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COHERENCE PROPERTIES IN BI-AMALGAMATED MODULES.

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Abstract. In this paper, we introduce and investigate the transfer of the property of coherence to the bi-amalgamation module $M \bowtie^{\phi,\psi} (JN, J'P)$. We provide necessary and sufficient conditions for $M \bowtie^{\phi,\psi} (JN, J'P)$ to be a coherent module.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. Let A be a ring and let I be an ideal of A. In [2], the authers introduced the following subring:

$$A \bowtie I := \{(a, a+i) | a \in A, i \in I\}$$

of the direct product $A \times A$ and named it *the amalgamated duplication of A along I*. This ring constrction aries as a sort of variant of the Nagata idealization $A(+)X$ of an A-module X: it is defined by endowing the product $A \times X$ with the ring structure where addition is induced componentwise from additions of A and X and multiplication is defined by setting

$$(a, x)(b, y) := (ab, ay + bx), \text{ for all } (a, x), (b, y) \in A(+)X,$$

in such a way the ring A is identified as a subring of $A(+)X$ and the A-module X is isomorphic to the ideal $0 \times X$ of $A(+)X$. It can be proved that if I is a nilpotent ideal of intex 2, then the duplication $A \bowtie I$ is canonically isomorphic to the idealization $A(+)I$, but in general, this is not the case, since if A is a reduced ring, then $A(+)I$ is a reduced ring, while $(0) \times X$ is a nilpotent ideal of intex 2 of $A(+)X$, whenever $X \neq 0$. Among motivations for studying duplications of rings along ideals, it is worth noting that in [1] the author showed that the amalgamated duplication of an algebroid curve along a regular multiplicative ideal always yields an algebroid curve.

As it is immediately seen, whenever I is a nonzero ideal of A, the ring $A \bowtie I$ is not an integral domain. A new ring construction, which is more general than the amalgamated duplication and can an integral domain, was introduced in [3] as follows: starting from a ring homomorphim $f : A \longrightarrow B$ and from an ideal J

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of B, the auther defined *the amalgamation of A and B along J, with respect to f* to be the following subring:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

of the direct product $A \times B$. Obviously, if $f = Id_A : A \longrightarrow A$ and J is an ideal of A, then $A \bowtie^f J$ is the classical amalgamated duplication. This new ring construction, whose genesis is in some sense related to the procedure due to Dorroh of endowing a ring (without unity) with a unity (see, for instance, [2, Remark 2.2]), is a powerful tool for unifying several classical classes of rings, as $D+m$ domains, ring of the type $A+XB[X]$, $A+XB[[X]]$, CPI extensions. The fact that rings of the form $A \bowtie^f J$ arise with a peculiar fiber product structure allowed to provide a complete description of the order theoretic and topological properties of its prime spectrum and furthermore to find bounds for its Krull dimension [4].

A new class of rings, which covers that of amalgamated algebras as a particular case, was recently introduced in [11]: given ring homomorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$ and given ideals J of B and J' of C such that $f^{-}(J) = g^{-}(J')$, the subring

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j')|a \in A, (j, j') \in J \times J'\}$$

of $B \times C$ is called *the bi-amalgamtion of A with (B, C) along (J, J') with respect to (f, g)*.

Let A be a ring, I an ideal of A, and M an A-module. The authors of [6] introduced *the duplication of the A-module M along the ideal I* denoted by $M \bowtie I$ and gave the following definition

$$M \bowtie I = \{(m, m') \in M \times M|m - m' \in IM\}$$

which is an $A \bowtie I$-module with the multiplication given by

$$(a, a + i).(m, m') = (am, (a + i)m'),\text{where } a \in A, i \in I, \text{ and } (m, m') \in M \bowtie I.$$

If $M = A$, then the duplication of the A-module A along the ideal I coincides with the amalgamated duplication of the ring A along the ideal I. In their article, they studied some basic properties of the duplication of an A-module M along an ideal I. More precisely, they studied when $M \bowtie I$ is a Noetherian, an Artinian or an (Nil_*)-coherent $A \bowtie I$-module. They also investigated some basic homological properties of $M \bowtie I$: when $M \bowtie I$ is an injective module, a projective module or a flat module.

In [12], recently introduced the generalization of the duplication of modules along an ideal called *amalgamated modules along ideal*. Let $f : A \longrightarrow B$ be a ring homomorphism, J be an ideal of B, M be an A-module, N be an B-module (which is an A-module induced naturally by f) and $\phi : M \longrightarrow N$ be an A-module homomorphism. They defined the amalgamation of M and N along J with respect to ϕ by

$$M \bowtie^\phi JN := \{(m, \phi(m) + n)|m \in M, n \in JN\}.$$

It can be seen that $M \bowtie^\phi JN$ is an $A \bowtie^f J$-module by the following scalar product

$$(a, f(a) + j)(m, \phi(m) + n) := (am, \phi(am) + f(a)n + j\phi(m) + jn.$$

Note that $\phi(am) = f(a)\phi(m)$, since ϕ is an A-module homomorphism. If $M = A$, $N = B$ and $\phi = f$, then the amalgamation of the A-module A and the B-module B and B along J with respect to ϕ coincides with the amalgamation of rings A and B along J with respect to f. In their article, they studied some basic properties of the amalgamation modules along ideal. More precisely, they studied when $M \bowtie^{\phi} JN$ is a Noetherian, coherent $A \bowtie^{f} J$-module.

