \ e_I = \epsilon_{i-1}(\Res_D(W)) \ for \ i = 2,\ldots,a. \ The \ string \ (e_1(W),\ldots,e_k(W)) \ with \ e_i(W) = 0 \ for \ all \ a < i \leq k \ is \ called \ the \ maximal \ degree \ sequence \ of \ W. \ Note \ that \ e_1(W) \geq \min\{\deg(W),n_{i_1}\} \ and \ that \ e_1(W) \geq n_{i_1} \ if \ Y \ is \ concise \ for \ W. \ Note \ that \ \epsilon_j(W) \geq \min\{n_{i_1},z-e_1(W)\cdots-e_{j-1}(W)\} \ for \ all \ j > 1. \ The \ degree \ string \ of \ W \ is \ the \ ordered \ set \ (\deg(\pi_1(W)),\ldots,\deg(\pi_k(W))) \ of \ k \ positive \ integers. \ The \ injectivity \ string \ of \ W \ is \ the \ ordered \ set \ (t_1(W),\ldots,t_k(W)) \ of \ k \ positive \ integers, \ where \ t_i(W) \ is \ the \ maximal \ integer \ such \ that \ \pi_{i+1}(W) \ is \ an \ embedding. \ Now \ we \ define \ another \ family \ of \ ordering \ \hat{j}_1,\ldots,\hat{j}_k \ of \ \{1,\ldots,k\} \ \ and \ schemes \ W^{\hat{j}_1,\ldots,\hat{j}_k} \subset W. \ Let \ j_1 \ be \ the \ any \ integer \ j \in \{1,\ldots,k\} \ such \ that \ there \ is \ D \in |\mathcal{O}_Y(\epsilon_j)| \ with \ f_{j_1}(W) := \deg(W \cap D) > 0, \ i.e. \ o \in D, \ and \ minimal \ for \ any \ possible \ choices \ of \ j \ and \ D \ containing \ o. \ Then \ we \ call \ \{j_1,\ldots,j_k\} \ a \ minimal \ degree \ sequence \ of \ W \ if \ f_{j_2}(W) \cdots f_{j_k}(W) \ is \ a \ minimal \ sequence \ of \ \Res_D(W) \ for \ the \ multiprojective \ space \ Y_{j_1}. \ We \ call \ \{f_{j_1}(W),\ldots,f_{j_k}(W)\} \ a \ minimal \ degree \ sequence \ of \ W.

Remark 7.1. The schemes \ W_t, \ t \in \mathbb{Z}, \ only \ depends \ on \ the \ isomorphism \ class \ of \ the \ scheme \ W, \ not \ on \ its \ inclusion \ W \hookrightarrow Y. \ The \ scheme \ W(i_1,\ldots,i_a) \subset W \ depends \ on \ the \ inclusion \ W \hookrightarrow Y, \ but \ if \ Y' \hookrightarrow Y \ is \ a \ multiprojective \ subspace \ of \ Y, \ then \ computing \ W(i_1,\ldots,i_a) \ using \ Y \ or \ using \ Y' \ gives \ the \ same \ scheme. \ Indeed, \ there \ are \ linear \ subspaces \ L_i \subset \mathbb{P}^{n_i}, \ 1 \leq i \leq k, \ such \ that \ Y' = L_1 \times \cdots \times L_k; \ of \ course, \ we \ need \ to \ allow \ that \ some \ L_i \ may \ be \ points. \ Indeed, \ for \ any \ D \in |\mathcal{O}_Y(\epsilon_j)| \ we \ have \ D \cap Y' \subset |\mathcal{O}_{Y'}(\epsilon_j)| \ and \ (D \cap Y') \cap W' = D \cap W' \ for \ any \ W' \subset W. \ Thus \ \Res_D(W') = \Res_{D'}(W') \ where \ Res_D \ is \ taken \ in \ Y \ and \ Res_{D'} \ is \ taken \ in \ Y'. \ Moreover, \ for \ each \ D' \in |\mathcal{O}_{Y'}(\epsilon_j)| \ there \ is \ D \in |\mathcal{O}_Y(\epsilon_j)| \ such \ that \ D \cap Y' = D'.

Remark 7.2. Assume \ f_i(W) \geq 2 \ for \ all \ i. \ Then \ f_i(W) \leq 1 \ for \ all \ i.

Proof of Theorem 1.2: \ Let \ Y \ be \ the \ minimal \ multiprojective \ space \ containing \ W. \ Set \ k := w(Y). \ Assume \ k \leq \deg(W) - 1. \ Since \ f_i(W) \leq 1 \ for \ all \ i. \ Thus \ there \ is \ i \leq k \ with \ f_i(W) + \cdots + f_i(W) = \deg(W) - 1. \ The \ proof \ of \ Proposition 6.1 \ gives \ a \ contradiction. \ \qed
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REFERENCES

1. E. Ballico, Linear dependent subsets of Segre varieties, J. Geom. 111, Issue 2, 1 August 2020, Article number 23.
2. E. Ballico, Linearly dependent and concise subsets of a Segre variety depending on $k$ factors, arXiv:2002.09720

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