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# ON SOME POWERFUL IDEALS OF SEMIGROUPS

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Abstract. Let $S$ be a non trivial additive cancellation commutative semigroup having a zero 0 with the quotient group $G = \{s_1 - s_2 \mid s_1, s_2 \in S\}$. A prime ideal $P$ of $S$ is called strongly prime if for $a$, $b \in G$, whenever $a + b \in P$, we get $a \in P$ or $b \in P$. We show that a prime ideal of $S$ is strongly prime $\Leftrightarrow$ it is powerful. Finally, we prove that if $O$ is an oversemigroup of $S$ and $S$ and $O$ have the nonzero ideal $I$ where $I$ is powerful in $O$, then $3I$ is a powerful ideal of $S$.

## 1. Introduction and preliminaries

Let $S$ be an additive commutative cancellation semigroups with a zero 0 and $S \neq 0$. The set $G = \{s_1 - s_2 \mid s_1, s_2 \in S\}$ is a torsion-free commutative group which is called the quotient group of $S$ and $S$ is a subsemigroup of $G$. The nonempty set $O$ is called an oversemigroup of $S$ if $O$ is a subsemigroup of $G$ which contains $S$. Suppose $G$ is the quotient group of $S$. For each $a \in G$, $a$ is called an integral element over $S$ if $na \in S$ for some positive integer $n$. An oversemigroup $O$ is called an integral semigroup over $S$ if every element $a$ of $O$ is an integral element over $S$. If $S'$ is the set of all integral elements over $S$ in $G$, we term $S'$ as the integral closure of $S$. A nonempty subset $I$ of a semigroup $S$ is called an ideal of $S$ if $I + S \subset I$, that is, $x + s \in I$ for each $x \in I$ and $s \in S$. A proper ideal $P$ of $S$ is called a prime ideal of $S$ if $a + b \in P$ with $a$, $b \in S$ implies either $a \in P$ or $b \in P$. Let $N$ be the set of positive integers. The radical of an ideal $I$ is $rad(I) = \{s \in S \mid ns \in I \text{ for some } n \in N\}$. A semigroup $S$ is a valuation semigroup if $s \in G$, then $s \in S$ or $-s \in S$. We call $S$ a seminormal semigroup [3] if $2s$, $3s \in S$ for $s \in G$, we have $s \in S$. A commutative semigroup (with unit) which has only finitely many maximal ideals is called quasi-local semigroup and is denoted by $(S, X)$ where $X$ is a maximal ideal. An element $a \in S$ is called a unit if $a + b = 0$ for some $b \in S$. Hedstrom and Houston[2] defined and studied strongly prime ideals of integral domains. Ayman Badawi and Evan Houston[1] introduced and investigated powerful ideals of integral domains as a generalization of strongly prime ideals of integral domains.

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## 2. Semigroups and Powerful Ideals

We start with introducing the following definitions.

**Definition 2.1.** A prime ideal $L$ of $S$ is called strongly prime if for any $a$, $b \in G$, whenever $a + b \in L$, we get $a \in L$ or $b \in L$.

**Definition 2.2.** A semigroup $S$ is called a pseudo-valuation semigroup if every prime ideal of $S$ is strongly prime.

**Definition 2.3.** A non-zero ideal $T$ of $S$ is called powerful if for any $a$, $b \in G$, we have $a \in S$ or $b \in S$, whenever $a + b \in T$.

**Theorem 2.4.** *Every valuation semigroup is a pseudo-valuation semigroup.*

*Proof.* Suppose $S$ is a valuation semigroup, and suppose $P$ is a prime ideal of $S$. Let $a+b \in P$ where $a$, $b \in G$, the quotient group of $S$. If $a$,$b \in S$, there is nothing to prove. Let $a \notin S$. We obtain $-a \in S$ since $S$ is a valuation semigroup. Hence $b = (a + b) - a \in P$. □

To prove Theorem 2.6, we need the following Lemma.