In this paper, we introduce the generalization of the amalgamation of modules along ideals. Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms, J be an ideal of B and J' be an ideal of C and let M be an A-module, N be an B-module (which is an A-module induced naturally by f) and P be an C-module (which is an A-module induced naturally by g) and $\phi : M \longrightarrow N, \psi : M \longrightarrow P$ are two A-module homomorphisms. we define the bi-amalgamation of M with (N, P) along $(JN, J'P)$ with respect to (ϕ, ψ) with $I := \phi^{-1}(JN) = \psi^{-1}(J'P)$ and defined by

$$M \bowtie^{\phi,\psi} (JN, J'P) := \{(\phi(m) + n, \psi(m) + p)| \in M, n \in JN, p \in J'P\}$$

It can be seen that $M \bowtie^{\phi,\psi} (JN, J'P)$ is an $A \bowtie^{f,g} (J, J')$-module by the following scalar product $(f(a) + j, g(a) + j')(\phi(m) + n, \psi(m) + p) :=$ $(\phi(am)+f(a)n+j\phi(m)+jn, \psi(am)+g(a)p+j'\psi(m)+j'p)$. Note that $\phi(am) = f(a)\phi(m)$ and $\psi(am) = g(a)\psi(m)$, since ϕ and ψ are A-module homomorphisms.

In this paper, we study some basic properties of the bi-amalgamation of modules. More precisely, we study when $M \bowtie^{\phi,\psi} (JN, J'P)$ is a Noetherian or a coherent $A \bowtie^{f,g} (J, J')$-module. The new results generalize some known on the bi-amalgamation of rings and the amalgamation of module along an ideal.

2. Bi-amalgamation of modules along ideal

In this section, we introduce the definition and pullback of bi-amalgamation of M with (N, P) along $(JN, J'P)$ with respect to (ϕ, ψ), $M \bowtie^{\phi,\psi} (JN, J'P)$.

Definition 2.1. Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms, J be an ideal of B and J' be an ideal of C and let M be an A-module, N be an B-module (which is an A-module induced naturally by f) and P be an C-module (which is an A-module induced naturally by g) and $\phi : M \longrightarrow N, \psi : M \longrightarrow P$ are two A-module homomorphisms. we define the bi-amalgamation of M with (N, P) along $(JN, J'P)$ with respect to (ϕ, ψ) with $I := \phi^{-1}(JN) = \psi^{-1}(J'P)$ and defined by

$$M \bowtie^{\phi,\psi} (JN, J'P) := \{(\phi(m) + n, \psi(m) + p)| \in M, n \in JN, p \in J'P\}$$

Recall that, the following diagram of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\iota_2} & B \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_1} \\ C & \xrightarrow{\iota_1} & D \end{array}$$

is called a *pullback* (or *fiber product*) if the homomorphism $\iota_2 \times \mu_2 : A \longrightarrow B \times C$ induces an isomorphism of A onto the subring of $B \times C$ given by

$$\mu_1 \times \iota_1 := \{(b, c) | \mu_1(b) = \iota_1(c)\}$$

That of fiber product is the typical algebraic structure intervening in the classical operation of attaching spectral spaces. Thus this type of construction has a prominent role to produce rings whose prime spectra have to satisfy prescribed conditions. For a deeper understanding of this circle of ideas see, for instance, [9].

Proposition 2.2. *Consider the module homomorphisms* $\alpha : \phi(M)+JN \longrightarrow M/I$ *and is defined by* $\alpha(\phi(m) + n) = m + I$ *and* $\beta : \psi(M) + J'P \longrightarrow M/I$ *and is defined by* $\beta(\psi(m) + p) = m + I$. *Then, the bi-amalgamation is determined by the following pullback*

$$\begin{tikzcd}
M \bowtie^{\phi,\psi} (JN, J'P) \arrow[r] \arrow[d] & \phi(M) + JN \arrow[d, "\alpha"] \\
\psi(M) + J'P \arrow[r, "\beta"] & M/I
\end{tikzcd}$$

that is $M \bowtie^{\phi,\psi} (JN, J'P) = \alpha \times_{M/I} \beta$

Proof. Clearly the maps α and β are well defined since $I := \phi^{-1}(JN) = \psi^{-1}(J'P)$ and are module homomorphisms. we observe that the inclusion $M \bowtie^{\phi,\psi} (JN, J'P) \subseteq \alpha \times_{M/I} \beta$ is trivial. Now we enough to prove that $\alpha \times_{M/I} \beta \subseteq M \bowtie^{\phi,\psi} (JN, J'P)$. The pullback can be written as follows

$$\alpha \times_{M/I} \beta = \{(\phi(m_1) + n, \psi(m_2) + p) | m_1, m_2 \in M, (n, p) \in JN \times J'P, \alpha(m_1) = \beta(m_2)\}$$

