ON THE X-RANKS ASSOCIATED TO A LINEAR SUBSPACE

EDOARDO BALlico

Abstract. Let $X \subset \mathbb{P}^N$ be an integral non-degenerate variety. For any $q \in \mathbb{P}^N$ the $X$-rank $r_X(q)$ of $q$ is the minimal cardinality of a set $S \subset X$ whose linear span contains $q$. For any $r$-dimensional linear subspace $V \subset \mathbb{P}^N$ the $X$-rank $r_X(V)$ of $V$ is the minimal cardinality of a set $S \subset X$ whose linear span contains $V$. We define several invariants related to the $X$-rank, e.g. the open rank of $V$, $r + 1$ integers $r_X(q_i)$, $0 \leq i \leq r$, with $r_X(q_i) \leq r_X(q_j)$ for all $i < j$ and $q_0, \ldots, q_r$ a basis of $V$ and minimal $\eta_X(V) := r_X(q_0) + \cdots + r_X(q_r)$. We always have $\eta_X(V) \geq r_X(V)$ and we give an upper bound for $\eta_X(V)$, which ensures that we may compute $r_X(V)$ with knowledge of $q_0, \ldots, q_r$ and in particular check if $\eta_X(V) = r_X(V)$ (the general case, but not always). The integer $r_X(q_t)$ is the minimal integer $t$ such that $V$ is spanned by points of $X$-rank $\leq t$. Many of these invariants of $V$ give lower bounds for the invariants of any linear subspace of $V$.

1. The main definitions and the main results

Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate variety over $\mathbb{C}$ (but we also consider the real field $\mathbb{R}$ in section 3, while Proposition 1.3 and the definitions of the integers $\eta_X(V)$, $r_X,i(V)$, $0 \leq i \leq \dim(V)$ work over an arbitrary field). For each $q \in \mathbb{P}^N$ the $X$-rank $r_X(q)$ of $q$ is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S \rangle$, where $\langle \rangle$ denotes the linear span. When $N + 1 = \prod_{i=1}^{k}(n_i + 1)$, $k \geq 2$, and $X$ is the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ the $X$-rank $r_X(q)$ of $q$ is the tensor rank of the tensor of format $(n_1 + 1, \ldots, n_k + 1)$ associated to $q$. When $k = 2$ and $X$ is the Segre embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ the tensor rank $r_X(q)$ is just the rank of the $(n_1 + 1) \times (n_2 + 1)$ matrix associated to $q$, but even in this classical case this notion leads to many research topics if (as in this paper) instead of a single point $q$ we consider a linear space $V \subset \mathbb{P}^N$. Rank metric codes can be seen as a vector space of $n_1 \times n_2$ matrices where the entries are from a finite field and the rank distance is the rank of the difference between two matrices ([17], [21], [26], [30], [30], [32], [34]). The motivation of many of these works came from network codings ([27]). For linear spaces of matrices over a field with an upper/lower bound of their rank or of constant rank, see [4], [10], [17], [19], [28], [29]. When $N + 1 = \binom{n+d}{n}$ with $n$ a positive integer, $d \geq 2$, and $X$ is the order $d$ Veronese embedding of $\mathbb{P}^n$, then $r_X(q)$ is the minimal number of addendum of the

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* Corresponding author.

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degree \(d\) homogeneous polynomial \(f(x_0, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n]\) associated to \(q\) in a decomposition \(f(x_0, \ldots, x_n) = \sum_i \ell_i(x_0, \ldots, x_n)^d\), where each \(\ell_i(x_0, \ldots, x_n)\) is a linear form; for this set-up the papers [15], [18], [23], [33, 1.2] look at the rank of a general \(r\)-dimensional linear subspace. Abo and Wan considered the case of several anti-symmetric forms ([1]).

For any integer \(t > 0\) the \(t\)-secant variety \(\sigma_t(X)\) of \(X\) is the closure in \(\mathbb{P}^N\) of the set of all \(q \in \mathbb{P}^N\) with \(r_X(q) = t\); here over \(\mathbb{C}\) we may take the closure either in the euclidean topology or the Zariski topology and get the same set \(\sigma_t(X)\). The set \(\sigma_1(X)\) is an irreducible projective variety, which contains all \(q \in \mathbb{P}^N\) with \(r_X(q) \leq t\) (but often it contains points with higher \(X\)-rank). We have \(\sigma_1(X) = X\) and \(\sigma_1(X) \subset \sigma_{t+1}(X)\) for all \(t > 0\). Let \(r_{X,\text{gen}}\) be the minimal integer \(x\) such that \(\sigma_x(X) = \mathbb{P}^N\), i.e. the only integer \(x\) such that \(r_X(q) = x\) for all \(q\) in a non-empty open subset of \(\mathbb{P}^N\) (either for the euclidean or the Zariski topology of \(\mathbb{P}^N\)).

In this paper we look at properties related to the \(X\)-rank, but for linear subspaces \(V \subset \mathbb{P}^N\). For each linear space \(V \subset \mathbb{P}^N\) the \(X\)-rank \(r_X(V)\) of \(X\) is the minimal cardinality of a finite set \(S \subset X\) such that \(V \subset \langle S \rangle\). Obviously \(r_X(\{q\}) = r_X(q)\) for all \(q \in \mathbb{P}^N\).

Adapting the proofs in [12] we get the following statement.

**Proposition 1.1.** Let \(V \subset \mathbb{P}^N\) be an \(r\)-dimensional linear subspace. Let \(\rho\) be the minimal value of all \(r_X(o), o \in V\). Then:

(i) \(r_X(V) \leq \min\{(r + 2)r_{X,\text{gen}}, (r + 1)r_{X,\text{gen}} + \rho\};

(ii) If \(u_1, \ldots, u_k \in V, k \geq 2\), are linearly independent, then \(r_X(V) \leq (r + 2 - k)r_{X,\text{reg}} + r_X(u_1) + \cdots + r_X(u_k)\).

