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SOME SUBMANIFOLDS OF GENERALIZED KENMOTSU MANIFOLDS

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ABSTRACT. In this paper, invariant submanifolds of a generalized Kenmotsu manifold are studied and given some properties. An example is constructed for an invariant submanifold of a generalized Kenmotsu manifold. In addition, integrabilities of invariant distribution is investigated, and some theorems are given related to curvature tensor and the second fundamental form in invariant submanifolds of a generalized Kenmotsu manifold. Moreover, semi-parallel and 2-semi-parallel invariant submanifolds of a generalized Kenmotsu manifold are studied. Necessary and sufficient conditions are given on semi-parallel and 2-semi-parallel invariant submanifolds of a generalized Kenmotsu manifold to be totally geodesic.

1. INTRODUCTION AND PRELIMINARIES

Almost contact and contact structure in differential geometry are very important parts. Fundamentally, contact structure has more important applications in Physics. Yano started the first work on contact structure in [19], and he introduced an f-structure on differentiable manifolds. In 1970, Blair studied the structure group to $U(n) \times O(s)$ on differentiable manifolds [2]. Later, Nakagawa in [12], [13] defined globally framed f -manifold which is called an f -manifold. After Goldberg and Yano investigated these manifolds [6], [7], [8]. In 1970, Blair defined K-structure which class of f -manifolds [2]. A K-manifold is said to be an S manifold if $d\eta^i = \Phi$, where Φ is the fundamental 2-form.

In [10] , K. Kenmotsu has studied different types of almost contact manifolds. He has shown neither Sasakian nor quasi Sasakian of this type of almost contact manifolds. This manifold is called a Kenmotsu manifold. Present authors have introduced Kenmotsu manifolds on globally frame f -manifolds. We have called generalized Kenmotsu manifolds [16].

In 1985 , Deprez investigated a semi-parallel immersion and semi-parallel hypersurfaces [4],[5]. Let M and $\bar{M}$ be Riemannian manifolds. $x : M \to \bar{M}$ be an isometric immersion, $\bar{R}$ curvature tensor and h the second fundamental form, then x is said to be semi-parallel if $\bar{R}.h = 0$. In 2000 , K. Arslan and colleagues

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defined, a submanifold to be 2-semi-parallel [1]. Many authors studied the geometric properties of some manifolds [9, 11, 14, 15, 17].

The present paper deals with invariant submanifolds of generalized Kenmotsu manifolds. In our first review, in section 2, we give some basic information and notions about generalized Kenmotsu manifolds. In section 3, we define the invariant submanifolds of a generalized Kenmotsu manifold and demonstrate of an example of submanifold. We study integrabilities of invariant distribution. In addition, some theorems are given related to curvature tensor and second fundamental form in invariant submanifolds of a generalized Kenmotsu manifold. Finally, necessary and sufficient conditions are given on semi-parallel and 2-semi-parallel invariant submanifolds to be totally geodesic.

2. Generalized Kenmotsu Manifolds

In [6], differentiable manifold M is said to be a metric $f-$manifold if there exist a Riemannian metric g, s vector fields $\xi_1, \ldots, \xi_s$, an $(1,1)$ type tensor field ϕ and s 1-forms $\eta^1, \ldots, \eta^s$ such that

$$\phi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \qquad \eta^i(\xi_j) = \delta_{ij} \tag{2.1}$$

$$g(\phi V, \phi X) = g(V, K) - \sum_{i=1}^{s} \eta^i(V)\eta^i(K) \tag{2.2}$$

for any $V, K \in \Gamma(TM)$. Moreover, we obtain

$$\eta^i(K) = g(K, \xi_i), \quad \phi\xi_i = 0, \quad g(V, \phi K) = -g(\phi V, K), \eta^i \circ \phi = 0 \tag{2.3}$$

On the other hand, let Φ is a 2-form. Then Φ is called fundamental 2-form if $\Phi(V, K) = g(V, \phi K)$ for any $V, K \in \Gamma(TM)$.

Moreover, let $[\phi, \phi]$ is Nijenhuis tensor field of M. Then M is said to be normal if

$$[\phi, \phi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0.$$

Definition 2.1. A metric f-manifold is said to be an almost s-contact metric structure such that $\eta^1 \wedge \ldots \wedge \eta^s \wedge \Phi^n \neq 0$, where 1-forms η^i and fundamental 2-form Φ [18].