**Lemma 2.5.** *Let $T$ be an ideal of $S$. For each $a \in G \setminus S$, we have that $T + (-a) \subseteq S$ if and only if $T$ is powerful.*

*Proof.* Let $T + (-a) \subseteq S$, for each $a \in G \setminus S$. Assume $b + c \in T$, $b$; $c \in G$. Let $b \notin S$. Then $c = (-b) + (b + c) \in (-b) + T \subseteq T$. Conversely, let $T$ be a powerful ideal of $S$, and assume $a \in G \setminus S$. Then we have $[a + (-a)] + x = x \in T$, for $x \in T$ and so $(-a) + x \in S$. □

**Theorem 2.6.** *Let $P$ be a proper subset of $T$ where $P$ is a prime ideal of $S$ and $T$ is a powerful ideal of $S$. Then $T/P$ is powerful in $S/P$.*

*Proof.* Suppose $\varphi : S \to S/P$ be the canonical homomorphism. Let $a = \varphi(b) - \varphi(c)$ is a member of the quotient group of $S/P$ and $a \notin S/P$. Therefore, $b-c \notin S$. So if $x \in T$, we obtain $(c-b)+x \in S$ and then it implies that $(\varphi(c)-[\varphi(b)+\varphi(x)] \in S/P$. Hence $S/P \supseteq (-a) + (T/P)$ by Lemma 2.5. □

**Theorem 2.7.** *A prime ideal of $S$ is strongly prime if and only if it is powerful.*

*Proof.* $\Rightarrow$ It is trivial.
$\Leftarrow$ Let $T$ be a powerful prime ideal of $S$. Suppose $a + b \in T$ for some $a$, $b \in G$. So $(2a) + (2b) \in T$. Therefore, we may now consider $a \notin S$ and $b \in S$. If $2a \in S$, then $(2a) + (2b) \in T$ and since $a \notin S$, $2a \notin T$, it follows that $2b \in T$, so $b \in T$. Again, since $[2b - (a + b)] + 2a \in T$, so if $2a \notin S$ then we have $2b - (a + b) \in S$. Hence $2b = [2b - (a + b)] + (a + b) \in T$, and we again obtain $b \in T$.

□

**Definition 2.8.** Let $L$ be a strongly prime ideal of $S$ and $I$ be an ideal of $S$. We say $L$ and $I$ are comparable if $L \subset I$ or $I \subset L$.

**Theorem 2.9.** *Suppose that $T$ is a powerful ideal of $S$. Then:*

(i) *Assume that $I$ is an ideal of $S$. Then $I \subseteq T$ or $T + T \subseteq I$.*

(ii) *Assume that $I$ is a prime ideal of $S$. Then $T$ and $I$ are comparable.*
(iii) *The prime ideals of $S$ which are contained in $Rad(T)$ are linearly ordered.*

*Proof.* (i) Let $I$ be an ideal of $S$ and $I \subsetneq T$. We choose $x \in I \setminus T$, and suppose $y$, $z \in T$. Then $[(y+z)-x]+(x-y) \in T$, and because $T$ is powerful such that $(x-y) \notin S$, we have $(y+z)-x \in S$. Hence $(y+z) \in x+S \subseteq I$.
(ii) Easily follows from (i)
(iii) Let $X$, $Y$ be prime ideals which are properly contained in $Rad(T)$. Then $X \subset T$ and $Y \subset T$ and so both $X$ and $Y$ are powerful. Hence they are comparable by (ii). □

**Theorem 2.10.** *A semigroup $S$ contains a unique greatest powerful ideal if it contains a powerful ideal.*

*Proof.* Let $\{T_\beta\}$ be a class of powerful ideals of $S$. Then by Lemma 2.5, $(-a)+T_\beta \subseteq S$ for each $\beta$ if $a \in G \setminus S$. Therefore $(-a)+\sum_\beta T_\beta \subseteq S$, and we obtain $\sum_\beta T_\beta$ is powerful by the same Lemma 2.5. □

**Theorem 2.11.** *Suppose $T$ is a powerful ideal of $S$. Let $a$, $b \in G$ and $a+b \in Rad(T)$, then there exists a positive integer $n$ such that either $na \in T$ or $nb \in T$.*

*Proof.* Obviously, $n(a+b) \in T$ for some $n > 0$. Therefore $(3n)a-(na+nb)+(3n)b-(na+nb) = na+nb \in T$. Now we have either $[3na-(na+nb)] \in S$ or $[3nb-(na+nb)] \in S$ since $T$ is powerful. Hence either $3na \in T$ or $3nb \in T$. □

The radical of a powerful ideal need not be powerful. We illustrate it by the following example.

**Example 2.12.** Suppose $E = G+I$ is a rank one discrete valuation semigroup, where $G$ is a group and $I = s+E$ is the maximal ideal of $E$. Further suppose $a+b \in 3I$, with $a$, $b \in G$. Express $a = x+ns$, $b = y+ms$, where $x$, $y$ are units of $E$ and $n$, $m$ are integers. Since $a+b \in 3I$, we obtain $n+m \geq 3$. Therefore either $n \geq 2$ or $m \geq 2$. Let $n \geq 2$. Then $a = x+ns \in 2I \subseteq S$. Hence $3I$ is a powerful ideal of $S$. However, $Rad(3I) = I+I = 2I$ is not powerful in $S$ since $2s \in 2I$ but $s \notin S$.

