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CURVILINEAR SUBSCHEMES OF SEGRE VARIETIES AND THE CACTUS RANK

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Abstract. We study zero-dimensional linearly dependent subschemes W of the Segre variety with $\deg(W) \le 5$. If W is connected and curvilinear with arbitrary degree we give a strong restriction on the number of factors of the concise Segre containing W.

1. Introduction

Our main aim is to study linearly dependent zero-dimensional subschemes W with low degree contained in a Segre variety and linearly dependent. As a first crucial step one can classify the subschemes W such that all proper subschemes of W are linearly independent. With this restriction the case $\deg(W) \le 3$ is obvious (Remark 5.2) and we prove the case $\deg(W) = 4$ (Theorem 5.1) and $\deg(W) = 5$, W connected and curvilinear (Theorem 6.1). In [1, 2] we did the case in which W is reduced, i.e. a finite set of points, up to the case $\#W = 6$ using [6]. As in [1, 2] we also introduce concepts which allow to measure the complexity of linearly dependent zero-dimensional subschemes of a Segre variety. We also prove several related results.

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k > 0$, $n_i > 0$ for all i, be a multiprojective space. Let $\nu : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^{k}(n_i + 1)$, be the Segre embedding. Set $X := \nu(Y)$. For any $q \in \mathbb{P}^r$ the tensor rank of q or the X-rank $r_X(q)$ of q is the minimal cardinality of a set $A \subset X$ such that $q \in \langle A \rangle$, where $\langle\ \rangle$ denote the linear span. The cactus rank $cr_X(q)$ of q is the minimal degree of a zero-dimensional scheme $W \subset X$ such that $q \in \langle A \rangle$. Obviously $cr_X(q) \le r_X(q)$. Let $\mathcal{S}(Y, q)$ denote the set of all finite sets $S \subset Y$ such that $\#S = r_X(q)$ and $q \in \langle \nu(S) \rangle$. Let $\mathcal{Z}(Y, q)$ denote the set of all zero-dimensional schemes $W \subset Y$ such that $\deg(W) = cr_X(q)$ and $q \in \langle \nu(W) \rangle$. A very useful result on the tensor rank is that if $Y' \subset Y$ is a multiprojective subspace and $q \in \langle \nu(Y') \rangle$, then $r_X(q) = r_{\nu(Y')}(q)$ and every $S \in \mathcal{S}(Y, q)$ is contained in Y' ([11, Proposition 3.1.3.1]).

Question 1.1. Does concision hold for the cactus rank of tensors and/or the cactus rank of homogeneous polynomials? Are all zero-dimensional schemes evincing the cactus rank of a tensor or of a homogeneous polynomial linearly independent?

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As a justification for the cactus rank, see [7, 8, 9, 10].

As in [2] we consider the width of a multiprojective space and of a zero-dimensional subscheme of a multiprojective space. For any multiprojective subspace $Y' \subseteq Y$ let $w(Y')$ denote the number of positive-dimensional factors of Y'. For instance, $w(Y) = k$, because we assumed $n_i > 0$ for all i. For any zero-dimensional scheme $W \subset Y$ set $w(W) = w(Y')$, where Y' is the minimal multiprojective subspace of Y containing W. It is straightforward to compute Y' knowing W (Remark 3.1). We recall that a curvilinear zero-dimensional scheme of a smooth quasi-projective variety is a zero-dimensional scheme whose connected components are either reduced or with one-dimensional Zariski tangent space. We prove the following result in which $\pi_i : Y \to \mathbb{P}^{n_i}$ is the projection of Y onto its i-th factor.

Theorem 1.2. *Let W be a connected curvilinear scheme. Set $o = (o_1, \dots, o_k) := W_{\mathrm{red}}$. Assume $\deg(\pi_i^{-1}(o_i) \cap W) = 1$ for all i and that $\nu(W)$ is linearly dependent. Then $w(W) \le \deg(W) - 2$.*

The assumptions in Theorem 1.2 are very strong, but also its thesis is very strong (compare it to the thesis of [2, Theorem 1.1]).

2. Notation

Let M be an integral projective variety. For all integers $t > 0$ and $a_1 \ge \cdots \ge a_t > 0$ let $A(M, a_1, \dots, a_t)$ denote the set of all zero-dimensional and curvilinear schemes $Z \subset M_{\mathrm{reg}}$ with t connected components $Z_1, \dots, Z_t$ with $\deg(Z_i) = a_i$ for all i.

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space. Fix $i \in \{1, \dots, k\}$. Set $Y_i := \prod_{h \ne i} \mathbb{P}^{n_h}$. Let ν_i denote the Segre embedding of the multiprojective space Y_i. Let $\pi_i : Y \to \mathbb{P}^{n_i}$, $i = 1, \dots, k$, denote the projection onto its i-th factor. Let $\eta_i : Y \to Y_i$ be the surjection which forget the i-th coordinate of any $p = (p_1, \dots, p_k) \in Y$. For any $i = 1, \dots, k$ let $\mathcal{O}_Y(\epsilon_i)$ (resp. $\mathcal{O}_Y(\hat{\epsilon}_i)$) denote the line bundle on Y with multidegree $(a_1, \dots, a_k)$ with $a_i = 1$ and $a_h = 0$ for all $h \ne i$ (resp. $a_i = 0$ and $a_h = 1$ for all $h \ne i$).

3. Preliminary observations and lemmas

Remark 3.1. Let $W \subset Y$ be a zero-dimensional subscheme. The minimal multiprojective space containing W is the multiprojective subspace $\prod_{i=1}^k \langle \pi_i(W) \rangle$, where $\pi_i(W)$ denote the scheme-theoretic image.

Remark 3.2. Let $E \subset Y$ be a zero-dimensional scheme such that $\deg(E) \le 3$ and $\dim \nu(E) \le \deg(E) - 2$. Since ν is an embedding, we have $\deg(E) = 3$ and $\langle \nu(E) \rangle$ is a line. Since X is scheme-theoretical cut out by quadrics, we have $\langle \nu(E) \rangle \subseteq \nu(Y)$, i.e. there are $i \in \{1, \dots, k\}$ and $o \in \mathbb{P}^{n_i}$ such that $E \subset \eta_i^{-1}(o)$. In particular $\nu(A)$ is linearly independent for any degree 2 scheme $A \subset Y$.