Definition 2.2. An almost $s-$contact metric manifold $(M, \phi, \xi_i, \eta^i, g)$ is called a generalized Kenmotsu manifold (GKM) such that M is normal,

$$\eta^i \text{ are closed } (or\ d\eta^i = 0)$$

and

$$d\Phi = 2\sum_{i=1}^{s} \eta^i \wedge \Phi$$

where $i \in \{1, \ldots, s\}$ [16].

Theorem 2.3. *An almost s-contact metric manifold* (M,ϕ,ξ_i,η^i,g) *to be a generalized Kenmotsu manifold if and only if*

$$\left(\bar{\nabla}_V\phi\right)K=\sum_{i=1}^{s}\left\{g(\phi V,K)\xi_i-\eta^i(K)\phi V\right\} \tag{2.4}$$

for all $V,K\in\Gamma(TM)$, [16].

Theorem 2.4. *Let* (M,ϕ,ξ_i,η^i,g) *be a generalized Kenmotsu manifold. Then we have*

$$\bar{\nabla}_V\xi_j=-\phi^2V \tag{2.5}$$

for all $V\in\Gamma(TM), i\in\{1,2,...,s\}$ [16].

Theorem 2.5. *Let* (M,ϕ,ξ_i,η^i,g) *be a generalized Kenmotsu manifold. Then we have*

$$\bar{R}(V,K)\xi_i=\sum_{j=1}^{s}\{\eta^j(X)\phi^2V-\eta^j(V)\phi^2K\} \tag{2.6}$$

$$\bar{R}(V,\xi_j)\xi_i=\phi^2V \tag{2.7}$$

$$\bar{R}(\xi_j,V)K=\sum_{j=1}^{s}\{g(V,\phi^2K)\xi_j-\eta^j(K)\phi^2V\} \tag{2.8}$$

$$\overline{S}(K,\xi_i)=-2n\sum_{i=1}^{s}\eta^i(K) \tag{2.9}$$

where $\overline{S}$ *is the Ricci tensor of GKM and for all* $V,K\in\Gamma(TM)$ [16].

3. Invariant Submanifolds of a Generalized Kenmotsu Manifold

In this section we state and prove some results regarding an invariant submanifold of generalized Kenmotsu manifold. Let's start by defining the invariant submanifold of generalized Kenmotsu manifold.

Let $\left(\bar{M},\phi,\xi_i,\eta^i,g\right)$ be a generalized Kenmotsu manifold. M is a submanifold of $\bar{M}$ and ξ_i are tangent to the submanifold M. On the other hand, let D is orthogonal distribution to $\xi_i, i\in\{1,2,...,s\}$ in TM. Therefore we can denote

$$TM=D\oplus sp\{\xi_1,\xi_2,\dots\xi_s\}$$

where $D=\{K\in\Gamma(TM)\mid\eta(K)=0\}$.

If $\phi(T_vM))\subset T_vM$, for any point $v\in M$ and ξ_i are tanget to M, then M is called an invariant submanifold of $\bar{M}$.

Example 3.1. The consider $(\mathbb{R}^{2n+s},\varphi,\eta,g,\xi_i)$ a generalized Kenmotsu manifold with its usual generalized Kenmotsu structure given by

$$\varphi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i})+\sum_{\alpha=1}^{s}Z_\alpha\frac{\partial}{\partial z_\alpha})=\sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x_i}-X_i\frac{\partial}{\partial y_i})+\sum_{\alpha=1}^{s}\sum_{i=1}^{n}Y_iy_i\frac{\partial}{\partial z_\alpha},$$

$$\eta^\alpha=dz_\alpha,\qquad \xi_\alpha=\frac{\partial}{\partial z_\alpha}$$

$$g = \frac{1}{f_1^2 + f_2^2} \sum_{i=1}^{n} (dx_i \otimes dx_i + dy_i \otimes dy_i) + \sum_{\alpha=1}^{s} dz_\alpha \otimes dz_\alpha$$

where f_1 and f_2 are given by

$$\begin{aligned} f_1(t_1, t_2, t_3) &= c_2 e^{-(t_1+t_2+t_3)} \cos(t_1 + t_2 + t_3) \\ &\quad -c_1 e^{-(t_1+t_2+t_3)} \sin(t_1 + t_2 + t_3), \\ f_2(t_1, t_2, t_3) &= c_1 e^{-(t_1+t_2+t_3)} \cos(t_1 + t_2 + t_3) \\ &\quad +c_2 e^{-(t_1+t_2+t_3)} \sin(t_1 + t_2 + t_3) \end{aligned}$$

for nonzero constant c_1, c_2. Now,we denote a submanifold M of $\mathbb{R}^7$ such that

$$M = X(u, v, t_1, t_2, t_3) = (f_1 u - f_2 v, f_1 u + f_2 v, 0, 0, t_1, t_2, t_3).$$

