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EXPLICIT REPRESENTATIONS OF SOME GENERALIZED AND MIXED RELATIVES OF THE BESSEL FUNCTIONS

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Abstract. Special functions can be defined in different ways such as Rodrigue's formulae, generating functions, summation formulae, integral representations et cetera, but it is usually shown to be expressible as a series, because this is frequently the most practical way to obtain numerical values for the functions. In this paper, the explicit representations of certain mixed special functions related to the Bessel functions are obtained. A Laurent type hypergeometric generating relation is derived using series rearrangement technique. Some special cases are obtained as generating function of the known mixed type relatives of the Bessel functions. Finally explicit forms of these mixed type special functions are obtained as applications.

1. Introduction and preliminaries

The analytical and numerical study of generalized Bessel functions has revealed their interesting properties, which in some sense can be regarded as an extension of the properties of Bessel functions to a two-dimensional domain. In this connection, the relevance of generalized Bessel functions and their multi-variable extension in mathematical physics has been emphasized, since they provide analytical solutions to partial differential equations such as the multi-dimensional diffusion equation, the Schrödinger and Klein-Gordon equations. The algebraic structure underlying generalized Bessel functions can be recognized in full analogy with Bessel functions, thus providing a unifying view to the theory of both Bessel and generalized Bessel functions. Hence the interest for the generalized Bessel functions is justified, for details see [3] and references therein.

The 2-variable Bessel functions $J_n(x,y)$ are defined by the following generating function [3, p. 332 (2.7)]:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right]=\sum_{n=-\infty}^{\infty}J_n(x,y)t^n. \tag{1.1}$$

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Also, the generalized Bessel functions $J_n^{(m)}(x,y)$ possess the generating function [4, p. 117, Eq. (3.3b)]:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^m-\frac{1}{t^m}\right)\right]=\sum_{n=-\infty}^{\infty}J_n^{(m)}(x,y)t^n. \tag{1.2}$$

Very recently, Sheffer-Bessel functions ${}_sJ_n(x,y)$ are introduced in [7] and discussed their special case as Hermite-Bessel functions ${}_HJ_n(x,y)$.

The following generating function for the 2-variable Hermite-Bessel functions ${}_HJ_n(x,y)$ is obtained [7, p. 277, Eq. (4.4)]:

$$\exp\left[x\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)-\frac{1}{4}\left(t-\frac{1}{t}\right)^2\right]=\sum_{n=-\infty}^{\infty}{}_HJ_n(x,y)t^n. \tag{1.3}$$

Dattoli *et al.* [5, p.405, Eq. (36)] generated the Hermite-Bessel functions associated with the Bell-type polynomials $H_n^{(3,2)}(x,w,y)$ by the following generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{w}{4}\left(t-\frac{1}{t}\right)^2+\frac{y}{8}\left(t-\frac{1}{t}\right)^3\right]=\sum_{n=-\infty}^{\infty}{}_{H^{(3,2)}}J_n(x,w,y)t^n. \tag{1.4}$$

The explicit form of ${}_{H^{(3,2)}}J_n(x,w,y)$ are given as:

$$ {}_{H^{(3,2)}}J_n(x,w,y)=\sum_{r=0}^{\infty}\frac{(-1)^r\ H_{n+2r}^{(3,2)}(x,w,y)}{r!\ (n+r)!\ 2^{n+2r}}, \tag{1.5}$$

where $H_n^{(3,2)}(x,w,y)$ are defined by

$$H_n^{(3,2)}(x,w,y)=n!\sum_{r=0}^{[n/3]}\frac{y^r\ H_{n-3r}^{(2)}(x,w)}{r!\ (n-3r)!} \tag{1.6}$$

and the 2-variable Hermite Kampé de Fériet polynomials 2VHKdFP $H_n(x,w)$ [1] are defined as:

$$H_n^{(2)}(x,w)=\sum_{r=0}^{[\frac{n}{2}]}\frac{x^{n-2r}w^r}{r!(n-2r)!}. \tag{1.7}$$

The classical Hermite polynomials $H_j(z)$ are defined as [8, p. 187 and p. 191]:

$$H_j(z) = (2z)^j\ {}_2F_0\left[\begin{array}{ccc}\frac{-j}{2},\ \frac{-j+1}{2} & ; & \\ & & -\frac{1}{z^2}\\ — & ; & \end{array}\right],\quad j\in\mathbb{N}_0 \tag{1.8}$$

$$= \sum_{k=0}^{[\frac{j}{2}]}\frac{(-1)^k\ j!\ (2z)^{j-2k}}{k!\ (j-2k)!},\quad j\in\mathbb{N}_0. \tag{1.9}$$