Remark 3.3. Assume $\deg(A) = 3$ and that $A \in \mathcal{Z}(Y, q)$ for some tensor q. Since A evinces the cactus rank of q, it is not a so-called fat point or 2-point of a plane. Thus A is curvilinear. Either A is connected or A has two connected component, one of degree 2 and one of degree 1, or it is reduced, i.e. the union of 3 different points. Since $q \in \sigma_3(X) \setminus \sigma_2(X)$, in the latter case we have $r_X(q) = 3$. In all cases the linear space $\langle \nu(A)\rangle$ is a plane not contained in X, because $q \notin X$ and any line $L \subset \mathbb{P}^r$ with $\deg(L \cap X) \ge 3$ is contained in X.

Lemma 3.4. *Fix* $u \in \mathbb{P}^r$ *and any* $E \in \mathcal{Z}(Y,u)$ *with* $\deg(E) \le 3$. *Then* $\dim\langle \nu(E)\rangle = \deg(E) - 1$ *and* $\eta_{i|E}$ *is an embedding for* $i = 1,\dots,k$.

Proof. The first assertion is true for any degree 2 scheme (since ν is an embedding) and it is true if $\deg(E) = 3$, because any line $L \subset \mathbb{P}^r$ containing a degree 3 subscheme of X is contained in X. Assume that $\eta_{i|E}$ is not an embedding for some i, say for $i = k$. Since the scheme-theoretic fibers of η_k are projective schemes embedded as linear subspaces by ν, there is a degree 2 scheme $v \subseteq E$ such that $\eta_i(v)$ is a point and $\langle \nu(v)\rangle$ is a line contained in X. Since $\deg(v) = 2$ and E evinces a cactus rank, we have $E \ne v$ and hence $\deg(E) = 3$. Thus $\langle \nu(E)\rangle$ is a plane containing $\nu(E)$ and the line $\langle \nu(v)\rangle \subset X$. Since X is cut out by quadrics, either $\langle \nu(E)\rangle \subset X$ (excluded because $cr_X(u) = 3$) or $X \cap \langle\langle \nu(E)\rangle$ is the union of two different lines. Thus all $u' \in \langle \nu(E)\rangle$ have $r_X(u') \le 2$, contradicting the assumption $cr_X(u) = 3$. □

Lemma 3.5. *Let* $E \subset Y$ *be a zero-dimensional scheme such that* $\deg(E) \le 3$. *There is* $i \in \{1,\dots,k\}$ *such that* $h^1(\mathcal{I}_E(\hat{\epsilon}_i)) > 0$ *if and only if either there is* $v \subseteq E$ *such that* $\deg(v) = 2$ *and* $\deg(\eta_i(v)) = 1$ *or* $\deg(E) = 3$ *and there is* $j \in \{1,\dots,k\} \setminus \{i\}$ *such that* $\deg(\pi_h(E)) = 1$ *for all* $h \in \{1,\dots,k\} \setminus \{i,j\}$.

Proof. We have $h^1(\mathcal{I}_E(\hat{\epsilon}_i)) > 0$ if and only if either $\eta_{i|E}$ is not an embedding or $\deg(E) = \deg(\eta_i(E))$ and $h^1(Y_i, \mathcal{I}_{\eta_i(E)}(1,\dots,1)) > 0$. $\eta_{i|E}$ is not an embedding if and only if there is $v \subseteq E$ such that $\deg(v) = 2$ and $\deg(\eta_i(v)) = 1$. Now assume that $\eta_{i|E}$ is an embedding, i.e. assume $\deg(\eta_i(E)) = \deg(E)$. Since $\eta_{i|E}$ is an embedding, we have $h^1(\mathcal{I}_E(\hat{\epsilon}_i)) = h^1(Y_i, \mathcal{I}_{\eta_i(E)}(1,\dots,1))$. Since $\mathcal{O}_{Y_i}(1,\dots,1)$ is very ample, the structure of lines on the Segre variety $\nu_i(Y_i)$ gives that $h^1(Y_i, \mathcal{I}_{\eta_i(E)}(1,\dots,1)) > 0$ if and only if there is $j \in \{1,\dots,k\} \setminus \{i\}$ such that $\deg(\pi_h(E)) = 1$ for all $h \in \{1,\dots,k\} \setminus \{i,j\}$. □

4. Cactus rank for matrices

In this section we consider the case $k = 2$, i.e. $Y = \mathbb{P}^m \times \mathbb{P}^n$, $m > 0$, $n > 0$.

Proposition 4.1. *Assume* $k = 2$, *i.e. assume* $Y = \mathbb{P}^m \times \mathbb{P}^n$ *for some* $m > 0$, $n > 0$. *Fix* $q \in \mathbb{P}^r$ *and set* $a := r_X(q)$. *Let* $Y' \subseteq Y$ *be the minimal multiprojective subspace of* Y *such that* $q \in \langle \nu(Y')\rangle$ *Then:*

(i) $cr_{\nu(Y)}(q) = cr_{\nu(Y')}(q) = a$ *and* $\mathcal{Z}(Y,q) = \mathcal{Z}(Y',q)$.

(ii) For all integers $t > 0$ *and* $a_1 \ge \dots \ge a_t > 0$ *such that* $a_1 + \dots + a_t = a$ *we have* $Z(Y', a_1,\dots,a_t) \cap \mathcal{Z}(Y',q) \ne \emptyset$.

Proof. By the classification of matrices we have $Y' \cong \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$, all matrices with rank a are equivalent and $\mathcal{S}(Y,q) = \mathcal{S}(Y',q)$. Thus $cr_{\nu(Y)}(q) \le cr_{\nu(Y')}(q) \le a$. Take a zero-dimensional scheme $Z \subset Y$ such that $q \in \langle \nu(Z)\rangle$. Since $\nu(Y')$ is the minimum (not just minimal, by concision for the tensor decomposition) multiprojective space with $q \in \langle \nu(Y')\rangle$, we have $\langle \nu(Z)\rangle \supseteq \langle \pi_i(Y')\rangle$ for all $i = 1,\dots,k$. Thus $cr_{\nu(Y)}(q) \ge a$, concluding the proof of all assertions of part (i).