There are two notions related to the \(X\)-rank and for which the proof of Proposition 1.1 may be extended with some effort. These two notions are related. They capture what happens if a part of \(X\) cannot be used for the interpolation, i.e. the finite set \(S\) used in the definition of \(X\)-rank must avoid a proper closed subset of \(X\). We fix a proper closed subset \(T \subset X\) for the Zariski topology. For any \(q \in \mathbb{P}^N\) (resp. linear subspace \(V \subset \mathbb{P}^N\)) let \(r_{X \setminus T}\) (resp. \(r_{X \setminus T}(V)\)) be the minimal cardinality of a finite set \(S \subset X \setminus T\) such that \(q \in \langle S \rangle\) (resp. \(V \subset \langle S \rangle\)). Obviously \(r_{X \setminus T}(q) \geq r_X(q)\) and \(r_{X \setminus T}(V) \geq r_X(V)\). The proof of Proposition 1.1 and of Proposition 3.1 below works verbatim using \(r_{X \setminus T}\) and \(r_{X \setminus T,\mathbb{R}}\) instead of \(r_X\) and \(r_{X,\mathbb{R}}\), respectively. The notion of open rank is more subtle (it was introduced by J. Jelsiejew in [25], inspired by a related notion for affine varieties introduced in [8] and [9]). The open rank \(ora_X(q)\) of \(q \in \mathbb{P}^N\) is the minimal integer \(t > 0\) such that for every Zariski closed proper subset \(T \subset X\) there is a set \(S_T \subset X \setminus T\) with \(\sharp(S_T) \leq t\) and \(q \in \langle S_T \rangle\). Obviously \(ora_X(q) \geq r_X(q)\) and \(ora_X(q) \geq r_{X \setminus T}(q)\) for any closed \(T \subset X\), but often \(ora_X(q) > r_X(q)\) (always if \(r_X(q) = 1\), i.e. if \(q \in X\)). There is a case for the symmetric tensor rank in which the maximal value of the rank is strictly less than the maximal value of the open rank ([6], i.e. trivariate homogeneous polynomials of degree 4). There are cases in which the generic rank is not the generic open rank, i.e. there is no non-empty open subset \(U \subset \mathbb{P}^N\) (for the Zariski or the euclidean topology) such that \(ora_X(q) = r_{X,\text{gen}}\) for all \(q \in U\). Take any non-defective \(X \subset \mathbb{P}^N\) such that \(a := (N + 1)/(\dim(X) + 1)\) is an integer; when \(a := (N + 1)/(\dim(X) + 1) \in \mathbb{N}\), \(X\) is non-defective if and only
if $\sigma_a(X) = \mathbb{P}^N$, i.e. if and only if $a = r_{X,\text{gen}}$. By assumption and a dimensional count there is a non-empty open subset $U \subset \mathbb{P}^N$ for the Zariski topology such that $r_X(q) = a$ for all $q \in U$ and for each $q \in U$ there are only finitely many $S \subset X$ with $\sharp(S) = a$ and $q \in \langle S \rangle$. We claim that $\text{ora}_X(q) > a$ for every $q \in U$. Fix $q \in U$ and take the finitely many sets $S_1, \ldots, S_k$ such that $\sharp(S_i) = a$ and $q \in \langle S_i \rangle$. Set $T := S_1 \cup \cdots \cup S_k$. By the definition of $T$ there is no $S \subset X \setminus T$ with $\sharp(S) = a$ and $q \in \langle S \rangle$; if $\dim(X) > 1$ we may even take instead of $S_1 \cup \cdots \cup S_k$ an irreducible $T_1 \subset X$ with $T_1 \supset S_1 \cup \cdots \cup S_k$. Hence the generic open rank is at least $a + 1$. We may find many examples $X \subset \mathbb{P}^N$ as above among the Veronese embeddings of a projective space $\mathbb{P}^n$ with respect to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$, $d > 2$, i.e. in the set up of the rank 1 decomposition of homogeneous polynomials; in this case $N + 1 = \binom{n + d}{n}$ and it is sufficient to assume $\binom{n + d}{n} \equiv 0 \pmod{n + 1}$ and that $(n, d)$ is not in the Alexander-Hirschowitz list of defective Veronese embeddings of degree $> 2$ ([2], [3], [13]), i.e., with the assumption $\binom{n + d}{n} \equiv 0 \pmod{n + 1}$ and $d > 2$, if and only if $(n, d) \notin \{ (2, 4), (4, 3) \}$. For instance if $n = 2$ it is sufficient to take any integer $d \geq 5$ with $d \equiv 1, 2 \pmod{3}$.

For any linear subspace $V \subset \mathbb{P}^N$ the open rank of $V$ is the minimal integer $t > 0$ such that for every Zariski closed proper subset $T \subset X$ there is a set $S_T \subset X \setminus T$ with $\sharp(S_T) \leq t$ and $q \in \langle S_T \rangle$.

In section 2 we adapt the proof of Proposition 1.1 to the open rank and get the following result.

**Theorem 1.2.** Let $V \subset \mathbb{P}^N$ be an $r$-dimensional linear subspace. Let $\rho$ be the minimal value of all $\text{ora}_X(o)$, $o \in V$. Then

(i) $\text{ora}_X(V) \leq \min\{(r + 2)r_{X,\text{gen}}, (r + 1)r_{X,\text{gen}} + \rho\}$.

(ii) If $u_1, \ldots, u_k \in V$, $k \geq 2$, are linearly independent, then $r_X(V) \leq (r + 2 - k)r_{X,\text{reg}} + \text{ora}_X(u_1) + \cdots + \text{ora}_X(u_k)$.

As in [12] we may extend this type of results over $\mathbb{R}$ with very mild assumptions (see section 3).