Also, it is given that [8, p. 188, Eq. (4)]:

$$H_j(-z)=(-1)^j\ H_j(z). \tag{1.10}$$

The Gould-Hopper generalizations of the Hermite polynomials are defined as [6, p.58, Eqn. (6.2)]:

$$g_n^m(x,h) = \sum_{k=0}^{[n/m]} \frac{n!}{k!\,(n-mk)!} h^k\, x^{n-mk} \tag{1.11}$$

$$= x^n\ {}_mF_0\left[\begin{array}{ll} \Delta(m;-n) & ; \\ & h\left(\frac{-m}{x}\right)^m \\ \text{—} & ; \end{array}\right], \tag{1.12}$$

where $\Delta(m;-n)$ abbreviates the array of m parameters given as:

$$\frac{-n}{m},\ \frac{-n+1}{m},\ \frac{-n+2}{m},\cdots\frac{-n+m-1}{m};\quad m\geq 1.$$

It should be noted that:

$$g_n^2(2x,-1) = H_n(x).$$

There have been tremendous studies on special polynomials and holomorphic functions (see, [2, 9] and references therein). There has been a great revival of interest in the study of hypergeometric functions in the last two decades. This newfound interest comes from the connections between hypergeometric functions and many areas of mathematics such as representation theory, algebraic geometry and Hodge theory, combinatorics, D-modules, number theory, mirror symmetry, etc. The integral representations played an important role in the study of the hypergeometric functions. The celebrated Euler's integral for the Gauss functions ${}_2F_1$ was probably the first among them.

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{ll} (\alpha_p); & \\ & z \\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{ll} \alpha_1,\alpha_2,\ldots,\alpha_p; & \\ & z \\ \beta_1,\beta_2,\ldots,\beta_q; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!} \tag{1.13}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\ ;\ j=1,2,\ldots,q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ is denoted by (α_p) with similar interpretation for others throughout this paper.

Supposing that none of numerator parameters is zero or a negative integer and for $\beta_j\neq 0,-1,-2,\ldots;\ j=1,2,\ldots,q$, we note that the ${}_pF_q$ series defined by equation (1.13):

(i) converges for $|z|<\infty$, if $p\leq q$,

(ii) converges for $|z|<1$, if $p=q+1$ and
(iii) diverges for all z, $z\neq 0$, if $p>q+1$.

Fox-Wright generalized hypergeometric function of one variable [14, p. 50-51]

$$ {}_p\Psi_q\left[\begin{array}{lll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ; & \\ & & z\\ (\beta_1,B_1),\cdots,(\beta_q,B_q) & ; & \end{array}\right]=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+nA_1)\cdots\Gamma(\alpha_p+nA_p)}{\Gamma(\beta_1+nB_1)\cdots\Gamma(\beta_q+nB_q)}\ \frac{z^n}{n!}, \tag{1.14}$$

$$ {}_p\Psi_q^\star\left[\begin{array}{lll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ; & \\ & & z\\ (\beta_1,B_1),\cdots,(\beta_q,B_q) & ; & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{nA_1}\cdots(\alpha_p)_{nA_p}}{(\beta_1)_{nB_1}\cdots(\beta_q)_{nB_q}}\ \frac{z^n}{n!}, \tag{1.15}$$

where $A_1,\cdots,A_p,\ B_1,\cdots,B_q$ are positive real numbers; subject to the convergence conditions:

(i) $1+\sum_{i=1}^{q}B_i-\sum_{i=1}^{p}A_i>0$ and $0<|z|<\infty$; $\quad z\neq 0$

(ii) $1+\sum_{i=1}^{q}B_i-\sum_{i=1}^{p}A_i=0$ and $0<|z|<A_1^{-A_1}\cdots A_p^{-A_p}B_1^{B_1}\cdots B_q^{B_q}$.

For $A_1=A_2=\cdots=A_p=B_1=B_2=\cdots=B_q=1$, ${}_p\Psi_q^\star$ in equation (1.15) reduces to ${}_pF_q$ defined by equation (1.13).

About five decades ago, Srivastava and Daoust [10] first considered the two-variable case of their multiple hypergeometric function [11, p.454]; see also [12]). For the sake of ready reference, we choose to recall here their definition only in the two-variable case as follows [10, p.199, Equation (2.1)]:

$$ F^{A:B;B'}_{C:D;D'}\left(\begin{array}{ll}[(a_A):\theta,\phi]:[(b_B):\psi];[(b'_{B'}):\psi'] ; & \\ & x,\,y\\ [(c_C):\xi,\eta]:[(d_D):\zeta];[(d'_{D'}):\zeta'] ; & \end{array}\right)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m\theta_j+n\phi_j}}{\prod_{j=1}^{C}(c_j)_{m\xi_j+n\eta_j}}\frac{\prod_{j=1}^{B}(b_j)_{m\psi_j}}{\prod_{j=1}^{D}(d_j)_{m\zeta_j}}\frac{\prod_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod_{j=1}^{D'}(d'_j)_{n\zeta'_j}}\frac{x^m\,y^n}{m!\,n!}, \tag{1.16}$$