Fix integers $t > 0$ and $a_1 \ge \cdots \ge a_t > 0$. Fix t distinct points $p_1,\dots,p_t \in \mathbb{P}^1$ Fix a basis of $|\mathcal{O}_{\mathbb{P}^1}(a-1)|$ and use it to define an embedding $f_1 : \mathbb{P}^1 \to \mathbb{P}^{a-1}$ with a rational normal curve as its image. Since $f_1(\mathbb{P}^1)$ is a rational normal curve and $a_1 + \cdots + a_t = a$, $f_1(a_1p_1 + \cdots + a_tp_t)$ is a linearly independent degree a zero-dimensional scheme spanning $\mathbb{P}^{a-1}$. Let $f : \mathbb{P}^1 \to Y' = \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$ be the morphism $f = (f_1, f_1)$. The scheme $Z := f(a_1p_1 + \cdots + a_tp_t)$ is an element of $A(Y', a_1,\dots,a_t) \subseteq A(Y, a_1,\dots,a_t)$ and its images by the projections onto the factors of Y' span the factors. Thus Y' is the minimal multiprojective space containing Z. Take a general $q' \in \langle \nu(Z)\rangle$. Since $Z \subset Y'$, $r_X(q') \le a$. Since all $m \times n$ matrices with rank a are projectively equivalent, to prove part (ii) it is sufficient to prove $r_{\nu(Y')}(q') \ge a$. Assume $r_{\nu(Y')}(q') \le a-1$ and take $A \subset Y$ such that $\#A \le a-1$ and $q' \in \langle \nu(A)\rangle$. We identify Y' with $\mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$. Set $B := A \cup Z$. Since Z is curvilinear, it has only finitely many proper subschemes. Since q' is general in $\langle \nu(Z)\rangle$ and $\nu(Z)$ is linearly independent, there is no $Z' \subsetneq Z$ such that $q' \in \langle \nu(Z')\rangle$. In particular $A \nsubseteq Z$. Since $q' \in \langle \nu(A)\rangle \cap \langle \nu(Z)\rangle$, we have $h^1(\mathcal{I}_B(1,1)) > 0$. Since $\#A \le a-1$, there is $H \in |\mathcal{O}_{Y'}(1,0)|$ containing A. Since $\deg(H \cap \mathbb{P}^{a-1} \cap f(\mathbb{P}^1)) = a-1 < \deg(Z)$, we have $\mathrm{Res}_H(B) \neq \emptyset$. Thus $h^1(\mathcal{I}_{\mathrm{Res}_H(B)}(0,1)) > 0$ ([5, Lemma 5.1]). This is absurd, because $\mathrm{Res}_H(B) \subseteq Z$, $\pi_{2|Z}$ is an embedding and $\pi_2(Z)$ is linearly independent. □

5. $\deg(W) \le 4$

The aim of this section is the proof of the following result.

Theorem 5.1. *Let $W \subset Y$ be a zero-dimensional scheme such that $\deg(W) = 4$, Y is the minimal multiprojective space containing W, $\nu(W)$ is linearly dependent and $\nu(W')$ is linearly independent for all $W' \subsetneq W$. Then either $k = 1$ and $n_1 = 2$ or $k = 2$, $n_1 = n_2 = 1$ and W is as in one of the Examples 5.3 and 5.4.*

Remark 5.2. Let $W \subset W$ be a zero-dimensional scheme such that $\deg(W) \le 3$ and $\nu(W)$ is linearly dependent. Since ν is an embedding, $\deg(W) = 3$ and $\langle \nu(W)\rangle$ is a line. Since $\nu(Y)$ is scheme-theoretically cut out by quadrics and $W \subseteq \langle \nu(E)\rangle \cap \nu(Y)$, then $\langle \nu(E)\rangle \subset Y$. The structure of linear subspaces of a Segre variety shows the existence of $i \in \{1,\dots,k\}$ such that $\deg(\pi_h(W)) = 1$ for all $h \neq i$, $\pi_{i|W}$ is an embedding and $\pi_i(W) \subseteq \mathbb{P}^{n_i}$ is a line.

Example 5.3. Let $W \subset \mathbb{P}^n$ be a zero-dimensional scheme such that $\deg(W) = 4$, W spans $\mathbb{P}^n$, W is linearly dependent, but all proper subschemes are linearly independent. Obviously $n = 2$. Since $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 6$ and $\deg(L \cap W) \le 3$ for all lines $L \subset \mathbb{P}^2$, $\dim |\mathcal{I}_W(2)| = 2$ and either W is the complete intersection of 2 conics or $\deg(L \cap W) = 3$ for exactly one line L and there is $o \in \mathbb{P}^2$ such that all $E \in |\mathcal{I}_W(2)|$ is of the form $E = L \cup R$ with $R \in |\mathcal{I}_o(1)|$. In both cases there are non-curvilinear W's. In the latter case the latter case $o \in L$ and W is the union of the fat point $2o$ and some $p \in L \setminus \{o\}$.

Example 5.4. Let $W \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a degree 4 scheme. Since $h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) = 4$, $h^1(\mathcal{I}_W(1,1)) > 0$ if and only if there is $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ containing W. If $\nu(W')$

is linearly independent for all $W' \subsetneq W$, then D is unique, because we have $h^1(\mathcal{I}_W(1,1)) = 1$ in this case.

Lemma 5.5. *Let $W \subset Y$ be a zero-dimensional scheme such that $\deg(W) = 4$, Y is the minimal multiprojective space containing W, $\nu(W)$ is linearly dependent, $\nu(W')$ is linearly independent for all $W' \subsetneq W$ and $\#W_{\mathrm{red}} = 3$. Then either $k = 1$ and $n_1 = 2$ or $k = 2$ and $n_1 = n_2 = 1$.*

Proof. Since the case $k = 1$ is obvious, we may assume $k \geq 2$. Write $W = v \cup \{a, b\}$ with v connected and $\deg(v) = 2$.