For a linear subspace $V$ with $r := \dim(V) > 0$ many other invariants may be defined in the following way. For any integer $r$ such that $0 \leq r \leq N$ let $G(r, N)$ denote the Grassmannian of all $r$-dimensional linear subspaces of $\mathbb{P}^N$. Let $V \subset \mathbb{P}^N$ be a linear subspace. Set $r_{X,\text{max}}(V) := \max_{q \in V} r_X(q)$ and $r_{X,\text{min}}(V) := \min_{q \in V} r_X(q)$. Let $r_{X,\text{gen}}(V)$ denote the only integer $t$ such that there is a non-empty open subset $U \subset V$ with $r_X(q) = t$ for all $q \in U$ (it is called the generic $X$-rank of $V$). If $X$ is defined over $\mathbb{R}$ with $X_{\text{reg}}(\mathbb{R}) \neq \emptyset$, then $r_{X,\mathbb{R},\text{max}}(V) := \max_{q \in V} r_{X,\mathbb{R}}(q)$ and $r_{X,\mathbb{R},\text{min}}(V) := \min_{q \in V} r_{X,\mathbb{R}}(q)$. In this case we may also define the $X$-typical ranks of $V$, i.e. the integers $t$ for which there is a non-empty euclidean open set $E \subset V$ with $r_{X,\mathbb{R}}(q) = t$ (see section 3). In certain situations related to coding theory small $r_{X,\text{min}}(V)$ is bad, while in other situations small $r_{X,\text{min}}(V)$ is good. For instance, software programs which compute the tensor rank of tensors and the rank of homogeneous polynomials are more effective for low rank tensors. By Proposition 1.1 small $r_{X,\text{min}}(V)$ helps to get an upper bound for $r_X(V)$. For a general $V \subset G(r, N)$ the invariant $r_{X,\text{min}}(V)$ is computed in terms of the dimensions of the secant varieties $\sigma_t(V)$, $t > 0$. 

For any $V \in G(r, N)$ let $\eta_X(V)$ be the minimal integer $x$ such that there is $S \subset V$ with $S$ spanning $V$ and $x = \sum_{q \in S} r_X(q)$. We obviously have

$$r_X(V) \leq \eta_X(V) \leq (r + 1) r_{X, \min}(V)$$

Define the flag $U_0 \subset \cdots \subset U_h$, $h \geq 0$, and the integers $\ell_{X,0}(V), \ldots, \ell_{X,h}(V)$, $r_{X,0}(V) \leq \cdots \leq r_{X,r}(V)$, in the following way. Let $U_0$ be the linear span of all $v \in V$ with $r_X(v) = r_{X,\min}(V)$. Set $\ell_{X,0}(V) := \dim(U_0)$ and $r_{X,i}(V) = r_{X,\min}(V)$ for all $i = 0, \ldots, \ell_{X,0}(V)$. If $U_0 = V$, i.e. if $\ell_{X,0}(V) = r$, then we take $h = 0$ and stop. Now assume $\ell_{X,0}(V) < r$. Let $r_{X,\ell_{X,0}(V)+1}(V)$ be the minimal integer $r_X(v)$ for some $v \in V \setminus U_0$. Let $U_1 \subseteq V$ be the linear span of the union of $U_0$ and all $v \in V \setminus U_0$ with $r_X(v) = r_{X,\ell_{X,0}(V)+1}(V)$. Set $\ell_{X,1}(V) = \dim(U_1)$ and $r_{X,i}(V) = r_{X,\ell_{X,0}(V)+1}(V)$ for all $x \in \{\ell_{X,0}(V) + 1, \ldots, \ell_{X,1}(V)\}$. If $U_1 = V$, i.e. if $\ell_{X,1}(V) = r$, then we stop. Now assume $\ell_{X,1}(V) < r$. Let $r_{X,\ell_{X,1}(V)+1}(V)$ be the minimal integer $r_X(v)$ for some $v \in V \setminus U_1$. Let $U_2 \subseteq V$ be the linear span of the union of $U_1$ and all $v \in V \setminus U_1$ with $r_X(v) = r_{X,\ell_{X,1}(V)+1}(V)$. Set $\ell_{X,2}(V) = \dim(U_2)$ and $r_{X,i}(V) = r_{X,\ell_{X,1}(V)+1}(V)$ for all $x \in \{\ell_{X,1}(V) + 1, \ldots, \ell_{X,2}(V)\}$. And so on.

**Proposition 1.3.** Fix $s \in \{0, \ldots, r\}$ and let $W \subseteq V$ be any $s$-dimensional linear subspace. Then $\eta_X(W) \leq \sum_{i=0}^s r_{X,i}(V)$.

Using Proposition 1.3 we get the following recipe to compute all integers $\eta_X(V)$, $\ell_{X,j}(V)$ and $r_{X,i}(V)$. Take $q_0 \in V$ such that $r_X(q_0) = r_{X,\min}(V)$. If $r = 0$, then stop. If $r > 0$, then take $q_1 \in V \setminus \{q_0\}$ with minimal rank. If $r = 1$, then set $\eta_X(V) = r_X(q_0) + r_X(q_1)$ and then stop. Now assume $r > 1$. Assume to have defined linearly independent points $q_0, \ldots, q_s \in V$, $s < r$, such that for $h = 1, \ldots, s - 1$, $r_X(q_h)$ is the minimal $X$-rank of a point of $V \setminus \{(q_0, \ldots, q_{h-1})\}$. Since $s < r$, we have $V \supseteq \{(q_0, \ldots, q_s)\}$. Let $q_{s+1}$ be any point of $V \setminus \{(q_0, \ldots, q_s)\}$ with minimal $X$-rank. And so on. We obtain in this way $r + 1$ points $q_0, \ldots, q_r$ with non-decreasing $X$-rank and spanning $V$.

**Claim:** We have $r_{X,i}(V) = r_X(q_i)$ for all $i, q_i \in U_e$ if and only if $i \leq \ell_{X,e}(V)$ and each $U_j$ is spanned by $U_j \cap \{q_0, \ldots, q_r\}$.