The condition $\alpha(m_1) = \beta(m_2)$ means that $\phi(m_2 - m_1) \in JN$ and $\psi(m_2 - m_1) \in J'P$. Now we get $\psi(m_2)+p = \psi(m_1)-\psi(m_1)+\psi(m_2)+p = \psi(m_1)+(p+\psi(m_2-m_1))$ with $p + \psi(m_2 - m_1) \in J'P$. Hence $\alpha \times_{M/I} \beta \subseteq M \bowtie^{\phi,\psi} (JN, J'P)$. □

It is easily observed that the bi-amalgamation $M \bowtie^{\phi,\psi} (JN, J'P)$ is isomorphic to the fiber product of the module homomorphisms $i_{\phi,\psi}$ and π (that make the following diagram commute)

$$\begin{tikzcd}
M \bowtie^{\phi,\psi} (JN, J'P) \arrow[r] \arrow[d] & M/I \arrow[d, "i_{\phi,\psi}"] \\
N \times P \arrow[r, "\pi"] & \frac{N}{JN} \times \frac{P}{J'P}
\end{tikzcd}$$

where $\pi : N\times P \longrightarrow \frac{N}{JN}\times\frac{P}{J'P}$ is the canonical projection and $i_{\phi,\psi} : M/I \longrightarrow \frac{N}{JN}\times\frac{P}{J'P}$ is the canonical embeding defined by $i_{\phi,\psi}(m+I) := (\phi(m)+JN, \psi(m)+J'P)$, for any $m+I \in M/I$.

Remark 2.3. (1) Let $f(A)+J$ be a subring of B. Thus N is an $f(A)+J$-module. It is clear that $\phi(M)+JN$ is an $f(A)+J$-submodule of N. Hence, $\phi(M)+JN$ is an $A\bowtie^{f,g}(J,J')$-module via $P_B : A\bowtie^{f,g}(J,J') \longrightarrow f(A)+J$ defined by $P_B(f(a)+j, g(a)+j') = f(a)+j$.

(2) Let $g(A)+J'$ be a subring of C. Thus P is an $g(A)+J'$-module. It is clear that $\psi(M)+J'P$ is an $g(A)+J'$-submodule of P. Hence, $\psi(M)+J'P$ is an $A\bowtie^{f,g}(J,J')$-module via $P_C : A\bowtie^{f,g}(J,J') \longrightarrow g(A)+J'$ defined by $P_C(f(a)+j, g(a)+j') = g(a)+j'$.

(3) $\pi_N : M\bowtie^{\phi,\psi}(JN,J'P) \longrightarrow \phi(M)+JN$ defined by $\pi_N(\phi(m)+n, \psi(m)+p) = \phi(m)+n$.

(4) $\pi_P : M\bowtie^{\phi,\psi}(JN,J'P) \longrightarrow \psi(M)+J'P$ defined by $\pi_P(\phi(m)+n, \psi(m)+p) = \psi(m)+p$.

(5) We have the following exact sequences of $A\bowtie^{f,g}(J,J')$-module and $A\bowtie^{f,g}(J,J')$ homomorphisms:

$$0 \longrightarrow JN\times 0 \xrightarrow{\imath} M\bowtie^{\phi,\psi}(JN,J'P) \xrightarrow{\pi_N} \phi(M)+JN \longrightarrow 0$$

where $\imath : JN\times 0 \longrightarrow M\bowtie^{\phi,\psi}(JN,J'P)$ given by $\imath(n,0) = (n,0)$ and

$$0 \longrightarrow 0\times J'P \xrightarrow{\imath'} M\bowtie^{\phi,\psi}(JN,J'P) \xrightarrow{\pi_P} \psi(M)+J'P \longrightarrow 0$$

where $\imath' : 0\times J'P \longrightarrow M\bowtie^{\phi,\psi}(JN,J'P)$ given by $\imath'(0,p) = (0,p)$

Proposition 2.4. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms, J be an ideal of B and J' be an ideal of C and let M be an A-module, N be an B-module and P be an C-module and $\phi : M \longrightarrow N, \psi : M \longrightarrow P$ are two A-module homomorphisms with $\phi^{-1}(JN) = \psi^{-1}(J'P)$. Then the following hold:*

(1) $\frac{M\bowtie^{\phi,\psi}(JN,J'P)}{0\times J'P} \cong \phi(M)+JN$

(2) $\frac{M\bowtie^{\phi,\psi}(JN,J'P)}{JN\times 0} \cong \psi(M)+J'P$

Proof. (1) If $\phi(m)+n = 0$ for some $m\in M$ and $n\in JN$, then $\psi(m)+p\in J'P$ for any $p\in J'P$. So the kernel of the surjective canonical homomorphism $M\bowtie^{\phi,\psi}(JN,J'P) \twoheadrightarrow \phi(M)+JN$ coincides with $0\times J'P$. Hence, the result follows by the first isomorphism theorem.