(a) Assume $n_i \geq 2$ for some i, say $n_1 \geq 2$. Thus there is $H \in |\mathcal{I}_v(\epsilon_1)|$. Since Y is he minimal multiprojective space containing W, $W \nsubseteq H$ and hence $h^1(\mathcal{I}_{H\cap W}(1,\dots,1)) = 0$. The residual exact sequence of H gives the inequality $h^1(\mathcal{I}_{\mathrm{Res}_H(W)}(\hat{\epsilon_1})) > 0$. Thus $H \cap W = v$ and $\eta_1(a) = \eta_1(b)$. Thus there is $D \in |\mathcal{O}_Y(\epsilon_2)|$ containing $\{a, b\}$. The residual exact sequence of D gives $D\cap W = \{a, b\}$ and $\deg(\eta_2(v) = 1$. Thus there is $M \in |\mathcal{O}_Y(\epsilon_1)|$ with $M \supseteq \{v, a\}$. The residual exact sequence of M gives a contradiction.

(b) Now assume $k \geq 3$ and $n_i = 1$ for all i. Take $H \in |\mathcal{O}_Y(\epsilon_1)|$ with $H\cap W \neq \emptyset$. Easy modifications of the proof of step (a) gives $\deg(H \cap W) = 1$. Take $M \in |\mathcal{O}_Y(\epsilon_2)|$ with $M \cap \mathrm{Res}_H(W)) \neq \emptyset$. As in step (a) we get $\deg(\mathrm{Res}_{H\cup M}(W)) = 2$ and $h^1(\mathcal{I}_{\mathrm{Res}_{H\cup M}(W)}(0,0,1,\dots,1)) > 0$. Thus there is $H' \in |\mathcal{O}_Y(\epsilon_3)|$ such that $\deg(H' \cap W) \geq 2$. Using H' instead of H we get a contradiction. □

Lemma 5.6. *Let $W \subset Y$ be a zero-dimensional scheme such that $\deg(W) = 4$, Y is the minimal multiprojective space containing W, $\nu(W)$ is linearly dependent and W has a connected component of degree 3 which is not curvilinear. Then there is $W' \subsetneq W$ such that $\nu(W')$ is linearly dependent.*

Proof. Call A the degree 3 connected component of W and write $W = A\cup\{a\}$. Set $\{o\} := A_{\mathrm{red}}$. Since A is not curvilinear there is a 2-dimensional linear subspace U of the tangent space T_oY (a vector space of dimension $n_1 + \cdots + n_k$) such that $A = (2o, U)$ is the 2-fat point of U with o as its reduction. Since ν is an embedding, $\dim\langle\nu(A)\rangle = 2$. Since $A = (2o, U)$, the line $\langle\{\nu(o), \nu(a)\}$ has degree 3 intersection with $\nu(W)$. Thus $\nu(W)$ has a degree 3 linearly dependent subscheme. □

Lemma 5.7. *Let $E \subset Y$ be a degree 3 curvilinear scheme and $o \in Y \setminus E_{\mathrm{red}}$. Assume $h^1(\mathcal{I}_E(1,\dots,1)) = 0$, $h^1(\mathcal{I}_{E\cup\{o\}}(1,\dots,1)) > 0$ and that Y is the minimal multiprojective space containing $E \cup \{o\}$. Then either $k = 1$ or $k = 2$ and $n_1 = n_2 = 1$.*

Proof. Since $h^1(\mathcal{I}_E(1,\dots,1)) = 0$, $\langle\nu(E)\rangle$ is a plane containing $\nu(o)$. Take $H \in |\mathcal{O}_Y(\epsilon_1)|$ containing o. By Lemma 3.5 either there is $v \subseteq \mathrm{Res}_H(E)$ with $\deg(v) = 2$ and $\deg(\eta_1(v)) = 1$ or there is $i > 1$ such that $\deg(\pi_h(E)) = 1$ for all $h \notin \{1, i\}$. In the second case the minimal multiprojective space Y' containing E is contained in a multiprojective space isomorphic to $\mathbb{P}^{n_1} \times \mathbb{P}^{n_i}$. Since $\nu(o) \in \langle\nu(Y')\rangle$, concision for the tensor decomposition ([11, Proposition 3.1.3.1]) applied to the tensor decomposition of $\nu(o)$ gives $Y' = Y$. Hence $k = 2$ and $n_1 \leq 2$ and $n_2 \leq 2$. Assume $(n_1, n_2) \neq (1, 1)$, say $n_1 = 2$. Take $H \in |\mathcal{I}_o(\epsilon_1)|$. Since $E \nsubseteq H$, [5, 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_H}(1,0)) > 0$ and hence $h^1(\mathcal{I}_E(1,0))$. The definition of Y' gives $\langle \pi_1(E)\rangle = \mathbb{P}^2$. Thus $h^1(\mathcal{I}_E(1,0)) = 0$, a contradiction. □

As in Lemma 5.5 we get the following lemma.

Lemma 5.8. *Take $v, w \subset Y$ degree 2 connected zero-dimensional scheme such that $v \cap w = \emptyset$, $\nu(v\cup w)$ is linearly dependent and $\nu(W')$ is linearly independent for all $W' \subsetneq v \cup w$. Then the minimal multiprojective space Y' containing $v\cup w$ is either $\mathbb{P}^m$, $m \le 2$, or $\mathbb{P}^1\times\mathbb{P}^1$.*

Example 5.9. Take $Y = \mathbb{P}^1\times\mathbb{P}^1$. Fix $o \in Y$ and set $\{D_i\} := |\mathcal{I}_o(\epsilon_i)|$, $i = 1, 2$, and $T := D_1 \cup D_2$. In the plane $M := \langle \nu(T)\rangle$ take any conic $E \subset M$ such that $\nu(o) \in \mathrm{Sing}(E)$, $\nu(D_1) \nsubseteq E$ and $\nu(D_2) \nsubseteq E$. Thus $B := E \cap \nu(T)$ degree 4 zero-dimensional scheme, which is the complete intersection of two conics. Thus $h^0(M, \mathcal{I}_W(2)) = 2$, i.e. the scheme B is the scheme-theoretic base locus of the pencils of all conics of M singular at $\nu(o)$. Thus B is independent of the choice of T and E. Since $B \subset \nu(T)$, there is a unique scheme $W \subset T \subset Y$ such that $B = \nu(W)$. Since $\deg(W) = 4$ and $W \subset T$, $\nu(W)$ is linearly dependent. Since B is the complete intersection of two conics singular at $4\nu(o)$ the connected degree 4 schemes B and W are Gorenstein, i.e. there are unique $W' \subset W$ and $B' \subset B$ such that $\deg(B') = \deg(W') = 3$. The uniqueness of B' gives $\nu(W') = B'$. Since B is the complete intersection of 2 conics of M, $\deg(B)\cap D_i) \ge 2$ for $i = 1, 2$, i.e. B contains the first infinitesimal neighborhood $2\nu(o)$ of $\nu(o)$ in M. Since $\deg(\nu(o)) = 3$, $2n(o) = B'$. Since $2\nu(o)$ spans M, B' is linearly independent. Thus each $\nu(W'')$, $W'' \subsetneq W$, is linearly independent. The construction shows that W is the only connected and not curvilinear degree 4 zero-dimensional scheme Z such that $Z_{\mathrm{red}} = \{o\}$, $\nu(Z)$ is linearly dependent and each $\nu(Z')$, $Z' \subsetneq Z$, is linearly independent.