**Proof of the Claim:** Since $q_0, \ldots, q_r$ are linearly independent, the second and the third statement of the Claim are equivalent. The case $i = 0$ of the first statement is true, because $r_X(q_0) = r_{X,\min}(V)$. Moreover $q_0 \in U_0$ by the definitions of $U_0$ and of $q_0$. Now assume $i > 0$ and that $r_{X,j}(V) = r_X(q_j)$ for all $j < i$. Set $U_{i-1} := \emptyset$. Let $e$ be the minimal non-negative integer such that $i \leq \ell_{X,e}(V)$. If $i - 1 \leq \ell_{X,e-1}$, then set $f := e - 1$, otherwise set $f := e$. By the inductive assumption we have $q_0, \ldots, q_{i-1} \in U_f$. First assume $f < e$, i.e. $f = e - 1$, i.e. $r_{X,i}(V) > r_{X,j-1}(V)$. In this case we have $U_f = \{q_0, \ldots, q_{i-1}\}$. Since $q_i \notin U_f$, $U_f = \{q_0, \ldots, q_{i-1}\}$ and $r_{X,i}(V)$ is the minimal $X$-rank of an element of $V \setminus U_f$, we have $r_X(q_i) \geq r_{X,i}(V)$. Since $\{q_0, \ldots, q_{i-1}\} = U_f$, our choice of $q_i$ gives $r_X(q_i) \leq r_{X,i}(V)$. Now assume $f = e$. In this case we have $r_{X,i}(V) = r_{X,i-1}(V)$. Set $t := \ell_{X,e-1}(V)$. We have $U_{e-1} = \{q_0, \ldots, q_t\}$. There is an $\ell_{X,e}(V)$-dimensional linear subspace $U_e$ of $V$ spanned by $U_{e-1}$ and $\ell_{X,e}(V) - \ell_{X,e-1}(V)$ linearly independent elements with $X$-rank $r_{X,i}(V) = r_{X,i-1}(V)$, while $U_e \setminus U_{e-1}$ has no element with $X$-rank $< r_{X,i}(V)$. By the inductive assumption we also
have \( q_j \in U_e \) for all \( j < i \) and \( U_{e-1} = \langle \{ q_0, \ldots, q_t \} \rangle \). The definition of \( q_i \) gives \( r_X(q_i) = r_{X,i}(V) \).

This procedure shows that the integer \( \delta_X(V) := r_X(q_t) \) is an invariant of \( X \) and the linear space \( V \). The integer \( \delta_X(V) \) is the minimal integer \( t \) such that \( V \) is spanned by the points of \( V \) with \( X \)-rank at most \( t \). If \( V \) is an \( s \)-dimensional linear subspace of \( V \), then the Claim implies \( \delta_X(W) \geq r_{X,s}(V) \). Obviously \( \langle \{ q_0, \ldots, q_s \} \rangle \) is an \( s \)-dimensional subspace of \( V \) with \( \delta_X(\langle \{ q_0, \ldots, q_s \} \rangle) = r_{X,s}(V) \).

Take any \( V \in G(r,N) \) and assume that the set of all \( q \in V \) with \( r_X(q) = r_{X,\min}(V) \) spans \( V \). In this case we obviously have \( r_X(V) \leq (r + 1)r_{X,\min}(V) \). Can we improve this upper bound for \( r_X(V) \)? If \( r_{X,\min}(V) = 1 \), i.e. if \( V \cap X \neq \emptyset \), then this bound cannot be improved, because \( r_X(V) \geq r + 1 \) for all \( V \in G(r,N) \). Now assume that \( V \) is general in \( G(r,N) \) and set \( z := r_{X,\min} \) and \( e := \deg(\sigma_z(X)) \) and \( g := N - \dim(\sigma_z(X)) \).

**Proposition 1.4.** Set \( n := \dim(X) \), \( a := r_{X,\text{gen}} \). For any \( i \in \{ 1, \ldots, a \} \) set \( \sigma(i) := \dim(\sigma_i(X)) \). Set \( \sigma(0) = -1 \) and \( \sigma_0(X) = \emptyset \). Fix a general \( V \in G(r,N) \).

(i) We have \( r_{X,\text{gen}}(V) = a \).

(ii) If \( r < N - \sigma(a - 1) \), then \( r_{X,\min}(V) = a \).

(iii) For any \( i < a \) there is \( q \in V \) with \( r_X(q) = i \) if and only if \( r \geq N - \sigma(i) \).

(iv) Assume \( r \geq N - \sigma(i) \). The integer \( z := r_{X,\min}(V) \) is the positive integer \( \leq a - 1 \) such that \( N - \sigma(z) \leq r < N - \sigma(z - 1) \).

(v) We have \( r_{X,i}(V) = r_{X,\min}(V) \) for all \( i = 0, \ldots, r \), and hence \( r_X(V) \leq \eta_X(V) = (r + 1)r_{X,\min}(V) \).

In general the inequality \( r_X(V) \leq \eta_X(V) \) is very rough. If we are only interested in the case of a general \( V \in G(r,N) \) in some important cases (e.g. if \( X \subset \mathbb{P}^N \) is a Veronese embedding of a projective space as in [15], [18], [23]), this integer is the minimal integer \( t \) such that \( \sigma_t(X) = \mathbb{P}^N \), where \( X \subset \mathbb{P}^N \) is a certain embedding of a variety \( \tilde{X} \) constructed from the embedding \( X \subset \mathbb{P}^N \). For an arbitrary \( X \) we know a general result for the uniqueness of the set evincing the \( X \)-rank of a point \( q \in \mathbb{P}^N \) ([14, Theorem 1.18] is stated for the symmetric tensor rank, but its proof works in the general case). For these type results it was introduced the following invariant of the embedding \( X \subset \mathbb{P}^N \). Let \( \rho(X) \) be the maximal positive integer \( t \) such that any set \( S \subset X \) with \( z(S) \leq t \) is linearly independent. We have \( 2 \leq \rho(X) \leq N + 1 \). It is easy to check that \( \rho(X) = N + 1 \) if and only if \( X \) is a rational normal curve of \( \mathbb{P}^N \). We have \( \rho(X) = d + 1 \) if \( X \subset \mathbb{P}^N \), \( N = -1 + \binom{n+d}{n} \), is the order \( d \) Veronese embedding of \( \mathbb{P}^n \). In section 2 we prove the following result.