Then local frame of TM

$$\begin{aligned} e_1 &= f_1(t_1, t_2, t_3)\frac{\partial}{\partial x} + f_2(t_1, t_2, t_3)\frac{\partial}{\partial y} \\ e_2 &= -f_2(t_1, t_2, t_3)\frac{\partial}{\partial x} + f_1(t_1, t_2, t_3)\frac{\partial}{\partial y} \\ e_3 &= \frac{\partial}{\partial t_1}, \qquad e_4 = \frac{\partial}{\partial t_2}, \qquad e_5 = \frac{\partial}{\partial t_3}. \end{aligned}$$

We can easily that $\varphi(e_1) = -\, e_2$, then M is an invariant submanifold.

We define $\bar{\nabla}$ and ∇ Levi-Civita connection on $\bar{M}$ and M respectively, g be induced metric on M. Therefore for any $V, K \in \Gamma(TM)$ and $N \in \Gamma(TM)^\perp$ we get,

$$\bar{\nabla}_V K = \nabla_V X + h(V, K) \tag{3.1}$$

$$\bar{\nabla}_V N = \nabla^\perp_V N - A_N V \tag{3.2}$$

the last two equation is said to be Gauss and Weingarten formulas respectively. In addition, h is the second fundamental from, $\nabla^\perp$ is the connection in the normal bundle and A_N is the Weingarten endomorphism associated with N. Moreover, shape operator A and the second fundamental form h connected by

$$g(h(V, X), N) = g(A_N V, X). \tag{3.3}$$

Then using equation (3.1) and (3.2), we obtain for all vector fields V, W, K tanget to M

$$\begin{aligned} \bar{R}(V, W)K &= R(V, W)K - A_{h(W,K)}(V) + A_{h(V,K)}(W) \\ &\quad +(\nabla_V h)(W, K) - (\nabla_W h)(V, K) \end{aligned} \tag{3.4}$$

where $\bar{R}$ and R are Riemannian curvature tensor $\bar{M}$and M, respectively.

From now on we will denote a generalized Kenmotsu manifold by $\bar{M}$.

Theorem 3.2. *If M is an invariant submanifold of $\bar{M}$ then we have*

$$(\nabla_V \phi)\, K = \sum_{i=1}^{s} \left\{ g(\phi V, K)\xi_i - \eta^i(X)\phi V \right\}, \tag{3.5}$$

$$h(V, \phi K) = \phi h(V, K) \tag{3.6}$$

for all $V, K \in \Gamma(TM)$.

Proof. Since $\bar{M}$ is a GKM, we have,

$$\left(\bar{\nabla}_V \phi\right) K = \sum_{i=1}^{s} \left\{ g(\phi V, K)\xi_i - \eta^i(K)\phi V \right\}.$$

Using (3.1), we arrive,

$$\left(\bar{\nabla}_V \phi\right) K = \left(\nabla_V \phi\right) K - h(V, \phi K) + \phi h(V, K).$$

Comparing the tangential and normal part of last equation, we get the desired results. □

From the above theorem, we have the following:

Corollary 3.3. *Let* M *be an invariant submanifold of* $\bar{M}$. *Then the second fundemantal form* h *of* M *satisfies,*

$$h(V, \phi K) = \phi h(V, K) = h(\phi V, K), \tag{3.7}$$

$$h(\phi V, \phi K) = -h(V, K) \tag{3.8}$$

for all $V, K \in \Gamma(TM)$.

Theorem 3.4. *If* M *is an invariant submanifold of* $\bar{M}$ *then we have*

$$\nabla_V \xi_j = -\phi^2 V, \tag{3.9}$$

$$h(V, \xi_j) = 0 \tag{3.10}$$

for all $V \in \Gamma(TM)$.

Proof. If equation (2.5) is used for a GKM, we have

$$\overline{\nabla}_V \xi_j = V - \sum_{i=1}^{s} \eta^i(V)\xi_i.$$

On the other hand using (3.1), then we get the following equations

$$\nabla_V \xi_j + h(V, \xi_j) = V - \sum_{i=1}^{s} \eta^i(V)\xi_i.$$

Therefore, we obtain that

$$\nabla_V \xi_j = V - \sum_{i=1}^{s} \eta^i(V)\xi_i$$

and

$$h(V, \xi_j) = 0.$$

□

Now, we examine weingarten edomorphism for invariant submanifold.