Lemma 5.10. *Let $W \subset Y$ be a degree 4 connected and not curvilinear subscheme such that $\nu(W)$ is linearly dependent, while all $\nu(W')$, $W' \subseteq W$ are linearly independent. Assume that Y is the minimal multiprojective subspace containing W. Then $k \le 2$ and $k = 2$ if and only if $n_1 = n_2 = 1$ and W is as in Example 5.9.*

Proof. Set $\{o\} := W_{\mathrm{red}}$. Write $o = (o_1, \dots, o_k)$. For each $i = 1, \dots, k$ set $L_i := \eta_i^{-1}(o_i)$. Note that $\nu(L_i)$ is an n_i-dimensional projective space and that $\nu(L_1), \dots, \nu(L_k)$ are the k maximal linear subspaces of the Segre variety $\nu(Y)$.

By assumption $\langle\nu(W)\rangle$ is a plane and hence the Zariski tangent space $T_oW \subseteq T_oY$ has dimension ≤ 2. Since W is not curvilinear, its Zariski tangent space has dimension > 1. Thus $\dim T_oW = 2$. Thus W contains the 2-fat point of o in T_oW, i.e. the degree 3 zero-dimensional scheme $(2o, T_oW) \subset Y \cap T_oY$. This implies $\deg(D\cap W) \ge 2$ for every effective divisor D of Y containing o.

Assume $k \ge 2$ and take $H \in |\mathcal{I}_o(\epsilon_k)|$. Thus $\deg(H\cap W) \ge 2$. Since Y is the minimal multiprojective space containing W, $W \nsubseteq H$. Thus $2 \le \deg(H\cap W) \le 3$.

(a) Assume $\deg(H\cap W) = 3$. If $\langle \nu(W)\rangle \subset \nu(Y)$, then $k = 1$, a contradiction. Thus $\langle\nu(W)\rangle \nsubseteq \nu(Y)$. Since $\nu(Y)$ is scheme-theoretically cut out by quadric, either $\langle\nu(W)\rangle \cap \nu(Y)$ is a conic or it is a complete intersection of of two conics.

If the intersection is a conic, then $cr_X(q) = 2$ and $\mathcal{Z}(Y, q)$ is infinite for a general $q \in \langle \nu(W) \rangle$. Lemma 5.8 gives $k = 2$ and $n_1 = n_2 = 1$. The last sentence of Example 5.9 gives that W is as in Example 5.9.

(b) Assume $\deg(H \cap W) = 2$. We have $h^1(\mathcal{I}_{H\cap W}(1, \dots, 1)) = 0$, because ν is an embedding. Using the residual exact sequence of H we obtain the inequality $h^1(\mathcal{I}_{\mathrm{Res}_H(W)}(\hat{\epsilon}_k)) > 0$. Since $\deg(\mathrm{Res}_H(W)) = 2$, Lemma 3.5 gives $\deg(\eta_1(\mathrm{Res}_H(W)) = 1$. Since $\mathrm{Res}_H(W) \subset W$, the see that $\deg(W \cap L_k) \geq 2$. Using $i \in \{1, \dots, k\}$ instead of k we get $\deg(W \cap L_i) \geq 2$ for all i. Since the linear spaces $\nu(L_i)$ are as general as possible in the $(n_1 + \cdots + n_k)$-dimensional Zariski tangent space $T_{\nu(o)}\nu(Y)$ with the only restriction that they contain $\nu(o)$ and they have prescribed dimensions whose sum is $\dim T_{\nu(o)}\nu(Y)$, W has embedding dimensions at least k. Thus $k = 2$. Since $\deg(\eta_i(W)) = 2$ for all i, we have $\deg(\pi_h(W)) \leq 2$ for all h. Since Y is the minimal multiprojective space containing W, $n_i = 1$ for all i. The last sentence of Example 5.9 gives that W is as in Example 5.9. □

Lemma 5.11. *Let $W \subset Y$ be a connected and curvilinear scheme such that* $\deg(W) = 4$, *Y is the minimal multiprojective space containing W andall $\nu(W')$, $W' \subsetneq W$ are linearly dependent. The scheme $\nu(W)$ is linearly dependent if and only if either $k = 1$ and $n_1 = 2$ or $k = 2$ and $n_1 = n_2 = 1$.*

Proof. Since the case $k = 1$ is obvious, we assume $k \geq 2$. Call i_1 the integer $j \in \{1, \dots, k\}$ such that there is $D_1 \in |\mathcal{O}_Y(\epsilon)_{i_1}|$ with $w_1 := \deg(D \cap W)$ maximal. Up to a permutation of the factors of of Y we may assume $i_1 = 1$. Note that $D_1 \cap W = W_{e_1}$ and $\mathrm{Res}_{D_1}(W) = W_{4-e_1}$. If $e_1 < 4$ by the minimality f Y. Let i_2 be the integer $j \in \{2, \dots, k\}$ such that there is $D_2 \in |\mathcal{O}_Y(\epsilon_{i_2})|$ with $e_2 : \deg(D_2 \cap W_{4-e_1}$ maximal. Permuting the last $k-1$ factors of Y we may assume $i_2 = 2$. Since $W_{4-e_1} \subseteq W$, the maximality property of i_1 implies $e_1 \geq e_2$.

(a) Assume $e_1 + e_2 = 4$. Thus either $e_1 = 3$ and $e_2 = 1$ or $e_1 = e_2 = 2$.