**Theorem 1.5.** Take any \( V \in G(r,N) \).

(i) If \( 2r_X(V) \leq \rho(X) + r \), then there is a unique set \( S \subset X \) such that \( z(S) \leq r_X(V) \) and \( V \subseteq \langle S \rangle \).

(ii) Assume \( 2\eta_X(V) \leq \rho(X) + r \). The integer \( r_V(V) \) is computed in the following way. Take \( q_0, \ldots, q_t \in V \) such that \( r_X(q_i) = r_{X,i}(V) \) and \( \langle \{ q_0, \ldots, q_t \} \rangle = V \). Take any \( S_i \subset X \) such that \( z(S_i) = r_X(q_i) \) and \( q_i \in \langle S_i \rangle \). Set \( S := \cup_{i=0}^t S_i \). Then \( r_X(V) = z(S) \) and \( S \) is the only subset \( A \subset X \) with \( z(A) = r_X(V) \) and \( V \subseteq \langle A \rangle \).
Now we shows why part (ii) of Theorem 1.5 shows (for any $r > 0$) how to construct examples $V, M \in G(r, N)$ with $r_X(V) = \eta_X(V)$, $r_X(M) < \eta_X(M)$ and $\eta_X(V) = \eta_X(M) < (\rho(X) + r)/2$. Fix any $r + 1$ linearly independent points $q_0, \ldots, q_r$ with $r_X(q_i) \leq r_X(q_j)$ for all $i \leq j$ and $\sum_{i=0}^r r_X(q_i) \leq 2\rho(X) + r$. Set $V := \langle \{q_0, \ldots, q_r\} \rangle$. Since $2r_X(q_i) \leq \rho(X)$, there is a unique $S_i \subset X$ such that $\sharp(S_i) = r_X(q_i)$ and $q_i \in \langle S_i \rangle$. Set $S := \bigcup_{i=1}^r S_i$. By part (ii) of Theorem 1.5 we have $r_X(V) = \sharp(S)$. By the Claim and Proposition 1.3 we have $\eta_X(V) = \sum_{i=0}^r r_X(q_i) = \sum_{i=0}^r \sharp(S_i)$. Hence $r_X(V) = \ell_X(V)$ if and only if $S_i \cap S_j$ for all $i \neq j$. We may also reverse the construction. We start with sets $S_i \subset X$, $i = 0, \ldots, r$, with $S_0 \neq \emptyset$, $\sharp(S_i) \leq \sharp(S_j)$ for all $i \leq j$ and $\sum_{i=0}^r \sharp(S_i) \leq 2\rho(X) + r$. This inequality implies that each $S_i$ is linearly independent and that if $p_i \in \langle S_i \rangle$, then $r_X(p_i) = \sharp(S_i)$ if and only if there is no $S' \subset S_i$ with $p_i \in \langle S' \rangle$. We also assume that for all $i = 1, \ldots, r$, then $S_i \not\subset \langle \bigcup_{h=0}^{i-1} S_h \rangle$. Take a general $q_i \in \langle S_i \rangle$. We saw that $r_X(q_i) = \sharp(S_i)$. Since $S_i \not\subset \langle \bigcup_{h=0}^{i-1} S_h \rangle$ for $i = 1, \ldots, r$ we have $q_i \notin \langle \{q_0, \ldots, q_{i-1}\} \rangle$. Hence we may easily construct $V, M$ with $r_X(V) = \eta_X(V)$ and $r_X(M) < \eta_X(M)$ (note that we may take arbitrary set $S_i \neq \emptyset$ with $\sum_{i=0}^r \sharp(S_i) \leq \rho(X) + r$).

In all cases with uniqueness (e.g. in Theorem 1.5) we obviously have $\text{ora}_X(V) > r_X(V)$ and $\text{ora}_X(q_i) > r_X(q_i)$ for all $i$.

2. The proofs

Proof of Proposition 1.1: Fix $o \in V$ with $r_X(o) = \rho$ and set $a := r_{X, \text{gen}}$. Take a non-empty open subset $U \subset \mathbb{P}^N$ (either for the euclidean or the Zariski topology of $\mathbb{P}^N$) such that $r_X(q) = a$ for all $q \in U$. Fix $q \in U$ and set $W := \{q\} \cup V$. We have $\dim(W) \leq r + 1$ and $W \supseteq V$. Since $q \notin W \cap U$, the set $U \cap W$ is a non-empty open subset of $W$ (for the euclidean or the Zariski topology). Hence there are $e_1, \ldots, e_{r+2} \subset U \cap W$ spanning $W$. Since $r_X(e_i) = q$, there is $S_i \subset X$ such that $\sharp(S_i) = a$ and $e_i \in \langle S_i \rangle$. Set $S := S_1 \cup \cdots \cup S_{r+2}$. We have $\sharp(S) \leq (r + 2)a$ and $W \subset \langle S \rangle$. Since $W \supseteq V$, we get $r_X(V) \leq (r + 2)a$. Since $o \in W$ and $U \cap W$ contains a non-empty open subset of $W$, there are $f_1, \ldots, f_{r+1} \subset U \cap W$ such that $\{o, f_1, \ldots, f_{r+1}\}$ spans $W$. As above we get $r_X(V) \leq r_X(o) + (r + 1)a$. In the same way we get part (ii). \hfill \Box

Proof of Theorem 1.2: Fix $o \in V$ with $r_X(o) = \rho$ and set $a := r_{X, \text{gen}}$. Fix a closed subset $T \subset X$. By the definition of open rank there are $A, A_i \subset X \setminus T$, $1 \leq i \leq k$, such that $\sharp(A) = \rho$, $\sharp(A_i) = \text{ora}_X(u_i)$ for all $i \in A$ and $u_i \in \langle A_i \rangle$ for all $i$.