(2) If $\psi(m)+p = 0$ for some $m\in M$ and $p\in J'P$, then $\phi(m)+n\in JN$ for any $n\in JN$. So the kernel of the surjective canonical homomorphism $M\bowtie^{\phi,\psi}(JN,J'P) \twoheadrightarrow \psi(M)+J'P$ coincides with $JN\times 0$. Hence, the result follows by the first isomorphism theorem. □

In [11], the authers determined the Noetherian property of the bi-amalgamated algebra $A\bowtie^{f,g}(J,J')$. We will now see when the bi-amalgamation of modules along ideals are Noetherian.

Proposition 2.5. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms, J be an ideal of B and J' be an ideal of C and let M be an A-module, N be an B-module and P be an C-module and $\phi : M \longrightarrow N, \psi : M \longrightarrow P$ are two*

A-module homomorphisms. Assume that M is a Noetherian A-module. The bi-amalgamation $M \bowtie^{\phi,\psi} (JN, J'P)$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module if and only if $\phi(M) + JN$ is a Noetherian $f(A) + J$-module and $\psi(M) + J'P$ is a Noetherian $g(A) + J'$-module.

Proof. ($\Rightarrow$) $\phi(M) + JN$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module by using the homomorphism $\pi_N : M \bowtie^{\phi,\psi} (JN, J'P) \longrightarrow \phi(M) + JN$. Hence $\phi(M) + JN$ is a Noetherian $f(A) + J$-module by using P_B. And also $\psi(M) + J'P$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module by using the homomorphism $\pi_P : M \bowtie^{\phi,\psi} (JN, J'P) \longrightarrow \psi(M) + J'P$. Hence $\psi(M) + J'P$ is a Noetherian $g(A) + J'$-module by using P_C.

($\Leftarrow$) Since $\phi(M) + JN$ is a Noetherian $f(A) + J$-module, then $\phi(M) + JN$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module. Hence JN is a Noetherian $A \bowtie^{f,g} (J, J')$-module. Therefore $M \bowtie^{\phi,\psi} (JN, J'P)$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module by Part 5 of 2.3. Further, Since $\psi(M) + J'P$ is a Noetherian $g(A) + J'$-module, then $\psi(M) + J'P$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module. Hence $J'P$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module. Therefore $M \bowtie^{\phi,\psi} (JN, J'P)$ is a Noetherian $A \bowtie^{f,g} (J, J')$-module by Part 5 of 2.3. Hence complete the proof. □

3. Coherent property

In this section, we will see when the bi-amalgamation of modules along ideals are coherent.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms, J be an ideal of B and J' be an ideal of C and let M be an A-module, N be an B-module and P be an C-module and $\phi : M \longrightarrow N, \psi : M \longrightarrow P$ are two A-module homomorphisms and let n and s be two positive integers. Consider $\phi^n : M^n \longrightarrow N^n$ defined by $\phi^n((m_i)_{i=1}^{i=n}) = (\phi(m_i)_{i=1}^{i=n})$ and $\psi^s : M^s \longrightarrow P^s$ defined by $\psi^s((m_i)_{i=1}^{i=s}) = (\psi(m_i)_{i=1}^{i=s})$. Obviously, ϕ^n and ψ^s are A-module homomorphisms, and $JN = (JN)^n$ is a submodule of N^n and $J'P = (J'P)^s$ is a submodule of P^s. This allows us to define $M^n \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$ and $M^s \bowtie^{\phi^s,\psi^s} ((JN)^s, (J'P)^s)$.

We recall that the B-module N is an A-module induced by f and C-module P is an A-module induced by g, it is the same for N^n and P^s respectively. Hence, $ax = f(a)x$ for all $a \in A$ and $x \in N^n$ and so

$$\phi^n(am) = a\phi^n(m) = f(a)\phi^n(m)$$

for all $a \in A$ and $x \in N^n$ and $ay = g(a)y$ for all $a \in A$ and $y \in P^s$ and so

$$\psi^s(am) = a\psi^s(m) = g(a)\psi^s(m)$$

for all $a \in A$ and $y \in P^s$. We will use this in the proof of the following proposition.

Proposition 3.1. *Let F be a submodule of M^n. Then the following hold:*

1. *Assume that F is a finitely generated A-module, JN is a finitely generated $(f(A)+J)$-module and $J'P$ is a finitely generated $(g(A)+J')$-module. Then $F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module.*
2. *Suppose that F is a finitely generated A-module, $\phi^n(F) \subseteq (JN)^n$ and $\psi^n(F) \subseteq (J'P)^n$. Then $F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$ is a finitely generated*

$A \bowtie^{f,g} (J, J')$-module if and only if JN is a finitely generated $(f(A)+J)$-module and $J'P$ is a finitely generated $(g(A)+J')$-module.