where, for convergence of the double hypergeometric series,

$$1+\sum_{j=1}^{C}\xi_j+\sum_{j=1}^{D}\zeta_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j\geq 0 \tag{1.17}$$

and

$$1+\sum_{j=1}^{C}\eta_j+\sum_{j=1}^{D'}\zeta'_j-\sum_{j=1}^{A}\phi_j-\sum_{j=1}^{B'}\psi'_j\geq 0, \tag{1.18}$$

with equality only when $|x|$ and $|y|$ are constrained appropriately (see, for details, [12]). Here, for the sake of convenience, $[(a_A):\theta,\phi]$ represents the set of "A" number of parameters $[a_1:\theta_1,\phi_1]$, $[a_2:\theta_2,\phi_2]$, ..., $[a_A:\theta_A,\phi_A]$. The values of positive real coefficients θ_1, θ_2, ..., θ_A may be equal or different with similar interpretation for coefficients ϕ_1, ϕ_2, ..., ϕ_A and others.

Remark 1.1. The positivity of these coefficients was assumed by Srivastava-Daoust [12, pp. 153-158] in order merely to facilitate their investigations of the region of convergence of the multiple hypergeometric series (1.16).

Remark 1.2. For notational purposes the coefficients ξ_j, ζ_j, θ_j, ψ_j, η_j, ζ'_j, ψ'_j are allowed to take on all real values including, for example, zero and negative integers, see [13, pp.270-272].

Many useful cases of reducibility of double hypergeometric functions are known to exist when the coefficients θ, ϕ, ψ, ψ', ξ, η, ζ and ζ' in equation (1.16) are chosen to be unity.

In this article, the following lemma and definition have been used in proving our main results:

Lemma 1.3. [14, p.102, Eqn(16)] *For positive integers* $m_1,\cdots,m_r \quad (r\geq 1)$,

$$\sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{\infty}\Theta(k_1,\cdots,k_r;n)=\sum_{n=0}^{\infty}\sum_{k_1,\cdots,k_r=0}^{m_1k_1+\cdots+m_rk_r\leq n}\Theta(k_1,\cdots,k_r;n-m_1k_1-\cdots-m_rk_r), \tag{1.19}$$

provided that concerned multiple series of both sides are absolutely convergent.

Definition 1.4. Gauss's Multiplication Theorem[14, p.23]
For every positive integer m, we have

$$(\lambda)_{mn}=m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n, \quad n=0,1,2,\cdots. \tag{1.20}$$

The explicit forms of the generalized and hybrid Bessel functions discussed above have not been studied so far. In this article, we obtain the series definitions of the special functions related Bessel functions by using series rearrangement technique.

2. General series identity

In this section, we derive a general series identity in the form of the following theorem:

Theorem 2.1. *Let* $\{\Omega_1(\ell)\}$, $\{\Omega_2(\ell)\}$, $\{\Omega_3(\ell)\}$, $\{\Omega_4(\ell)\}$, $\{\Omega_5(\ell)\}$ *and* $\{\Omega_6(\ell)\}$; $\ell \in \{1,2,3,\cdots\}$ *are six bounded sequences of arbitrary complex numbers and* $\Omega_i(0) \neq 0$ $(i = 1,2,3,4,5,6)$. *Then*

$$\sum_{v,\ell,m,i,j,k=0}^{\infty} \Omega_1(v)\Omega_2(\ell)\Omega_3(m)\Omega_4(i)\Omega_5(j)\Omega_6(k) \frac{(\beta t)^v \ (\gamma t^\theta)^\ell \ (\zeta t^\phi)^m \ \left(\frac{\lambda}{t}\right)^i \ \left(\frac{\mu}{t^\psi}\right)^j \ \left(\frac{\sigma}{t^\nu}\right)^k}{v! \ \ell! \ m! \ i! \ j! \ k!}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} \frac{\beta^{i+n^\star+n} \lambda^{i+n^\star}}{(i+n^\star+n)! \ (i+n^\star)!} \sum_{\ell,m=0}^{\theta\ell+\phi m \leq i+n^\star+n} \Omega_1(i+n^\star+n-\theta\ell-\phi m) \ \Omega_2(\ell) \ \Omega_3(m) \times$$