Claim 1: There is a non-empty open subset $U' \subset \mathbb{P}^N$ such that for each $q \in U'$ there is $S_q \subset X \setminus T$ with $\sharp(S_q) = a$ and $q \in \langle S_q \rangle$.

Proof of Claim 1: By assumption we have $\sigma_a(X) = \mathbb{P}^N$, i.e. $\mathbb{P}^N$ is the Zariski closure of the union of all linear spaces $\langle S \rangle$ with $S \subset X$, $S$ linearly independent and $\sharp(S) = a$. Any point of $X$ is a limit of a family of points of $X \setminus T$. Hence $\mathbb{P}^N$ is the Zariski closure of the union of all linear spaces $\langle S \rangle$ with $S \subset X \setminus T$, $S$ linearly independent and $\sharp(S) = a$. Let $Z \subset (X \setminus T)^a$ be the set of all $(b_1, \ldots, b_a) \in (X \setminus T)^a$ which are linearly independent. $Z$ is a non-empty Zariski open subset of $(X \setminus T)^a$. Let $\Pi \subset Z \times \mathbb{P}^N$ be the set of all pairs $(b, q)$ with $b = (b_1, \ldots, b_a)$ and $q \in \langle \{b_1, \ldots, b_a\} \rangle$. $\Pi$ is an irreducible variety of dimension $a \dim(X) - 1$. Let $\pi : \Pi \rightarrow \mathbb{P}^N$ the projection onto the last factor. By assumption
$\mathbb{P}^N$ is the Zariski closure of the image $\text{Im}(\pi)$ of $\pi$. By Chevalley’s theorem $\text{Im}(\pi)$ is a constructible subset of $\mathbb{P}^N$ for the Zariski topology. Hence the image of $\pi$ contains a non-empty open subset $U'$ for the Zariski topology. The set $U'$ satisfies the thesis of Claim 1.

With Claim 1 the proof of Proposition 1.1 goes verbatim for the open rank, using $U'$ instead of $U$.

Proof of Proposition 1.3: We use induction on the integer $r$ (applied to all linear subspaces of $\mathbb{P}^N$). For any fixed integer $r$ we use induction on the integer $s$. For any $r$ the case $s = 0$ is trivially true, because for any linear space $M$ with $\dim(M) = r$ we have $r_{X, \min}(M) = r_{X, \min}(N)$ if $N \supseteq M$. Hence we may assume that $r \geq s > 0$. Let $\mathcal{W}(0) \subset \cdots \subset \mathcal{W}(h') = \mathcal{W}$, $h' \geq 0$, be the flag of linear subspaces of $\mathcal{W}$ obtained as in the definition of the integer $h \geq 0$ and the flag $\mathcal{U}_0 \subset \cdots \subset \mathcal{U}_s = \mathcal{W}$ of $\mathcal{W}$ using $\mathcal{W}$ instead of $V$. Take a basis $\{q_0, \ldots, q_r\}$ of $\mathcal{W}$ with $\mathcal{W}(q_i) = \mathcal{W}_{X,i}(V)$ for all $i$ and each $\mathcal{U}_i$ is spanned by $\mathcal{U}_i \cap \{q_0, \ldots, q_r\}$. Take a basis $\{p_0, \ldots, p_s\}$ of $\mathcal{W}$ with $\mathcal{W}(p_i) = \mathcal{W}_{X,i}(W)$, $i = 0, \ldots, s$ and such that each $\mathcal{W}(i)$ is spanned by $\mathcal{W}(i) \cap \{p_0, \ldots, p_s\}$. Let $\mathcal{W}'$ be the linear span of $\{p_0, \ldots, p_{s-1}\}$. By the inductive assumption for each integer $j < s$ we have $\sum_{i=0}^j r_{X}(p_i) \geq \sum_{i=0}^j q_i$. If $\mathcal{W}(p_s) \geq \mathcal{W}(q_s)$, then we are done. Now assume $\mathcal{W}(p_s) < \mathcal{W}(q_s)$. Since $\mathcal{W}(p_s) \geq \mathcal{W}(p_i)$ for all $i \leq s$, we get that the linear span $\mathcal{T}$ of the set of all $v \in \mathcal{W}$ with $\mathcal{W}(v) < \mathcal{W}(q_s)$ has dimension at least $s$. Hence $\mathcal{W}_{X,s}(\mathcal{V}) < \mathcal{W}(q_s)$, a contradiction. □

Remark 2.1. The case $s = r$ of Proposition 1.3 implies that $\eta_{X}(V) = \sum_{i=0}^r \mathcal{W}_{X,i}(V)$.

Proof of Proposition 1.4: Since $V$ is general, it contains a general point of $\mathbb{P}^N$ and hence part (i) is trivial. For any integer $i \in \{1, \ldots, a-1\}$ set $\mathcal{W}_i(X) := \{q \in \mathcal{W}_{X} | r_{X}(q) = i\}$. The set $\mathcal{W}_i(X)$ contains a non-empty open subset of $\mathcal{W}_i(X)$ and in particular the closure of $\mathcal{W}_i(X) \cap \mathcal{W}_i(X)$ is a finite union of algebraic sets of dimension $< r_{X}(q)$. Since $V$ is general, for any $i > 0$ we have $V \cap \mathcal{W}_i(X) \neq \emptyset$ if and only if $r \geq N - r_{X}(q)$. Hence $r_{X, \min}(V) > i$ if $r < N - r_{X}(q)$. Now assume $r \geq N - r_{X}(q)$ and hence $V \cap \mathcal{W}_i(X) \neq \emptyset$. We get $\dim(V \cap \mathcal{W}_i(X))) = \sigma(i) + r - N$, $\dim(V \cap (\mathcal{W}_i(X) \setminus \mathcal{W}_i(X))) < r_{X}(q) + r - N$ and so $V \cap (\mathcal{W}_i(X) \setminus \mathcal{W}_i(X))) \neq \emptyset$.