Theorem 3.5. *If M is an invariant submanifold of $\bar{M}$ then we have*

$$A_N\xi_i = 0$$

for all $N \in \Gamma(TM^{\perp})$.

Proof. For all $K \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, using (3.3), (3.10), we have $g(A_N\xi_i, K) = g(h(\xi_i, K), N) = 0$. □

Theorem 3.6. *If M is an invariant submanifold of $\bar{M}$ then we have*

$$\phi(A_N V) = A_{\phi N}V = -A_N\phi V$$

for all$V \in \Gamma(TM)$, $N \in \Gamma(TM^{\perp})$.

Proof. For all $V, K \in \Gamma(TM)$, $N \in \Gamma(TM)^{\perp}$, using (2.3), (3.3) and (3.7), we have,

$$\begin{aligned} g(\phi(A_N V), K) &= -g(A_N V, \phi K) \\ &= -\ g(h(\phi V, K), N) \\ &= -\ g(\ A_N\phi V, K). \end{aligned}$$

Similary way, from (2.3), (3.3) and (3.7), we have,

$$\begin{aligned} g(A_{\phi N}V, K) &= g(h(V, K), \phi N) \\ &= -g(h(V, \phi K), N) \\ &= g(\phi(A_N V), K). \end{aligned}$$

□

Riemannian curvature and Ricci curvature, which is defined as the trace of Riemannian curvature tensor, has a major role in the Riemannian geometry. Now, we investigate on invariant submanifold with certain conditions related to the curvature.

Theorem 3.7. *If M is an invariant submanifold of $\bar{M}$, then curvature tensor R and Ricci tensor S of M satisfies,*

$$R(V, K)\xi_i = \sum_{j=1}^{s}\{\eta^j(K)\phi^2 V - \eta^j(V)\phi^2 K\} \tag{3.11}$$

$$S(K, \xi_i) = -2n\sum_{i=1}^{s}\eta^i(K) \tag{3.12}$$

for all $V, K \in \Gamma(TM)$.

Proof. For all $V, K \in \Gamma(TM)$, from (3.4), we have

$$\begin{aligned} \bar{R}(V, K)\xi_i &= R(V, K)\xi_i - A_{h(K,\xi_i)}(V) + A_{h(V,\xi_i)}(K) \\ &\quad +\nabla_V h(K, \xi_i) - h(\nabla_V K, \xi_i) - h(K, \nabla_V \xi_i) \\ &\quad \nabla_K h(V, \xi_i) + h(\nabla_K V, \xi_i) + h(V, \nabla_K \xi_i). \end{aligned}$$

By virtue of (3.8), (3.9), (3.10), we conclude that

$$\bar{R}(V, K)\xi_i = R(V, K)\xi_i.$$

Then from (2.6) and (2.9), we get the desired results. □

Therefore, we provide the following corollary.

Corollary 3.8. *If M is an invariant submanifold of $\bar{M}$ then we have*

$$R(\xi_j, V)\xi_i = -\phi^2 V \tag{3.13}$$

$$R(\xi_j, V)X = \sum_{j=1}^{s}\{g(V, \phi^2 X)\xi_j - \eta^j(X)\phi^2 V\} \tag{3.14}$$

for all $V \in \Gamma(TM)$.

Theorem 3.9. *If M is an invariant submanifold of $\bar{M}$, then $\bar{R}(V,K)\xi_j$ is tangent to M for any $V, K \in \Gamma(TM)$.*

Proof. For each $N_l \in \Gamma(TM)^{\perp}$ using (2.6) we have,

$$\bar{g}(\bar{R}(V,K)\xi_j, N_l) = \bar{g}(\sum_{i=1}^{s}\eta^i(K)\phi^2 V, N_l) + \bar{g}(\sum_{i=1}^{s}\eta^i(V)\phi^2 K, N_l).$$

Then from (2.1) we get,

$$\begin{aligned}\bar{g}(\bar{R}(V,K)\xi_j, N_l) &= \sum_{i,k=1}^{s}\{-\bar{g}(V, N_l) + \eta^i(K)\eta^k(V)\bar{g}(\xi_k, N_l) \\ &\quad -\bar{g}(X, N_l) + \eta^i(K)\eta^k(V)\bar{g}(\xi_k, N_l)\}\end{aligned}$$

which completes proof. □

Now, we give some conditions for integrability and totally geodesicity.