Proof. (1) Assume that $F = \sum_{i=1}^{m} Af_i$, where $f_i \in A$ for all $i \in \{1, 2, ..., m\}$. Also let $(JN)^n = \sum_{i=1}^{m}(f(A) + J)n_i$ where $n_i \in (JN)^n$ since JN is a finitely generated $(f(A) + J)$-module and let $(J'P)^n = \sum_{i=1}^{m}(g(A) + J')p_i$ where $p_i \in (J'P)^n$ since $J'P$ is a finitely generated $(g(A) + J')$-module. We will prove that $F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n) = \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(\phi^n(f_i), \psi^n(f_i)) + \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(n_i, 0) + \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(0, p_i)$.
Indeed, $\sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(\phi^n(f_i), \psi^n(f_i)) + \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(n_i, 0) + \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(0, p_i) \subseteq F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$ since $(\phi^n(f_i), \psi^n(f_i)) \in F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$, $(n_i, 0) \in F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$ and $(0, p_i) \in F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$ for all $i \in \{1, 2, ..., m\}$. Let $(\phi^n(x) + k, \psi^n(x) + k') \in F \bowtie^{\phi^n,\psi^n} ((JN)^n, (J'P)^n)$ where $x \in F$ and $k \in (JN)^n, k' \in (J'P)^n$. Then $x = \sum_{i=1}^{m} a_i f_i$, $k = \sum_{i=1}^{m}(f(b_i) + j_i)n_i$ and $k' = \sum_{i=1}^{m}(g(c_i) + j_i')p_i$ where $a_i \in A$, $(f(b_i) + j_i) \in f(A) + J$ and $(g(c_i) + j_i') \in g(A) + J'$ for all $i \in \{1, 2, ..., m\}$. Hence $(\phi^n(x) + k, \psi^n(x) + k')$

$$
\begin{aligned}
&= \left(\phi^n(\sum_{i=1}^{m} a_i f_i) + \sum_{i=1}^{m}(f(b_i) + j_i)n_i, \psi^n(\sum_{i=1}^{m} a_i f_i) + \sum_{i=1}^{m}(g(c_i) + j_i')p_i\right) \\
&= \left(\sum_{i=1}^{m} \phi^n(a_i)\phi^n(f_i) + \sum_{i=1}^{m}(f(b_i) + j_i)n_i, (\sum_{i=1}^{m} \psi^n(a_i)\psi^n(f_i) + \sum_{i=1}^{m}(g(c_i) + j_i')p_i\right) \\
&= \left(\sum_{i=1}^{m} f(a_i)\phi^n(f_i) + \sum_{i=1}^{m}(f(b_i) + j_i)n_i, (\sum_{i=1}^{m} g(a_i)\psi^n(f_i) + \sum_{i=1}^{m}(g(c_i) + j_i')p_i\right) \\
&= \left(\sum_{i=1}^{m} f(a_i)\phi^n(f_i), \sum_{i=1}^{m} g(a_i)\psi^n(f_i)\right) + \left(\sum_{i=1}^{m}(f(b_i) + j_i)n_i, 0\right) \\
&\quad + \left(0, \sum_{i=1}^{m}(g(c_i) + j_i')p_i\right) \\
&= \sum_{i=1}^{m}(f(a_i), g(a_i))(\phi^n(f_i), \psi^n(f_i)) + \sum_{i=1}^{m}(f(b_i) + j_i), 0)(n_i, 0) \\
&\quad + \sum_{i=1}^{m}(0, g(c_i) + j_i')(0, p_i) \\
&= \sum_{i=1}^{m}(f(a_i), g(a_i))(\phi^n(f_i), \psi^n(f_i)) + \sum_{i=1}^{m}(f(b_i) + j_i), g(b_i)(n_i, 0) \\
&\quad + \sum_{i=1}^{m}(f(c_i), g(c_i) + j_i')(0, p_i)
\end{aligned}
$$

Therefore, $(\phi^n(x) + k, \psi^n(x) + k') \in \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(\phi^n(f_i), \psi^n(f_i)) + \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(n_i, 0) + \sum_{i=1}^{m}(A \bowtie^{f,g} (J, J'))(0, p_i)$ and so

$F \bowtie^{\phi^n,\psi^n} ((JN)^n,(J'P)^n) = \sum_{i=1}^m (A \bowtie^{f,g} (J,J'))(\phi^n(f_i),\psi^n(f_i)) + \sum_{i=1}^m (A \bowtie^{f,g} (J,J'))(n_i,0) + \sum_{i=1}^m (A \bowtie^{f,g} (J,J'))(0,p_i)$ is a finitely generated $A \bowtie^{f,g} (J,J')$-module as desired.