Now we prove part (v). Set $z := r_{X, \min}(V)$. It is sufficient to prove that $V$ is spanned by $r+1$ points with $z$ as their $X$-rank. If $z = a$, then we are done, because a non-empty Zariski open subset of $V$ is formed by points $q$ with $r_{X}(q) = z$. Now assume $z < a$. By assumption $N - \sigma(z) \leq r < N - \sigma(z - 1)$. Let $W$ be a general linear subspace of $V$ with $\dim(W) = N - \sigma(z)$. Since $V$ is general, $W$ is a general element of $G(N - \sigma(z), N)$. Hence $W \cap \mathcal{W}_i(X)$ is finite and contained in $\mathcal{W}_i(X)$. Since $X$ is irreducible, $\mathcal{W}_i(X)$ is irreducible. Hence $\mathcal{W}_i(X) \cap M$ is an irreducible curve for a general $M \in G(N - \sigma(z) + 1, N)$. Since $W$ is general, we may take as $M$ a subspace containing $W$. Take any hyperplane $H \subset \mathbb{P}^N$. Since $\mathcal{W}_i(W)$ is reduced and connected, we have $h^1(\mathcal{P}^N, \mathcal{I}_{\mathcal{W}_i(X)}) = 0$. Thus the exact sequence $0 \to \mathcal{I}_{\mathcal{W}_i(X)} \to \mathcal{I}_{\mathcal{W}_i(X)}(1) \to \mathcal{I}_{\mathcal{W}_i \cap H}(1) \to 0$

gives that $H \cap \mathcal{W}_i(X)$ spans $H$. Taking a hyperplane of $H$ and iterating the trick several times we first obtain that the integral curve $\mathcal{W}_i(X) \cap M$ spans $M$ and then
that the finite set \( \sigma_z(X) \cap W \) spans \( W \). Since \( \sigma_z(X) \cap W = \sigma_\rho(X) \cap W \), \( W \) is spanned by points with \( X\)-rank \( z \). Varying \( W \) in \( V \) we get that \( V \) is spanned by points with \( X\)-rank \( z \).

Proof of Theorem 1.5: We first prove part (i). We use induction on \( r \). Since the case \( r = 0 \) is true by the proof of [14, Theorem 1.18], we may assume \( r > 0 \). Take \( A, S \subset X \) with \( \sharp(S) = r_X(V) \), \( \sharp(A) = r_X(V) \), \( V \subset \langle S \rangle \cap \langle A \rangle \) and \( S \neq A \). The definition of the integer \( r_X(V) \) gives \( \sharp(A) = \sharp(S) = r_X(V) \).

First assume \( A \cap S = \emptyset \). Take \( A_2 \subset A \) with \( \sharp(A_2) = A - r \) and \( V \cap \langle A_2 \rangle \) a single point. Since \( V \subset \langle S \rangle \) and \( A_2 \cap S = \emptyset \), \( A_2 \cup S \) is linearly dependent. Since \( \sharp(S \cup A_2) \leq \rho(X) \), we get a contradiction.

Now assume \( S \cap A \neq \emptyset \) and set \( A_1 := A \setminus A \cap S \), \( S_1 := S \setminus S \cap A \), \( e := \sharp(A \cap S) \) and \( W := V \cap \langle A_1 \rangle \cap \langle S_1 \rangle \). Since \( S \neq A \) and no proper subset of \( A \) spans \( V \), we have \( S_1 \neq A_1 \), \( S_1 \neq \emptyset \), \( A_1 \neq \emptyset \), \( A_1 \neq S_1 \) and \( \dim(W) = r - e \). Apply the first part to \( W \), \( A_1 \) and \( S_1 \).

Now we prove part (ii). Since part (ii) follows from part (i) when \( r = 0 \), we may assume that \( r > 0 \) and use induction on \( r \). We have \( \sharp(S) \leq \sum_{i=0}^{\infty} r_X(q_i) = \eta_X(V) \) and \( V \subset \langle S \rangle \). Hence \( r_X(V) \leq \sharp(S) \). Let \( S' \) be the minimal subset of \( S \) such that \( V \subset \langle S' \rangle \). Take any \( A \subset X \) with \( \sharp(A) = r_X(V) \) and \( V \subset \langle A \rangle \) and assume \( S \neq A \). Since \( \sharp(A) + \sharp(S) \leq 2\eta_X(V) \leq \rho(X) + r \), the proof of part (i) gives \( A = S' \). If \( S' = S \), then the proof of part (ii) is over. Now assume \( S' \not\subset S \). Set \( M := \{q_0, \ldots, q_{r-1}\} \). We have \( \dim(M) = r - 1 \). The definition of \( q_0, \ldots, q_{r-1} \) gives \( r_{X,i}(M) = r_{X,i}(V) \). Since \( \dim(M) = r - 1 \), the inductive assumption implies \( S_i \cap S' = S \) for all \( i < r \). Take \( N := \{q_1, \ldots, q_r\} \). The Claim in the introduction gives \( r_{X,i}(N) = r_{X,i+1}(V) \) for all \( i < r \). The inductive assumption gives \( S_i \cap S' = S_i \) for all \( i > 0 \) and in particular \( S_r \subset S' \). Thus \( S' = S \), a contradiction.