Theorem 3.10. *If M is an invarinat submanifold of $\bar{M}$ then distribution D is always integrable.*

Proof. Firstly for all $V, K \in \Gamma(D)$

$$g(K, \xi_i) = 0 \Rightarrow g(\nabla_V K, \xi_i) = -g(V, \nabla_K \xi_i).$$

Then, from (3.9), we get

$$g([V,K], \xi_i) = -g(K, \varphi^2 V) + g(V, \varphi^2 K)$$

which gives our assertion. □

From the above theorem, we have the following:

Corollary 3.11. *If M be an invarinat submanifold of $\bar{M}$ then distribution $\overline{\xi}$ is always integrable, where $\overline{\xi} = sp\{\xi_1, \xi_2, \ldots \xi_s\}$.*

Theorem 3.12. *Let M is an invariant submanifold of $\bar{M}$. The second fundamental form h is parallel if and only if M is totally geodesic.*

Proof. Firstly, h is parallel. Then, for all $V, K \in \Gamma(TM)$,

$$(\nabla_V h)(K, \xi_i) = 0.$$

By virtue of (3.10),(3.9) we conclude that

$$h(V, K) = 0.$$

Vice versa, from $h = 0$, for all $V, K \in \Gamma(TM)$

$$(\nabla_V h)(K, U) = \nabla_V h(K, U) - h(\nabla_V K, U) - h(K, \nabla_V U) = 0.$$

Thus we have

$$\nabla h = 0.$$

□

4. Semiparallel and 2-Semiparallel Invariant Submanifolds of A Generalized Kenmotsu Manifold

Let $\bar{M}$ be a Riemannian manifold and M is a submanifold of $\bar{M}$. An isometric immersion $i : M \to \bar{M}$ is *semi-parallel* if

$$\bar{R}(U, V)h = \bar{\nabla}_U(\bar{\nabla}_V h) - \bar{\nabla}_V(\bar{\nabla}_U h) - \bar{\nabla}_{[U,V]} h = 0$$

where h is the second fundemental from and $\bar{R}$ is curvature tensor of $\bar{\nabla}$ [3].

In [1], K. Arslan and colleagues defined that M is *2-semiparallel* submanifolds if

$$R(E, F)\nabla h = 0$$

for all E, F tanget to M.

$\bar{\nabla}$ is the connection in $TM \oplus TM^{\perp}$ build with ∇ and $\nabla^{\perp}$, where R (resp. $R^{\perp}$) denote curvature tensor of the connection ∇ (resp. $\nabla^{\perp}$). Then, we have

$$(\bar{R}(E, F)h)(U, L) = R^{\perp}(E, F)h(U, L) - h(R(E, F)U, L) - h(U, R(E, F)L) \quad (4.1)$$

for all vector fields E, F, L, U tanget to M [3]. In addition,

$$\begin{aligned}(\bar{R}(E, F)\bar{\nabla}h)(U, L, W) = {} & R^{\perp}(E, F)(\bar{\nabla}h)(U, L, W) - (\bar{\nabla}h)(R(E, F)U, L, W) \\ & -(\bar{\nabla}h)(U, W, R(E, F)L) \\ & -(\bar{\nabla}h)(U, L, R(E, F)W) \end{aligned} \quad (4.2)$$

or all vector fields E, F, U, L, W tanget to M where $(\bar{\nabla}h)(U, L, W) = (\bar{\nabla}_U h)(L, W)$ [1].

Theorem 4.1. *Let M is an invariant submanifold of $\bar{M}$. M is semiparallel if and only if M is total geodesic.*

Proof. Let's assume that M is semiparallel. Then, $\bar{R}(V, W)h = 0$ for all $V, W \in \Gamma(TM)$. Using (4.1), we get

$$R^{\perp}(V, W)h(L, K) - h(R(V, W)L, K) - h(L, R(V, W)K) = 0.$$

We take $V = \xi_i$ and $K = \xi_j$ then,

$$R^{\perp}(\xi_i, W)h(L, \xi_j) - h(R(\xi_i, W)L, \xi_j) - h(L, R(\xi_i, W)\xi_j) = 0.$$

From (3.10), we have

$$h(L, R(\xi_i, W)\xi_j) = 0.$$

By virtue of (3.13), we conclude that

$$h(L, W) = 0.$$

□

Theorem 4.2. *Let M is an invariant submanifold of $\bar{M}$. M is 2-semiparallel if and only if M is total geodesic.*