$$\times \ (-i-n^\star-n)_{\theta\ell+\phi m} \frac{\left(\frac{(-1)^\theta \gamma}{\beta^\theta}\right)^\ell}{\ell!} \frac{\left(\frac{(-1)^\phi \zeta}{\beta^\phi}\right)^m}{m!} \sum_{j,k=0}^{\psi j+\nu k \leq i+n^\star} \Omega_4(i+n^\star-\psi j-\nu k) \ \times$$

$$\times \Omega_5(j) \ \Omega_6(k) \ (-i-n^\star)_{\psi j+\nu k} \frac{\left(\frac{(-1)^\psi \mu}{\lambda^\psi}\right)^j}{j!} \frac{\left(\frac{(-1)^\nu \sigma}{\lambda^\nu}\right)^k}{k!} \ t^n, \tag{2.1}$$

where θ, ϕ, ψ *and* ν *are relatively prime positive integers and* $n^\star$ *is defined as:*

$$n^\star = \max\{0,-n\} = \begin{cases} -n, & \textit{when} \ \ n = \cdots,-3,-2,-1 \\ 0, & \textit{when} \ n = 0,1,2,\cdots, \end{cases} \tag{2.2}$$

provided that each of the multiple series involved is absolutely convergent.

Proof. Suppose the l.h.s. of equation (2.1) is denoted by Λ. Then, we have

$$\Lambda \ = \ \sum_{v,\ell,m=0}^{\infty} \Omega_1(v)\Omega_2(\ell)\Omega_3(m) \frac{\beta^v \gamma^\ell \zeta^m}{v! \ \ell! \ m!} \ t^{v+\theta\ell+\phi m} \sum_{i,j,k=0}^{\infty} \Omega_4(i)\Omega_5(j)\Omega_6(k) \frac{\lambda^i \mu^j \sigma^k}{i! \ j! \ k!} \ \frac{1}{t^{i+\psi j+\nu k}}. \tag{2.3}$$

On replacing v by $v-\theta\ell-\phi m$ and i by $i-\psi j-\nu k$ in equation (2.3) and then using Lemma 1.3, we obtain

$$\Delta \ = \ \sum_{v=0}^{\infty} \sum_{\ell,m=0}^{\theta\ell+\phi m \leq v} \Omega_1(v-\theta\ell-\phi m)\Omega_2(\ell)\Omega_3(m) \frac{\beta^{v-\theta\ell-\phi m}\gamma^\ell \zeta^m}{(v-\theta\ell-\phi m)! \ \ell! \ m!} \times$$

$$\times \sum_{i=0}^{\infty} \sum_{j,k=0}^{\psi j+\nu k \leq i} \Omega_4(i-\psi j-\nu k)\Omega_5(j)\Omega_6(k) \frac{\lambda^{i-\psi j-\nu k} \mu^j \sigma^k}{(i-\psi j-\nu k)! \ j! \ k!} \ t^{v-i}. \tag{2.4}$$

Putting $v = i + n$ in equation (2.4), we get

$$\begin{aligned}\Delta &= \sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\sum_{\ell,m=0}^{\theta\ell+\phi m\leq i+n}\sum_{j,k=0}^{\psi j+\nu k\leq i}\Omega_1(i+n-\theta\ell-\phi m)\Omega_2(\ell)\Omega_3(m)\Omega_4\times\\ &\quad\times\Omega_4(i-\psi j-\nu k)\Omega_5(j)\Omega_6(k)\frac{\beta^{i+n-\theta\ell-\phi m}\gamma^\ell\zeta^m}{(i+n-\theta\ell-\phi m)!\ \ell!\ m!}\frac{\lambda^{i-\psi j-\nu k}\mu^j\sigma^k}{(i-\psi j-\nu k)!\ j!\ k!}\ t^n\\ &= \sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{\beta^{i+n}\lambda^i}{(i+n)!\ i!}\times\\ &\quad\times\sum_{\ell,m=0}^{\theta\ell+\phi m\leq i+n}\sum_{j,k=0}^{\psi j+\nu k\leq i}(-1)^{(\theta\ell+\phi m)}\ (-1)^{(\psi j+\nu k)}\ (-i-n)_{\theta\ell+\phi m}\ (-i)_{\psi j+\nu k}\times\\ &\quad\times\Omega_1(i+n-\theta\ell-\phi m)\ \Omega_2(\ell)\ \Omega_3(m)\ \Omega_4(i-\psi j-\nu k)\ \Omega_5(j)\ \Omega_6(k)\times\\ &\quad\times\frac{\left(\frac{\gamma}{\beta^\theta}\right)^\ell\ \left(\frac{\zeta}{\beta^\phi}\right)^m\ \left(\frac{\mu}{\lambda^\psi}\right)^j\ \left(\frac{\sigma}{\lambda^\nu}\right)^k}{\ell!\ m!\ j!\ k!}\ t^n. \end{aligned}\tag{2.5}$$