We now investigate a seminormal semigroup in which the radical of a powerful ideal is powerful. For this, first we introduce the following definition.

**Definition 2.13.** A radical ideal $R$ of $S$ is called strongly radical if $a \in G$ and $na \in R$ for some $n > 0$ implies that $a \in R$.

**Theorem 2.14.** *Suppose $T$ is a proper powerful ideal of $S$. Then $Rad(T)$ is powerful if and only if $Rad(T)$ is strongly radical.*

*Proof.* Obviously a powerful radical ideal is a strongly radical. Let $Rad(T)$ be strongly radical, and suppose that $(a+b) \in Rad(T)$ where $a$, $b \in G$. Then using Theorem 2.11. we obtain $pa \in t$ or $pb \in T$ for some $p > 0$. So now we may let $pa \in T$. Then $pa \in Rad(T)$ and so $a \in S$. □

**Lemma 2.15.** *Let $T$ be a powerful ideal of $S$. If $a \in G$ and $na \in T$ for some $n > 0$, then $(n+m)a \in S$ for each $m \geq 0$.*

*Proof.* Suppose $l$ =min$\{p \geq 1 \mid pa \in S\}$. Further, suppose that $m$ is a positive integer. Express $m = r + l + s$ where $0 \leq s \leq l$. If $s = 0$, then it is obvious to realize that $(n+m)a \in S$. Let $s > 0$ and we obtain $(l-s)a + (r+l+n+s)a = na + [(r+1)+l]a \in T$. We obtain $(n+m)a = (r+l+n+s)a \in S$ since $(l-s)a \notin S$. □

**Theorem 2.16.** *Suppose $T$ is a powerful ideal of $S$ and $O$ is an oversemigroup of $S$. Then we have that $T+O$ is a powerful ideal of $O$. If $T+O=O$, then $O$ is a valuation semigroup.*

*Proof.* Suppose $a \in G \setminus O$. Then $a \notin S$, and so by Lemma 2.5, $(-a)+T \subseteq S$. This shows that $(-a)+T+O \subseteq O$. Therefore, again by the same Lemma 2.5, $T+O$ is powerful in $O$. Now it is straightforward to infer that $O$ is powerful as an ideal of $O \Leftrightarrow$ a semigroup $O$ is a valuation semigroup. □

**Lemma 2.17.** *Suppose $P$ is a prime ideal of a semigroup $S$ with the quotient group $G$. Then $(-a)+P \subset P$ if and only if $P$ is strongly prime for $a \in G-S$.*

*Proof.* Let $P$ be strongly prime. We have $p = p+(-a)+a \in P$ if $a \in G-S$ and $p \in P$, and so $p+(-a) \in P$ or $a \in P$. We obtain $p+(-a) \in P$ because $a \notin S$. Hence $(-a)+P \subset P$. Conversely, let $(-a)+P \subset P$ if $a \in G-S$, and suppose $x+y \in P$. If $x$, $y \in S$, it is trivial. Therefore, we suppose that $x \notin S$ and so $(-x)+P \subset P$ and $y = (-x)+(x+y) \in P$. □

**Theorem 2.18.** *Suppose $(S, X)$ is a quasi-local semigroup. Then the following assertions are equivalent:*

(i) *$S$ is a pseudo-valuation semigroup.*
(ii) *For each pair $P$, $Q$ of ideals of $S$, either $P \subset Q$ or $X+Q \subset X+P$.*
(iii) *For each pair of ideals $P$, $Q$ of $S$, either $P \subset Q$ or $X+Q \subset P$.*
(iv) *$X$ is strongly prime.*

*Proof.* $(i) \Rightarrow (ii)$. Let $P \subsetneq Q$ and choose $x$ contained in $P-Q$. We have $x-y \notin S$ for each $y \in Q$ such that $X+y \subset X+x \subset X+P$ and $(y-x)+X \subset X$. Hence $X+P \supset X+Q$.
$(ii) \Rightarrow (iii)$. This is straightforward.
$(iii) \Rightarrow (iv)$. Suppose $x$, $y \in S$ and $(x-y) \notin S$. We have $(x) \subsetneq (y)$ since $(x-y) \notin S$ and hence $X+y \subset (x)$ and $(X+y)-a \subset S$. We obtain $X = (S+x)-y$ and $(x-y) \in S$ if $(X+y)-a = S$. This leads to a contradiction. Therefore $X+(y-a) \subset X$.
$(iv) \Rightarrow (i)$. Suppose $a$ is a member of the quotient group $G$ of $S$, $a \notin S$, and assume that $P$ is a prime ideal. So, by Lemma 2.17, it suffices to prove that $(-a)+P \subset P$. Suppose $p \in P$ and because $P \subset X$, we obtain $(-a)+p \in X$. Therefore $(-a)+p+(-a) \in X$, and so $[(-a)+p]+[(-a)+p] = (-a)+p+(-a)+p \in P$. Thus we obtain $(-a)+p \in P$ because $P$ is prime and $(-a)+p \in S$. □