(a1) Assume $e_1 = 3$ and $e_2 = 1$. Since $e_1 < 4$, $h^1(\mathcal{I}_{W\cap D_1}(1, \dots, 1)) = 0$. The residual exact sequence of D_1 gives $h^1(\mathcal{I}_{W(1)}\hat{\epsilon}_1)) > 0$. Since $W(1) = \{o\}$ and $\mathcal{O}_Y(\hat{\epsilon}_1)$ is globally generated, $h^1(\mathcal{I}_{W_1}\hat{\epsilon}_1)) = 0$, a contradiction. (a) Assume $e_1 = e_2 = 2.e_1 < 4$, $h^1(\mathcal{I}_{W\cap D_1}(1, \dots, 1)) = 0$. The residual exact sequence of D_1 gives $h^1(\mathcal{I}_{W_2}\hat{\epsilon}_1)) > 0$. Lemma 3.5 gives $\deg(\eta_1(W_2) = 0$, i.e. $\pi_{i|W_2}$ is constant for all $i > 1$. Since $e_2 = e_1$, we may first use D_2 and get $h^1(\mathcal{I}_{W_2}\hat{\epsilon}_1)) > 0$. Thus $\deg(\mathcal{I}_h(W_2) = 1$ for all $h \neq 2$. Thus the inclusion $W_2 \subset Y$ is not an embedding by the universal property of the product, a contradiction.

(b) Assume $e_1 + e_2 < 4$. Let i_3 be the integer $j \in \{3, \dots, k\}$ such that there is $D_3 \in |\mathcal{O}_Y(\epsilon_{i_3})|$ with $e_3 : \deg(D_3 \cap W(4 - e_1 - e_2)$ maximal. Permuting the last $k-1$ factors of Y we may assume $i_3 = 3$. Either $e_1 = 2$ and $e_2 = e_3 = 1$ or $e_1 = e_2 = e_3 = 1$. First assume $e_1 = 2$. Since $(D_1 \cup D_2) \cap W = W(3)$, we have $h^1(\mathcal{I}_{W\cap(D_1\cup D_2)}(1, \dots, 1)) = 0$. The residual exact sequence of $D_1 \cup D_2$ gives $h^1(\mathcal{I}_{W_1}(0, 0, 1, \dots 1)) > 0$, a contradiction. If $e_1 = e_2 = e_3 = 1$ we use the residual exact sequence of $D_1 \cup D_2 \cup D_3$.

(c) Thus $k = 2$. Now we check that $n_1 = n_2 = 1$. Assume $n_1 + n_2 > 2$, say $n_1 \geq 2$. In this case there is $H \in |\mathcal{O}_Y(1, 0)|$ such that $\deg(H \cap W) \geq 2$. Since W is not contained in a proper multiprojective subspace of Y, $2 \leq \deg(W \cap H) \leq 3$.

Since $\nu(W \cap H)$ is linearly independent, the residual exact sequence of H gives a contradiction if $\deg(H \cap W) = 3$. Thus $\deg(H \cap W) = 2$ and hence we have $n_1 = 2$. We get $\deg(\eta_1(W(2)) = 1$. Thus there is $D \in |\mathcal{O}_Y(0,2)|$ (even if $n_2 = 1$) containing W_2. Since $h^1(\mathcal{I}_{W(2)}(1,1)) = 0$, the residual exact sequence of D gives $h^1(\mathcal{I}_{W_2}(1,0)) > 0$. This is absurd, because the assumption $\deg(H \cap W) < 3$ gives that $\pi_1(W_2)$ spans a line. □

Remark 5.12. Take a degree 4 zero-dimensional scheme $W \subset \mathbb{P}^1 \times \mathbb{P}^1$ with $\deg(W) = 4$. $\nu(W)$ is linearly dependent if and only if there is $D \in \mathcal{O}_Y(1,1)|$ containing W and if $D \supset W$ exists it is unique because $\mathcal{O}_Y(1,1) \cdot \mathcal{O}_Y(1,1) = 2$ (intersection number). If W is not curvilinear, then D must be reducible with W_{red} as its singular point.

Proof of Theorem 5.1: Since the case $k = 1$ is obvious, we may assume $k \ge 2$. The case W reduced is true by [2, Proposition 5.2]. The case W connected and not curvilinear is true by Lemma 5.10. The case W connected and curvilinear is true by Lemma 5.11. Lemma 5.8 proves the case in which W has 2 connected components of degree 2. Lemma 5.7 proves the case in which W has a curvilinear connected component of degree 3. Lemma 5.5 proves the case in which W has a non-curvilinear connected component of degree 3. Lemma 5.5 gives the case in which W has 3 connected components. □

6. $\deg(W) = 5$

We expect that the next result holds for non-connected curvilinear schemes and that the cases with $k = 2$ and $n_1 + n_2 = 3$ and $k = 3$, $n_1 = n_2 = n_3 = 1$ are only the ones described in [1, Example 5.7 and Lemma 5.8].

Theorem 6.1. *Let $W \subset Y$ be a connected and curvilinear scheme such that $\deg(W) = 5$ and Y is the minimal multiprojective space containing W and that all $\nu(W')$, $W' \subsetneq W$ are linearly dependent. The scheme $\nu(W)$ is linearly dependent if and only if Y is in one of the following cases:*

1. *$k = 1$, $n_1 = 3$;*
2. *$k = 2$, $n_1 = n_2 = 1$;*
3. *$k = 2$ and $n_1 + n_2 = 3$;*
4. *$k = 3$, $n_1 = n_2 = n_3 = 1$.*

Proof. Call i_1 the integer $j \in \{1, \dots, k\}$ such that there is $D_1 \in |\mathcal{O}_Y(\epsilon)_{i_1}|$ with $e_1 := \deg(D \cap W)$ maximal. Up to a permutation of the factors of of Y we may assume $i_1 = 1$. Note that $D_1 \cap W = W_{e_1}$ and $\mathrm{Res}_{D_1}(W) = W_{5-e_1}$. By the minimality of Y we have $e_1 < 5$. Let i_2 be the integer $j \in \{2, \dots, k\}$ such that there is $D_2 \in |\mathcal{O}_Y(\epsilon_{i_2})|$ with $e_2 := \deg(D_2 \cap W_{4-e_1}$ maximal. Permuting the last $k-1$ factors of Y we may assume $i_2 = 2$. Since $W_{4-e_1} \subseteq W$, the maximality property of i_1 implies $e_1 \ge e_2$. If $e_1 + e_2 = 5$, then write $g := 2$. Now assume $e_1 + e_2 < 5$. We continue until we find an integer $g \ge 3$ such that $e_1 + \cdots + e_g = 5$. Note that $g \le 5$ and that $g = 5$ if and only $e_1 = 1$, because $e_i \ge e_j$ for all $i < j$. We assume for the moment that this construction is possible, i.e. that $g \le k$. Note that $n_i = 1$ if $e_i = 1$ and $e_1 + \cdots + e_i < 5$ and that $n_i \le e_i$ if $e_1 + \cdots + e_i < 5$.