(2) If JN is a finitely generated $f(A)+J$-module and $J'P$ is a finitely generated $g(A)+J'$-module, then $F \bowtie^{\phi^n,\psi^n} ((JN)^n,(J'P)^n)$ is a finitely generated $A \bowtie^{f,g} (J,J')$-module by (1). Conversely, assume that $F \bowtie^{\phi^n,\psi^n} ((JN)^n,(J'P)^n)$ is a finitely generated $A \bowtie^{f,g} (J,J')$-module. Then there exists $f_i \in F$, $n_i \in (JN)^n$ and $p_i \in (J'P)^n$ for $i \in \{1,2,...,m\}$ such that

$$F \bowtie^{\phi^n,\psi^n} ((JN)^n,(J'P)^n) = \sum_{i=1}^m \left(A \bowtie^{f,g} (J,J')\right)(\phi^n(f_i)+n_i,\psi^n(f_i)+p_i).$$

On the otherhand, we claim that $(JN)^n = \sum_{i=1}^m (f(A)+J)(\phi^n(f_i)+n_i)$ and $(J'P)^n = \sum_{i=1}^m (g(A)+J')(\psi^n(f_i)+p_i)$. Indeed, let $k \in (JN)^n$ and $k' \in (J'P)^n$. Then $(k,k') = \sum_{i=1}^m (f(a_i)+j_i, g(a_i)+j_i')(\phi^n(f_i)+n_i,\psi^n(f_i)+p_i)$ for some $a_i \in A, j_i \in J$ and $j' \in J'$. Thus, $k = \sum_{i=1}^m (f(a_i)+j_i)(\phi^n(f_i)+n_i) \in \sum_{i=1}^m (f(A)+J)(\phi^n(f_i)+n_i)$ and $k' = \sum_{i=1}^m (g(a_i)+j_i')(\psi^n(f_i)+p_i) \in \sum_{i=1}^m (g(A)+J')(\psi^n(f_i)+p_i)$. So $(JN)^n \subseteq \sum_{i=1}^m (f(A)+J)(\phi^n(f_i)+n_i)$ and $(J'P)^n \subseteq \sum_{i=1}^m (g(A)+J')(\psi^n(f_i)+p_i)$. Clearly $\phi^n(f_i) \in (JN)^n$ for all $i \in \{1,2,...,m\}$ and so $(\psi^n(f_i)+n_i) \in (JN)^n$ for all $i \in \{1,2,...,m\}$ since $\phi^n(F) \subseteq (JN)^n$. Hence $\sum_{i=1}^m (f(A)+J)(\phi^n(f_i)+n_i) \subseteq (JN)^n$. Thus $(JN)^n = \sum_{i=1}^m (f(A)+J)(\phi^n(f_i)+n_i)$ is a finitely generated $f(A)+J$-module and so JN is a finitely generated $f(A)+J$-module. Similarly $(J'P)^n \subseteq \sum_{i=1}^m (g(A)+J')(\psi^n(f_i)+p_i)$. Since $\psi^n(F) \subseteq (J'P)^n$, it follows that $\psi^n(f_i) \in (J'P)^n$ for all $i \in \{1,2,...,m\}$ and so $(\psi^n(f_i)+p_i) \in (J'P)^n$ for all $i \in \{1,2,...,m\}$. Hence $\sum_{i=1}^m (g(A)+J')(\psi^n(f_i)+p_i) \subseteq (J'P)^n$. Thus $(J'P)^n = \sum_{i=1}^m (g(A)+J')(\phi^n(f_i)+p_i)$ is a finitely generated $g(A)+J'$-module and so $J'P$ is a finitely generated $g(A)+J'$-module. This completes the proof. $\square$

Recall that an A-module M is called a coherent A-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

Theorem 3.2. *Let A be a ring and let $0 \longrightarrow P \xrightarrow{\alpha} N \xrightarrow{\beta} M \longrightarrow 0$ be an exact sequence of A modules, then:*

1. *If N is a coherent module and P is a finitely generated module then M is a coherent module.*
2. *If any of the modules are coherent, so is the third.*

Proof. This was proved in [[10], Theorem 2.2.1]. $\square$

Theorem 3.3. *Let A and B be rings and let $\phi : A \longrightarrow B$ be a ring homomorphism making B a finitely generated A module. Let M be an B module which is coherent as an A module, then M is coherent as an B module.*

Proof. This was proved in [[10] Theorem 2.2.7]. $\square$

Remark 3.4. Let $g : A \longrightarrow B$ be a surjective ring morphism with kernel K. If T is an A-module annihitated by K, then T is canonically a B-module and $AX = BX$ for each subset of X of T (let us say that the A-module structure on T is essentially the same as the B-module structure on T).