\( \square \)

3. Over \( \mathbb{R} \)

Let \( X \subset \mathbb{P}^N \) be a geometrically integral variety defined over \( \mathbb{R} \). We write \( X(\mathbb{R}) \subset \mathbb{P}^N(\mathbb{R}) \) for the real points of \( X \) and \( X(\mathbb{C}) \subset \mathbb{P}^N(\mathbb{C}) \) for the complex points of \( X \) (we called them \( X \) in the other sections). Recall that “geometrically integral” means that the complex variety \( X_{\mathbb{C}} \) (the same real equations of \( X \), but seen in \( \mathbb{P}^N(\mathbb{C}) \)) is an integral complex variety. We require that \( X(\mathbb{R}) \) contains at least one smooth point of \( X(\mathbb{C}) \) (usually this is written down as \( X_{\text{reg}}(\mathbb{R}) \neq \emptyset \)). This assumption implies that \( X_{\text{reg}}(\mathbb{R}) \) spans the real projective space \( \mathbb{P}^N(\mathbb{R}) \) (linear span with real coefficients) and hence the complex projective space \( \mathbb{P}^N(\mathbb{C}) \) (linear span with complex coefficients). A typical \( X\)-rank is an integer \( t > 0 \) such that there is a non-empty open subset \( E \subset \mathbb{P}^N \) for the euclidean topology with \( r_{X,E}(q) = t \) for all \( q \in E \) ([7], [11], [12], [16]). The generic \( X\)-rank of \( X(\mathbb{C}) \subset \mathbb{P}^N(\mathbb{C}) \) is the minimum of all typical \( X\)-ranks and if \( x_1 \) and \( x_2 \) are typical \( X\)-rank, then every integer \( x \) with \( x_1 \leq x \leq x_2 \) is a typical \( X\)-rank ([7, Theorem 2.2]). Fix \( V \in G(r, \mathbb{N})(\mathbb{R}) \). For any \( S \subset \mathbb{P}^N(\mathbb{R}) \) let \( \langle S \rangle_{\mathbb{R}} \) denote its \( \mathbb{R}\)-linear span in the real projective space \( \mathbb{P}^N(\mathbb{R}) \) and let \( \langle S \rangle_{\mathbb{C}} \) denote its \( \mathbb{C}\)-linear span in the complex projective space \( \mathbb{P}^N(\mathbb{C}) \). We have \( \dim_{\mathbb{R}}(\langle S \rangle_{\mathbb{R}}) = \dim_{\mathbb{C}}(\langle S \rangle_{\mathbb{C}}) \) and
Let \( \langle S \rangle \subseteq \mathbb{P}^N(\mathbb{R}) \). The real \( X \)-rank \( r_{X,\mathbb{R}}(V) \) of \( V \) is the minimal cardinality of a finite set \( S \subseteq X(\mathbb{R}) \) such that \( V \subseteq \langle S \rangle \).

**Proposition 3.1.** Let \( V \subseteq \mathbb{P}^N(\mathbb{R}) \) be an \( r \)-dimensional \( \mathbb{R} \)-linear subspace. Let \( \rho \) be the minimal value of all \( r_{X,\mathbb{R}}(o) \), \( o \in V \). Then

(i) \( r_X(V) \leq \min\{(r + 2) r_{X,\gen}(r + 1) r_{X,\gen} + \rho \}. \)

(ii) If \( u_1, \ldots, u_k \in V \) are linearly independent, then \( r_{X,\mathbb{R}}(V) \leq (r + 2 - k) r_{X,\reg} + r_{X,\mathbb{R}}(u_1) + \cdots + r_{X,\mathbb{R}}(u_k) \).

**Proof.** Fix \( o \in V \) with \( r_X(o) = \rho \) and set \( a := r_{X,\gen} \). Take a non-empty open subset \( U \subseteq \mathbb{P}^N(\mathbb{R}) \) for the euclidean topology of \( \mathbb{P}^N \) such that \( r_{X,\mathbb{R}}(q) = a \) for all \( q \in U \). Fix \( q \in U \) and set \( W := \{q\} \cup V \). \( W \) is an \( \mathbb{R} \)-linear subspace, \( \dim(W) \leq r + 1 \) and \( W \supseteq V \). Since \( q \in W \cap U \), the set \( U \cap W \) is a non-empty open subset of \( W \) for the euclidean or the Zariski topology). Then we continue as in the proof of Proposition 1.1. \( \square \)

**Remark 3.2.** Take the the assumptions of Proposition 3.1, but look at the proof of Claim 1 in the proof of Theorem 1.2. Set \( n := \dim(X) \). We fix a proper Zariski closed subset \( T \subseteq X \). Take a non-empty open and connected subset \( E \subseteq X_{\reg}(\mathbb{R}) \) for the euclidean topology such that \( E \cap T(\mathbb{C}) = \emptyset \). \( E \) has euclidean dimension \( n \) and in particular it is a connected topological manifold of dimension \( n \). Let \( Z_E \) be the set of all \( b = (b_1, \ldots, b_a) \in E^a \) such that \( b_1, \ldots, b_a \) are linearly independent. \( Z_E \) is a topological manifold of dimension \( na \), but we do not claim that it is connected. Let \( \mathbb{I}_E \) be the set of all pairs \( (b, u) \) with \( b = (b_1, \ldots, b_a) \in Z_E \) and \( u \in \langle \{b_1, \ldots, b_a\} \rangle \). Let \( \pi_E : \mathbb{I}_E \to \mathbb{P}^N(\mathbb{R}) \) denote the projection onto the second factor. Since \( \pi \) is dominant, its complex differential has rank \( N \) for a Zariski dense open subset of \( \mathbb{I} \). This open subset meets \( \mathbb{I}_E \). Hence there is \( c \in \mathbb{I}_E \) at which the complex differential of \( \pi \) has rank \( N \). The differential of \( \pi \) and of \( \pi_E \) at \( c \) are given by the same real matrix. Hence the image of \( \pi_E \) contains a non-empty euclidean subset of \( \mathbb{P}^N(\mathbb{R}) \). With this observation Theorem 1.2 is extended to the open real rank.

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**References**


1 **DEPT. OF MATHEMATICS, UNIVERSITY OF TRENTO, 38123 POVO (TN), ITALY**

*E-mail address: ballico@science.unitn.it*