(a) Assume $g = 2$. Either $e_1 = 4$ and $e_2 = 1$ or $e_1 = 3$ and $e_2 = 2$.

If $e_1 = 4$, the residual exact sequence of D_1 gives a contradiction, because $h^1(\mathcal{I}_{D_1\cap W}(1,\dots,1)) = 0$ and $h^1(\mathcal{I}_{W_1}(\hat{\epsilon}_1)) = 0$, because the line bundle $\mathcal{O}_Y(\hat{\epsilon}_1)$ is globally generated.

Assume $e_1 = 3$ and $e_2 = 2$. The residual exact sequence of D_1 and Lemma 3.5 give $\deg(\eta_1(W_2)) = 1$. Since $\deg(W \cap D_2) \geq 3$, the residual exact sequence of D_2 and Lemma 3.5 give that either $\deg(\eta_2(W_2)) = 1$ or $D_2 \cap W) = W_2$ and there is $i \in \{1,\dots,k\}\setminus\{2\}$ such that $\deg(\pi_h(W_3)) = 1$ for all $h \notin \{1,\dots,k\}\setminus\{2,i\}$. First assume $\deg(\eta_2(W_2)) = 1$. Since $\deg(\eta_1(W_2)) = 1$, the inclusion $W_2 \subset Y$ is not an embedding, absurd. Now assume $k \geq 3$ and the existence of $i \in \{1,\dots,k\}\setminus\{2\}$ such that $\deg(\pi_h(W_3)) = 1$ for all $h \notin \{1,\dots,k\}\setminus\{2,i\}$. Since $e_3 = 1$, we get $n_1 = 1$. Taking $H \in |\mathcal{O}_Y(\epsilon_h)|$ containing W_3 we obtain $\deg(\eta_h(W_2)) = 1$. If we may take $h \neq 1$, then $\deg(\pi(W_3)) = 1$ and hence $\deg(\pi_1(W_2)) = 1$. Since $\deg(\pi_j(W_2)) = 1$ for all $j > 1$, $W_2 \subset Y$ is not an embedding, absurd. Thus $i = 1$. Since $e_1 = 3$ and $\deg(\pi_1(W_3)) = 1$, $n_1 = 1$. Since $\dim|\mathcal{O}_Y(\hat{\epsilon}_1)| \geq 3$, there is $G \in |\mathcal{I}_{W_3}(1,\dots,1)|$. If $\deg(G \cap W) = 4$, the residual exact sequence of G gives a contradiction. If $\deg(G\cap W) = 3$, then $h^1(\mathcal{I}_{W_2}(\hat{\epsilon}_3)) = 1$ and hence $\deg(\pi_j(W_2)) = 1$ for all $j \neq 3$. Since $W_2 \subset Y$ is an embedding, $\deg(\pi_3(W_2)) = 2$, contradicting the equality $\deg(\eta_1(W_2)) = 1$.

(b) Assume $g = 3$. Either $e_1 = 3$ and $e_2 = e_3 = 1$ or $e_1 = e_2 = 2$ and $e_3 = 1$. Thus $e_3 = 1$. We get a contradiction using the residual exact sequence of $D_1 \cup D_2$.

(c) Assume $g = 4$. Since $e + 1 + e_2 + e_3 \leq 4$ and $e_i \geq e_j$ for all $i < j$, we get $e_1 = 2$ and $e_2 = e_3 = e_4 = 1$. Since $\deg(\mathrm{Res}_{D_1\cup D_2\cup D_3}(W)) = 1$, the residual exact sequence of $D_1 \cup D_2 \cup D_3$ gives $h^1(\mathcal{I}_{W\cap(D_1\cup D_2\cup D_3}(1,\dots,1)) > 0$, contradicting the inequality $e_1 + e_2 + e_3 < 5$.

(d) Assume $g = 5$. Thus $e_1 = e_2 = e_3 = e_4 = e_5 = 1$. Therefore we have $\deg(\mathrm{Res}_{D_1\cup D_2\cup D_3\cup D_4}(W)) = 1$. Thus the residual exact sequence of $D_1 \cup D_2 \cup D_3 \cup D_4$ gives $h^1(\mathcal{I}_{W\cap(D_1\cup D_2\cup D_3\cup D_4}(1,\dots,1)) > 0$, contradicting the inequality $e_1 + e_2 + e_3 + e_4 < 5$.

(e) Now we discuss all cases in which g is not defined, i.e. $e_1 + \cdots + e_k < 5$. Thus $k \leq 4$. Assume $k = 4$. We get $e_1 = e_2 = e_3 = e_4 = 1$ and hence $1 = 5 - e_1 - e_2 - e_3 - e_4$. The residual exact sequence of $D_1 \cup D_2 \cup D_3$ gives a contradiction. Now assume $k = 3$. Either $e_1 = e_2 = e_3 = 1$ or $e_1 = 2$ and $e_2 = e_3 = 1$. In the latter case the residual exact sequence of $D_1 \cup D_2 \cup D_3$. Assume $k = 3$ and $e_1 = e_2 = e_3 = 1$. Thus $Y = (\mathbb{P}^1)^3$ and each $\pi_{i|W}$ is an embedding.

(f) Assume $k = 2$. Thus e_1 and e_2 are well-defined. We excluded the case with $g = 2$ and $e_1 = 4$, the case $e_1 = e_2 = 2$ and the case $e_1 = 3$ and $e_2 = 1$. Thus it is sufficient to handle the cases $e_1 = 1$ and $(e_1, e_2) = (3, 2)$.