**Theorem 2.19.** *Let $T$ be a powerful ideal of $S$ and $X = Rad(T)$ is a maximal ideal of $S$. Then:*

(i) *$S$ is quasilocal having a maximal ideal $X$.*
(ii) *$T+S' \subseteq X$, and so $T+S'$ is an ideal of $S$.*

*Proof.* (i) Immediately follows from Theorem 2.9. For (ii), suppose $a \in S' \setminus S$. Recall that $(-a) \notin S$ since $S'$ is integral over $S$. Therefore, by Lemma 2.5, $a+T \subseteq S$. If $a+T = S$, then $-a \in S$. So, indeed, $a+T \subseteq X$. This implies that $T + S' \subseteq X$.

□

**Theorem 2.20.** *Suppose $T$ is a powerful ideal of $S$, and suppose $O \neq G$ is an oversemigroup of $S$. Then both $S$ and $O$ have a common ideal that is powerful in both $S$ and $O$. We have the following statements:*

(i) *Let $X$ be the maximal ideal of $O$. If $T + O = O$, then $Q = X \cap S$ is a shared ideal which is powerful in both semigroups.*
(ii) *If $T + O \neq O$, then $2T + O$ is a common ideal, and $3T + O$ is powerful in both semigroups.*

*Proof.* (i) By Theorem 2.16, $O$ is a valuation semigroup and by Theorem 2.9, $T$ is comparable to $Q$. It follows that $Q \subsetneq T$, and so $Q$ is powerful and hence strongly prime in $S$. We recall that $Q + O$ is powerful in $O$ by Theorem 2.16. Suppose $a \in O \setminus S$. Obviously, $(-a) \notin Q$. Therefore we obtain $a + Q \subseteq Q$ since $(-a) + a + Q \subseteq Q$ and $Q$ is strongly prime.
(ii) Suppose $a \in O \setminus S$. If $-a \notin S$, then $a + 2T \subseteq a + T \subseteq S$ by Lemma 2.5. If $-a \in S$, then by the assumption, $-a \notin T$, and so $2T \subseteq (-a)+S$ by Theorem 2.9. Therefore, $a + 2T \subseteq S$. Hence $2T + O$ is an ideal of $S$. Now $3T + O$ is powerful in $S$ since $3T + O \subseteq T$ and by Theorem 2.16, $3T + O$ is powerful in $O$. □

**Theorem 2.21.** *Let $O$ be an oversemigroup of $S$ and that $S$ and $O$ have the nonzero ideal $I$. If $I$ is powerful in $O$, then $3I = I + I + I$ is a powerful ideal of $S$.*

*Proof.* Suppose $a \in G \setminus S$. We have that $(-a) + I \subseteq O$ by Lemma 2.5 if $a \notin O$. So we obtain $(-a) + 3I \subseteq 2I + O \subseteq S$. Let $a \in O$. By Theorem 2.9, we obtain $2I \subseteq a + O$ since $a \notin I$. Hence $(-a) + 3I \subseteq I + O = I \subseteq S$. □

If $O$ is a valuation semigroup, then $2I = I + I$ the second multiple of $I$ is powerful in $S$. But, for general $O$, the third multiple is best possible. We support it by the following example.

**Example 2.22.** Suppose $G$ is the quotient group of the form $Q(\sqrt{2}) = \{a+b\sqrt{2} \mid a,\, b \in Q\}$, where $Q$ is a set of rational number and suppose that $B = G[[Y]] = G+X$, $X = Y+G[[Y]]$. Now suppose that $O = Q+X$, $I = Y+O$, and $S = Q+I$. Then $S$ and $O$ have the ideal $I$, and since $O$ is a pseudo valuation semigroup, $I$ is powerful in $O$. However, $2I$ is not powerful in $S$, since $\sqrt{2}Y+\sqrt{2}Y = 2\sqrt{2}Y \in 2I$, but $\sqrt{2}Y \notin S$.

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