Since n varies from $-\infty$ to ∞ and i varies from 0 to ∞, therefore due to the presence of $(i+n)!$ in denominator of above equation, equation (2.5) can be modified in the following form:

$$\begin{aligned}\Delta &= \sum_{n=-\infty}^{\infty}\sum_{i=n^\star}^{\infty}\sum_{\ell,m=0}^{\theta\ell+\phi m\leq i+n}\sum_{j,k=0}^{\psi j+\nu k\leq i}\Omega_1(i+n-\theta\ell-\phi m)\Omega_2(\ell)\Omega_3(m)\Omega_4\times\\ &\quad\times\Omega_4(i-\psi j-\nu k)\Omega_5(j)\Omega_6(k)\frac{\beta^{i+n-\theta\ell-\phi m}\gamma^\ell\zeta^m}{(i+n-\theta\ell-\phi m)!\ \ell!\ m!}\frac{\lambda^{i-\psi j-\nu k}\mu^j\sigma^k}{(i-\psi j-\nu k)!\ j!\ k!}\ t^n\\ &= \sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{\beta^{i+n}\lambda^i}{(i+n)!\ i!}\times\\ &\quad\times\sum_{\ell,m=0}^{\theta\ell+\phi m\leq i+n}\sum_{j,k=0}^{\psi j+\nu k\leq i}(-1)^{(\theta\ell+\phi m)}\ (-1)^{(\psi j+\nu k)}\ (-i-n)_{\theta\ell+\phi m}\ (-i)_{\psi j+\nu k}\times\\ &\quad\times\Omega_1(i+n-\theta\ell-\phi m)\ \Omega_2(\ell)\ \Omega_3(m)\ \Omega_4(i-\psi j-\nu k)\ \Omega_5(j)\ \Omega_6(k)\times\\ &\quad\times\frac{\left(\frac{\gamma}{\beta^\theta}\right)^\ell\ \left(\frac{\zeta}{\beta^\phi}\right)^m\ \left(\frac{\mu}{\lambda^\psi}\right)^j\ \left(\frac{\sigma}{\lambda^\nu}\right)^k}{\ell!\ m!\ j!\ k!}\ t^n. \end{aligned}\tag{2.6}$$

On replacing i by $i + n^\star$ in equation (2.6), we get assertion (2.1). □

3. Hypergeometric generating relation

In this section, a Laurent type hypergeometric generating relation in terms of six generalized hypergeometric functions of type ${}_pF_q$ is established.

Theorem 3.1. *The following Laurent type hypergeometric generating relation holds true:*

$$
{}_AF_B\left[\begin{array}{ccc}(a_A) & ; & \\ & & \beta t\\ (b_B) & ; & \end{array}\right]{}_CF_D\left[\begin{array}{ccc}(c_C) & ; & \\ & & \gamma t^{\theta}\\ (d_D) & ; & \end{array}\right]{}_GF_H\left[\begin{array}{ccc}(g_G) & ; & \\ & & \zeta t^{\phi}\\ (h_H) & ; & \end{array}\right]{}_PF_Q\left[\begin{array}{ccc}(p_P) & ; & \\ & & \frac{\lambda}{t}\\ (q_Q) & ; & \end{array}\right]\times
$$

$$
\times{}_RF_S\left[\begin{array}{ccc}(r_R) & ; & \\ & & \frac{\mu}{t^{\psi}}\\ (s_S) & ; & \end{array}\right]{}_UF_W\left[\begin{array}{ccc}(u_U) & ; & \\ & & \frac{\sigma}{t^{\nu}}\\ (w_W) & ; & \end{array}\right]
$$

$$
=\sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{\prod_{\delta=1}^{A}(a_\delta)_{i+n^\star+n}\ \prod_{\delta=1}^{P}(p_\delta)_{i+n^\star}}{\prod_{\delta=1}^{B}(a_\delta)_{i+n^\star+n}\ \prod_{\delta=1}^{Q}(q_\delta)_{i+n^\star}}\frac{\beta^{i+n^\star+n}\ \lambda^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!}\times
$$

$$
\times F^{B+1:C;G}_{A\ \ :D;H}\left(\begin{array}{lll}[1-(b_B)-i-n^\star-n:\theta,\phi],\ [-i-n^\star-n:\theta,\phi] & : & [(c_C):1]\ ;\\ [1-(a_A)-i-n^\star-n:\theta,\phi] & : & [(d_D):1]\ ;\end{array}\right.
$$