Theorem 3.5. (1) *$\{0\} \times J'P$ is a finitely generated $A \bowtie^{f,g} (J,J')$-module if and only if $J'P$ is a finitely generated $g(A)+J'$-module.*

(2) *$JN \times \{0\}$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module if and only if JN is a finitely generated $f(A) + J$-module.*
(3) *If $M \bowtie^{\phi,\psi} (JN, J'P)$ is a coherent $A \bowtie^{f,g} (J, J')$-module and JN is a finitely generated $f(A)+J$-module, then $\psi(M)+J'P$ is a coherent $g(A)+J'$-module.*
(4) *If $M \bowtie^{\phi,\psi} (JN, J'P)$ is a coherent $A \bowtie^{f,g} (J, J')$-module and $J'P$ is a finitely generated $g(A) + J'$-module, then $\phi(M) + JN$ is a coherent $f(A) + J$-module.*

Proof. (1) Consider the second component projection $P_B : A \bowtie^{f,g} (J, J') \longrightarrow g(A)+J'$ defined by $P_B(f(a)+j, g(a)+j') = g(a)+j'$. Then, for each subset X of the $A \bowtie^{f,g} (J, J')$-module $\{0\} \times J'P$, one has $(A \bowtie^{f,g} (J, J'))X = (g(A) + J')X$.

(2) Consider the first component projection $P_A : A \bowtie^{f,g} (J, J') \longrightarrow f(A) + J$ defined by $P_A(f(a)+j, g(a)+j') = f(a)+j$. Then, for each subset X of the $A \bowtie^{f,g} (J, J')$-module $JN\times\{0\}$, one has $(A \bowtie^{f,g} (J, J'))X = (f(A)+J)X$.

(3) Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is a coherent $A \bowtie^{f,g} (J, J')$-module and JN is a finitely generated $f(A) + J$-module. Then $JN \times \{0\}$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module by (2). By Proposition 2.4, $\frac{M\bowtie^{\phi,\psi}(JN,J'P)}{JN\times\{0\}} \cong \psi(M) + J'P$ and we have the exact sequence

$$0 \longrightarrow JN \times 0 \longrightarrow M \bowtie^{\phi,\psi} (JN, J'P) \longrightarrow \psi(M) + J'P \longrightarrow 0.$$

In light of Theorem 3.2, $\psi(M) + J'P$ is a coherent $A \bowtie^{f,g} (J, J')$-module and since $g(A)+J'$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module $g(A)+J' \cong (\{0\} \times g(A) + j') = (A \bowtie^{f,g} (J, J'))(0, 1)$, then $\psi(M) + J'P$ is a coherent $g(A) + J'$-module by Theorem 3.3.

(4) Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is a coherent $A \bowtie^{f,g} (J, J')$-module and $J'P$ is a finitely generated $g(A) + J'$-module. Then $\{0\} \times J'P$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module by (1). By Proposition 2.4, $\frac{M\bowtie^{\phi,\psi}(JN,J'P)}{\{0\}\times J'P} \cong \phi(M) + JN$ and we have the exact sequence

$$0 \longrightarrow 0 \times J'P \longrightarrow M \bowtie^{\phi,\psi} (JN, J'P) \longrightarrow \phi(M) + JN \longrightarrow 0.$$

In light of Theorem 3.2, $\phi(M) + JN$ is a coherent $A \bowtie^{f,g} (J, J')$-module and since $f(A)+J$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module $f(A)+J \cong (f(A) + J \times \{0\}) = (A \bowtie^{f,g} (J, J'))(1, 0)$, then $\phi(M) + JN$ is a coherent $f(A) + J$-module by Theorem 3.3.

□

Theorem 3.6. *Let A be a ring and let I be a finitely generated ideal of A. Then an A/I module M is A/I coherent iff it is A coherent. In particular, for a ring A and an ideal I of A, we have:*

(1) *If A is a coherent ring and I is a finitely generated ideal, then A/I is a coherent ring.*
(2) *If A/I is a coherent ring and I is a coherent A module, then A is a coherent ring.*

Proof. This was proved in [[10] Theorem 2.4.1]. □

Theorem 3.7. (1) *Assume that $J'P$ is finitely generated $g(A)+J'$-module and M is coherent A-module. Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module, then $\phi(M)+JN$ is coherent $f(A)+J$-module.*

(2) *Assume that JN is finitely generated $f(A)+J$-module and M is coherent A-module. Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module, then $\psi(M)+J'P$ is coherent $g(A)+J'$-module.*

(3) *Assume that J' is a finitely generated ideal of $g(A)+J'$ and ψ is surjective. If $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module, then $\phi(M)+JN$ is coherent $f(A)+J$-module.*

(4) *Assume that J is a finitely generated ideal of $f(A)+J$ and ϕ is surjective. If $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module, then $\psi(M)+J'P$ is coherent $g(A)+J'$-module.*

(5) *Assume that JN and $J'P$ are finitely generated $f(A)+J$-module and $g(A)+J'$-module respectively. Also assume that M is a coherent A-module. Then $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module if and only if $\phi(M)+JN$ is a coherent $f(A)+J$-module and $\psi(M)+J'P$ is a coherent $g(A)+J'$-module.*

(6) *Assume that J and J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$ respectively and ϕ and ψ are surjective. Then $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module if and only if $\phi(M)+JN$ is a coherent $f(A)+J$-module and $\psi(M)+J'P$ is a coherent $g(A)+J'$-module.*