Proof. Suppose that M is 2-semiparallel. Then, $\bar{R}(E, F)\bar{\nabla}h = 0, \forall\, E, F, Z, U, K \in \Gamma(TM)$. Using (4.2), we have

$$R^{\perp}(E, F)(\bar{\nabla}h)(U, Z, K) - (\bar{\nabla}h)(R(E, F)U, Z, K)$$
$$-(\bar{\nabla}h)(U, K, R(E, F)Z) - (\bar{\nabla}h)(U, Z, R(E, F)V) = 0.$$

We take $E = \xi_i$ and $Z = \xi_j$ then, we have

$$R^{\perp}(\xi_i, F)(\bar{\nabla}h)(U, \xi_j, K) - (\bar{\nabla}h)(R(\xi_i, F)U, \xi_j, K)$$
$$-(\bar{\nabla}h)(U, R(\xi_i, F)\xi_j, K) - (\bar{\nabla}h)(U, \xi_j, R(\xi_i, F)K) = 0.$$

Moreover, we get

$$\begin{aligned}(\bar{\nabla}h)(U, \xi_j, K) &= (\bar{\nabla}_U h)(\xi_j, K) \\ &= \nabla^{\perp}_U(h(\xi_j, K)) - h(\nabla_U\xi_j, K) - h(\xi_j, \nabla_U K).\end{aligned}$$

Using (3.9) and (3.10) we arrive,

$$(\bar{\nabla}h)(U, \xi_j, K) = -h(U, K).$$

Therefore,

$$(\bar{\nabla}h)(R(\xi_i, W)U, \xi_j, K) = (\bar{\nabla}_{R(\xi_i, W)U}h)(\xi_j, K).$$

From (3.10) and (3.14), we get,

$$(\bar{\nabla}h)(R(\xi_i, F)U, \xi_j, K) = -h(-\phi^2 R(\xi_i, F)U, K).$$

By virtue of (2.1), (3.7) and (3.14) we obtain,

$$(\bar{\nabla}h)(R(\xi_i, F)U, \xi_j, K) = -\sum_{l=1}^{s}\eta^l(U)h(F, K).$$

Now

$$(\bar{\nabla}h)(U, R(\xi_i, F)\xi_j, K) = (\bar{\nabla}_U h)(R(\xi_i, F)\xi_j, K)$$

From (3.13), we get

$$(\bar{\nabla}h)(U, R(\xi_i, F)\xi_j, K) = \nabla^{\perp}_U(h(-\phi^2 F, K) - h(\nabla_U(-\phi^2 F), K) - h(-\phi^2 F, \nabla_U K).$$

Finally we get

$$(\bar{\nabla}h)(U, \xi_j, R(\xi_i, F)K) = (\bar{\nabla}_U h)(\xi_j, R(\xi_i, F)K)$$

From (3.9),(3.13) and (3.14), we have

$$(\bar{\nabla}h)(U, \xi_j, R(\xi_i, W)K) = -h(\nabla_U\xi_j, R(\xi_i, W)K).$$

Using (2.1),(3.9) and (3.14), we obtain

$$(\bar{\nabla}h)(U,\xi_j,R(\xi_i,F)K)=-\sum_{l=1}^{s}\eta^l(K)h(U,F).$$

Then, we conclude that

$$R^{\perp}(\xi_i,F)(-h(U,K))-(-\sum_{l=1}^{s}\eta^l(U)h(F,K))-(\nabla_U^{\perp}(h(-\phi^2F,K)$$
$$-h(\nabla_U(-\phi^2F),K)-h(-\phi^2F,\nabla_UK))-(-\sum_{l=1}^{s}\eta^l(K)h(U,F))=0.$$

So we take $K=\xi_k$ and using (2.1), (3.10), (3.9), we have

$$h(F,U)=0.$$

□

5. Conclusion

Differential geometry of submanifolds has an important place in mathematics, physics and engineering. In particuar, invariant submanifolds play an important role in solving many methods in theoretical physics and applied mathematics. In general, an invariant submanifold has all the properties of an upper manifold. Therfore, invariant submanifold makes an important contribution to the development of differential geometry. In this paper, the idea of examining submanifold of generalized Kenmotsu manifold is emphasized. We defined and studied invariant submanifolds of generalized Kenmotsu manifold. We investigated geometry of invariant submanifolds. The works on this subject will be useful tools for the applications of submanifolds with different manifolds.

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