$$
\left.\begin{array}{ll}[(g_G):1]\ ; & \\ & \left(\frac{(-1)^{(A+B+1)}}{\beta}\right)^{\theta}\gamma,\ \left(\frac{(-1)^{(A+B+1)}}{\beta}\right)^{\phi}\zeta\\ [(h_H):1]\ ; & \end{array}\right)\times
$$

$$
\times F^{Q+1:R;U}_{P\ \ :S;W}\left(\begin{array}{lll}[1-(q_Q)-i-n^\star:\psi,\nu],\ [-i-n^\star:\psi,\nu] & : & [(r_R):1]\ ;\\ [1-(p_P)-i-n^\star:\psi,\nu] & : & [(s_S):1]\ ;\end{array}\right.
$$

$$
\left.\begin{array}{ll}[(u_U):1]\ ; & \\ & \left(\frac{(-1)^{(P+Q+1)}}{\lambda}\right)^{\psi}\mu,\ \left(\frac{(-1)^{(P+Q+1)}}{\lambda}\right)^{\nu}\sigma\\ [(w_W):1]\ ; & \end{array}\right)\ t^n,\quad \beta,\lambda,t\neq 0, \tag{3.1}
$$

where θ,ϕ,ψ and ν are positive integers and $n^\star$ is defined by equation (2.2).

Proof. Taking

$$
\Omega_1(v)=\frac{\prod_{\delta=1}^{A}(a_\delta)_v}{\prod_{\delta=1}^{B}(b_\delta)_v},\quad \Omega_2(\ell)=\frac{\prod_{\delta=1}^{C}(c_\delta)_\ell}{\prod_{\delta=1}^{D}(d_\delta)_\ell},\quad \Omega_3(m)=\frac{\prod_{\delta=1}^{G}(g_\delta)_m}{\prod_{\delta=1}^{H}(h_\delta)_m},\quad \Omega_4(i)=\frac{\prod_{\delta=1}^{P}(p_\delta)_i}{\prod_{\delta=1}^{Q}(q_\delta)_i},
$$

$$
\Omega_5(j)=\frac{\prod_{\delta=1}^{R}(r_\delta)_j}{\prod_{\delta=1}^{S}(s_\delta)_j},\quad \Omega_6(k)=\frac{\prod_{\delta=1}^{U}(u_\delta)_k}{\prod_{\delta=1}^{W}(w_\delta)_k}
$$

in general series identity (2.1), applying some algebraic properties of Pochhammer symbols and after simplification, we obtain:

$$ {}_AF_B\left[\begin{array}{lll}(a_A) & ; & \\ & & \beta t\\ (b_B) & ; & \end{array}\right] {}_CF_D\left[\begin{array}{lll}(c_C) & ; & \\ & & \gamma t^{\theta}\\ (d_D) & ; & \end{array}\right] {}_GF_H\left[\begin{array}{lll}(g_G) & ; & \\ & & \zeta t^{\phi}\\ (h_H) & ; & \end{array}\right] {}_PF_Q\left[\begin{array}{lll}(p_P) & ; & \\ & & \frac{\lambda}{t}\\ (q_Q) & ; & \end{array}\right]\times $$

$$ \times {}_RF_S\left[\begin{array}{lll}(r_R) & ; & \\ & & \frac{\mu}{t^{\psi}}\\ (s_S) & ; & \end{array}\right] {}_UF_W\left[\begin{array}{lll}(u_U) & ; & \\ & & \frac{\sigma}{t^{\nu}}\\ (w_W) & ; & \end{array}\right] $$

$$ = \sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{\prod_{\delta=1}^{A}(a_\delta)_{i+n^\star+n}}{\prod_{\delta=1}^{B}(b_\delta)_{i+n^\star+n}}\frac{\prod_{\delta=1}^{P}(p_\delta)_{i+n^\star}}{\prod_{\delta=1}^{Q}(q_\delta)_{i+n^\star}}\frac{\beta^{i+n^\star+n}\lambda^{i+n^\star}}{(i+n^\star+n)!\,(i+n^\star)!}\times $$

$$ \times\sum_{\ell,m=0}^{\theta\ell+\phi m\leq i+n^\star+n}\frac{\prod_{\delta=1}^{B}(1-b_\delta-i-n^\star-n)_{\theta\ell+\phi m}}{\prod_{\delta=1}^{A}(1-a_\delta-i-n^\star-n)_{\theta\ell+\phi m}}(-i-n^\star-n)_{\theta\ell+\phi m}\times $$

$$ \times\frac{\prod_{\delta=1}^{C}(c_\delta)_\ell}{\prod_{\delta=1}^{D}(d_\delta)_\ell}\frac{\prod_{\delta=1}^{G}(g_\delta)_m}{\prod_{\delta=1}^{H}(h_\delta)_m}\frac{\left(\frac{(-1)^{(A+B+1)\theta}\gamma}{\beta^{\theta}}\right)^{\ell}}{\ell!}\frac{\left(\frac{(-1)^{(A+B+1)\phi}\zeta}{\beta^{\phi}}\right)^{m}}{m!}\times $$