(f1) Assume $e_1 = 1$. By th definition of e_1 each $\pi_{i|W}$ is an embedding.

(f2) Assume $e_1 = 3$ and $e_2 = 2$. Since $e_1 \geq n_1$, we have $n_1 \leq 3$.

Now assume $(n_1, n_2) = (3, 2)$.

(g) Assume $k = 3$. By step (e) it is sufficient to consider the case $e_1 = e_2 = e_3 = 1$. Since $e_1(W) \geq \max\{n_1, n_2, n_3\}$, $n_1 = n_2 = n_3 = 1$. Since $\deg(W) = 5$ and $h^0(\mathcal{O}_Y(1,1,1)) = 8$, the scheme $\nu(W)$ is linearly dependent if and only if

$h^0(\mathcal{I}_W(1,1,1)) \ge 4$. Since $h^1(\mathcal{I}_{W'}(1,1,1)) = 0$ for all $W' \subsetneq W$, $h^1(\mathcal{I}_W(1,1,1)) = 1$ and hence $h^0(\mathcal{I}_W(1,1,1)) = 4$. □

7. Refined invariants

Let $W \subset Y$ be a connected and curvilinear zero-dimensional scheme.

Set $z := \deg(W)$ and $\{o\} := W_{\mathrm{red}}$. Since W is connected and curvilinear, for each integer t such that $0 \le t \le z$ there is a unique degree t scheme $W_t \subseteq W$ with $\deg(W_t) = t$. Note that $W_z = W$, $W_1 = \{o\}$ and $W_0 = \emptyset$. For any integer $t > z$ (resp. $t < 0$) set $W_t := W$ (resp. $W_t = \emptyset$). For any integer $a \in \{1, \dots, k\}$ and any a-ple $(i_1, \dots, i_a)$ of distinct integers between 1 and k define the scheme $W(i_1, \dots, i_a) \subseteq W$ in the following way. Take $D \in |\mathcal{O}_Y(\epsilon_{i_1})|$ with maximal $\deg(D \cap W)$. The divisor D may be not unique, unless $n_{i_1} = 1$, but all of them have the same $\mathrm{Res}_D(W)$, because $\mathrm{Res}_D(W) = W_{z-\deg(W\cap D)}$ and the integer $\deg(D \cap W)$ only depends on W and i_1. Set $W(i_1) := \mathrm{Res}_D(W)$. If $a \ge 2$ define recursively $W(i_1, \dots, i_a)$ by the formula $W(i_1, \dots, i_a) := \mathrm{Res}_W(i_2, \dots, i_a)$. We also get integers $e_1(W), \dots, e_a(W)$ with $e_1(W) := \deg(D \cap W)$ and, if $a \ge 2$, $eI = e_{i-1}(\mathrm{Res}_D(W))$ for $i = 2, \dots, a$. The string $(e_1(W), \dots, e_k(W))$ with $e_i(W) = 0$ for all $a < i \le k$ is called he *maximal degree sequence* of W. Note that $e_1(W) \ge \min\{\deg(W), n_{i_1}\}$ and that $e_1(W) \ge n_{i_1}$ if Y is concise for W. Note that $e_j(W) \ge \min\{n_{i_j}, z - e_1(W) - \cdots - e_{j-1}(W)\}$ for all $j > 1$. The *degree string* of W is the ordered set $(\deg(\pi_1(W), \dots, \deg(\pi_k(W))$ of k positive integers. The *injectivity string* of W the ordered set $(t_1(W), \dots, t_k(W))$ of k positive integers, where $t_i(W)$ is the maximal integer t such that $\pi_{|W_t}$ is an embedding. Now we define another family of ordering $j_1, \dots, j_k$ of $\{1, \dots, k\}$ and schemes $W^{j_1,\dots,j_k} \subseteq W$. Let j_1 be the any integer $j \in \{1, \dots, k\}$ such that there is $D \in |\mathcal{O}_Y(\epsilon_j)|$ with $f_{j_1}(W) := \deg(W \cap D) > 0$, i.e. $o \in D$, and minimal for any possible choices of j and D containing o. Then we call $(j_1, \dots, j_k)$ a *minimal sequence* of W if $(j_2, \dots, j_k)$ is a minimal sequence of $\mathrm{Res}_D(W)$ for the multiprojective space Y_{j_1}. We call $(f_1(W), \dots, f_k(W))$ a *minimal degree sequence* of W.

Remark 7.1. The schemes W_t, $t \in \mathcal{Z}$, only depends on the isomorphism class of the scheme W, not on its inclusion $W \hookrightarrow Y$. The scheme $W(i_1, \dots, i_a) \subseteq W$ depends on the inclusion $W \hookrightarrow Y$, but if $Y' \subseteq Y$ is a multiprojective subspace of Y, then computing $W(i_1, \dots, i_a)$ using Y or using Y' gives the same scheme. Indeed, there are linear subspaces $L_i \subseteq \mathbb{P}^{n_i}$, $1 \le i \le k$, such that $Y' = L_1 \times \cdots \times L_k$; of course, we need to allow that some L_i may be points. Indeed, for any $D \in |\mathcal{O}_Y(\epsilon_i)|$ we have $D \cap Y' \in |\mathcal{O}_{Y'}(\epsilon_i)$ and $(D \cap Y) \cap W' = D \cap W'$ for any $W' \subseteq W$. Thus $\mathrm{Res}_D(W') = \mathrm{Res}_{D'}(W')$, where Res_D is taken in Y and $\mathrm{Res}_{D'}$ is taken in Y'. Moreover, for each $D' \in |\mathcal{O}_{Y'}(\epsilon_i)|$ there is $D \in |\mathcal{O}_Y(\epsilon_i)|$ such that $D \cap Y' = D'$.

Remark 7.2. Assume $t_i(W) \ge 2$ for all i. Then $f_i(W) \le 1$ for all i.

Proof of Theorem 1.2: Let Y be the minimal multiprojective space containing W. Set $k := w(Y)$. Assume $k \le \deg(W) - 1$. Since $t_i \ge 2$ for all i, we have $f_i(W) \le 1$ for all i. Thus there is $i \le k$ with $f_1(W) + \cdots + f_i(W) = \deg(W) - 1$. The proof of Proposition 6.1 gives a contradiction. □

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