Proof. (1) Assume that $J'P$ is finitely generated $g(A)+J'$-module. Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module. By [11, Proposition 4.1], (2), we have $\frac{A \bowtie^{f,g}(J,J')}{0\times J'} = f(A)+J$ and In light of Theorem 3.6, it is suffices to prove that $\phi(M)+JN$ is $A \bowtie^{f,g} (J, J')$-module and since $J'P$ is finitely generated $g(A)+J'$-module, then $0\times J'P$ is finitely generated $A \bowtie^{f,g} (J, J')$-module. Since $\frac{M\bowtie^{\phi,\psi}(JN,J'P)}{0\times J'P} = \phi(M)+JN$ and we have an exact sequence of modules

$$0 \longrightarrow 0\times J'P \to M \bowtie^{\phi,\psi} (JN, J'P) \to \phi(M)+JN \longrightarrow 0$$

According to Theorem 3.2, we have $\phi(M)+JN$ is coherent $f(A)+J$-module.

(2) Assume that JN is finitely generated $f(A)+J$-module. Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is coherent $A \bowtie^{f,g} (J, J')$-module. We have $\frac{A\bowtie^{f,g}(J,J')}{J\times 0} = g(A)+J'$ by [11, Proposition 4.1]. Hence, using Theorem 3.6 to prove that $\psi(M)+J'P$ is $g(A)+J'$-module it is suffices to prove that $\psi(M)+J'P$ is $A \bowtie^{f,g} (J, J')$-module and since JN is finitely generated $f(A)+J$-module, then $JN\times 0$ is finitely generated $A \bowtie^{f,g} (J, J')$-module. Hence, using Proposition 2.4 and Theorem 3.2 our result follows immediately.

(3) Assume that J' is a finitely generated ideal of $g(A)+J'$. Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is a coherent $A \bowtie^{f,g} (J, J')$-module. Using (1) and Theorem 3.6, to prove that $\phi(M)+JN$ is a coherent $f(A)+J$-module it suffices to prove that it is coherent as an $A \bowtie^{f,g} (J, J')$-module. We first show that $(0\times J')(M \bowtie^{\phi,\psi} (JN, J'P)) = 0\times J'P$. Clearly $(0\times J')(M \bowtie^{\phi,\psi} (JN, J'P)) \subseteq 0\times J'P$. Now, let $(0,p) \in 0\times J'P$. Since J' is finitely generated and ψ is surjective. Hence we have $p = \sum_{i=1}^{n} j'_i p_i$ where $j' \in J'$ and $p_i \in P$ and for each p_i there exsits m_i such

that $\psi(m_i) = p_i$. Hence $(0, p) = \sum_{i=1}^{n}(0, j')(\phi(m_i), \psi(m_i) \in (0 \times J')((M \bowtie^{\phi,\psi} (JN, J'P))$; so that $(0 \times J')(M \bowtie^{\phi,\psi} (JN, J'P)) = 0 \times J'P$. Thus, $0 \times J'P$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module since $(0 \times J')$ and $M \bowtie^{\phi,\psi} (JN, J'P)$ are finitely generated $A \bowtie^{f,g} (J, J')$-modules. Using Proposition 2.4 and Theorem 3.2, we have $\phi(M) + JN$ is a coherent $f(A) + J$-module.

(4) Assume that J is a finitely generated ideal of $f(A) + J$. Suppose that $M \bowtie^{\phi,\psi} (JN, J'P)$ is a coherent $A \bowtie^{f,g} (J, J')$-module. As in (1), using Theorem 3.6, to prove that $\psi(M) + J'P$ is a coherent $g(A) + J'$-module it suffices to prove that it is coherent as an $A \bowtie^{f,g} (J, J')$-module. We first show that $(J \times 0)(M \bowtie^{\phi,\psi} (JN, J'P)) = JN \times 0$. Clearly $(J \times 0)(M \bowtie^{\phi,\psi} (JN, J'P)) \subseteq JN \times 0$. Now, let $(n, 0) \in JN \times 0$. Since J is finitely generated and ϕ is surjective. Hence we have $n = \sum_{i=1}^{m} j_i n_i$ where $j \in J$ and $n_i \in N$ and for each n_i there exsits m_i such that $\phi(m_i) = n_i$. Hence $(n, 0) = \sum_{i=1}^{m}(j, 0)(\phi(m_i), \psi(m_i) \in (J \times 0)((M \bowtie^{\phi,\psi} (JN, J'P))$; so that $(J \times 0)(M \bowtie^{\phi,\psi} (JN, J'P)) = JN \times 0$. Thus, $JN \times 0$ is a finitely generated $A \bowtie^{f,g} (J, J')$-module since $(J \times 0)$ and $M \bowtie^{\phi,\psi} (JN, J'P)$ are finitely generated $A \bowtie^{f,g} (J, J')$-modules. It follows from Proposition 2.4 and Theorem 3.2, that $\psi(M) + J'P$ is a coherent $A \bowtie^{f,g} (J, J')$-module.

(5) Follows immediately from (1) and (2).

(6) Follows immediately from (3) and (4). □

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