$$ \sum_{j,k=0}^{\psi j+\nu k\leq i+n^\star}\frac{\prod_{\delta=1}^{Q}(1-q_\delta-i-n^\star)_{\psi j+\nu k}}{\prod_{\delta=1}^{P}(1-p_\delta-i-n^\star)_{\psi j+\nu k}}(-i-n^\star)_{\psi j+\nu k}\times $$

$$ \times\frac{\prod_{\delta=1}^{R}(r_\delta)_j}{\prod_{\delta=1}^{S}(s_\delta)_j}\frac{\prod_{\delta=1}^{U}(u_\delta)_k}{\prod_{\delta=1}^{W}(w_\delta)_k}\frac{\left(\frac{(-1)^{(P+Q+1)\psi}\mu}{\lambda^{\psi}}\right)^{j}}{j!}\frac{\left(\frac{(-1)^{(P+Q+1)\nu}\sigma}{\lambda^{\nu}}\right)^{k}}{k!}\,t^n. \tag{3.2} $$

On using definition of the Srivastava and Daoust hypergeometric function (1.16) in the r.h.s. of equation (3.2), we obtain assertion (3.1).

□

Remark 3.2. Taking $A=B=C=D=G=H=P=Q=R=S=U=W=0$ in equation (3.1), we deduce the following consequence of Theorem 3.1:

Corollary 3.3. *The following bilateral generating relation holds true:*

$$\exp\left[\beta t+\gamma t^{\theta}+\zeta t^{\phi}+\frac{\lambda}{t}+\frac{\mu}{t^{\psi}}+\frac{\sigma}{t^{\nu}}\right]$$
$$=\sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{\beta^{i+n^\star+n}\lambda^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!}\times$$
$$\times F^{1:0;0}_{0:0;0}\left(\begin{array}{lcl}[-i-n^\star-n:\theta,\phi] & : & —\ ;\ —\ ; \\ & & \\ — & : & —\ ;\ —\ ;\end{array}\ \left(\tfrac{-1}{\beta}\right)^{\theta}\gamma,\ \left(\tfrac{-1}{\beta}\right)^{\phi}\zeta\right)\times$$
$$\times F^{1:0;0}_{0:0;0}\left(\begin{array}{lcl}[-i-n^\star:\psi,\nu] & : & —\ ;\ —\ ; \\ & & \\ — & : & —\ ;\ —\ ;\end{array}\ \left(\tfrac{-1}{\lambda}\right)^{\psi}\mu,\ \left(\tfrac{-1}{\lambda}\right)^{\nu}\sigma\right)t^{n},\quad \beta,\lambda,t\neq 0,\tag{3.3}$$

where θ,ϕ,ψ and ν are positive integers and $n^\star$ is defined by equation (2.2).

Remark 3.4. Taking $\zeta=\mu=0$ in equation (3.3), we deduce the following consequence of Corollary 3.3:

Corollary 3.5. *The following bilateral generating relation holds true:*

$$\exp\left[\beta t+\gamma t^{\theta}+\frac{\lambda}{t}+\frac{\sigma}{t^{\nu}}\right]=\sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{\beta^{i+n^\star+n}\lambda^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!}\times$$
$$\times\ {}_1\Psi_0^\star\left[\begin{array}{ll}(-i-n^\star-n,\theta) & ; \\ & \\ — & ;\end{array}\ \left(\tfrac{-1}{\beta}\right)^{\theta}\gamma\right]\ {}_1\Psi_0^\star\left[\begin{array}{ll}(-i-n^\star,\nu) & ; \\ & \\ — & ;\end{array}\ \left(\tfrac{-1}{\lambda}\right)^{\nu}\sigma\right]t^{n}\tag{3.4}$$

or

$$\exp\left[\beta t+\gamma t^{\theta}+\frac{\lambda}{t}+\frac{\sigma}{t^{\nu}}\right]=\sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{\beta^{i+n^\star+n}\lambda^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!}\times$$
$$\times\ {}_{\theta}F_0\left[\begin{array}{ll}\Delta(\theta;-i-n^\star-n) & ; \\ & \\ — & ;\end{array}\ \left(\tfrac{-\theta}{\beta}\right)^{\theta}\gamma\right]\ {}_{\nu}F_0\left[\begin{array}{ll}\Delta(\nu;-i-n^\star) & ; \\ & \\ — & ;\end{array}\ \left(\tfrac{-\nu}{\lambda}\right)^{\nu}\sigma\right]t^{n}\tag{3.5}$$

or

$$\exp\left[\beta t+\gamma t^{\theta}+\frac{\lambda}{t}+\frac{\sigma}{t^{\nu}}\right]=\sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{g^{\theta}_{i+n^\star+n}(\beta,\gamma)\ g^{\nu}_{i+n^\star}(\lambda,\sigma)}{(i+n^\star+n)!\ (i+n^\star)!}\ t^{n}.\tag{3.6}$$

4. Applications

In this section, we obtain the explicit forms of certain generalized and mixed relatives of the Bessel functions in the form of following results:

Result 4.1. *The 2-variable Bessel functions $J_n(x,y)$ have the following explicit representation:*

$$J_n(x,y) = (-\omega)^n \left(\frac{y}{2}\right)^{\frac{n}{2}} \sum_{i=0}^{\infty} \frac{\left(\frac{\omega y}{2}\right)^{i+n^\star} H_{i+n^\star+n}\left(\frac{\omega x}{2\sqrt{2y}}\right) H_{i+n^\star}\left(\frac{x}{2\sqrt{2y}}\right)}{(i+n^\star+n)!\ (i+n^\star)!}, \quad \omega^2=-1. \tag{4.1}$$

Interpretation: Taking $\beta=\frac{x}{2}$, $\gamma=\frac{y}{2}$, $\theta=2$, $\zeta=0$ $\lambda=-\frac{x}{2}$, $\mu=-\frac{y}{2}$, $\psi=2$, $\sigma=0$ in Corollary 3.3, we get

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right] = \sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{i+n^\star+n}\left(-\frac{x}{2}\right)^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!} \times$$

$$\times \sum_{\ell=0}^{2\ell\le i+n^\star+n} \sum_{j=0}^{2j\le i+n^\star} (-i-n^\star-n)_{2\ell}\ (-i-n^\star)_{2j} \frac{\left(\frac{2y}{x^2}\right)^\ell}{\ell!} \frac{\left(-\frac{2y}{x^2}\right)^j}{j!}\ t^n$$

$$= \sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{i+n^\star+n}\left(-\frac{x}{2}\right)^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!} \times$$

$$\times\ {}_2F_0\left[\begin{array}{ccc} \frac{-(i+n^\star+n)}{2}, & \frac{-(i+n^\star+n)+1}{2} & ; \\ & & \frac{8y}{x^2} \\ \text{———} & & ; \end{array}\right] {}_2F_0\left[\begin{array}{ccc} \frac{-(i+n^\star)}{2}, & \frac{-(i+n^\star)+1}{2} & ; \\ & & -\frac{8y}{x^2} \\ \text{———} & & ; \end{array}\right] t^n$$

$$= \sum_{n=-\infty}^{\infty}\sum_{i=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{i+n^\star+n}\left(-\frac{x}{2}\right)^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!} \times$$

$$\times \left(\frac{\sqrt{2y}}{\omega x}\right)^{i+n^\star+n} H_{i+n^\star+n}\left(\frac{\omega x}{2\sqrt{2y}}\right) \left(\frac{\sqrt{2y}}{x}\right)^{i+n^\star} H_{i+n^\star}\left(\frac{x}{2\sqrt{2y}}\right) t^n \tag{4.2}$$

On comparison of equations (4.2) and (1.1), Result 4.1 follows.

Result 4.2. *The 2-variable Hermite-Bessel functions ${}_HJ_n(x,y)$ have the following explicit representation:*

$$ {}_HJ_n(x,y) = e^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{x^{i+n^\star+n}(-x)^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!} \left(\frac{\sqrt{\frac{1}{4}-\frac{y}{2}}}{2x}\right)^{i+n^\star+n} H_{i+n^\star+n}\left(\frac{x}{\sqrt{\frac{1}{4}-\frac{y}{2}}}\right) \times$$

$$\times \left(\frac{\sqrt{\frac{1}{4}+\frac{y}{2}}}{2x}\right)^{i+n^\star} H_{i+n^\star}\left(\frac{x}{\sqrt{\frac{1}{4}+\frac{y}{2}}}\right). \tag{4.3}$$

Interpretation: Multiplying by $e^{\frac{1}{2}}$ on both sides of equation (3.3) and then taking $\beta = x$, $\gamma = \frac{y}{2} - \frac{1}{4}$, $\theta = 2$, $\zeta = 0$, $\lambda = -x$, $\mu = -\frac{y}{2} - \frac{1}{4}$, $\psi = 2$, $\sigma = 0$, we get

$$\exp\left[x\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^2 - \frac{1}{t^2}\right) - \frac{1}{4}\left(t - \frac{1}{t}\right)^2\right] = e^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} \frac{x^{i+n^\star+n}(-x)^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!} \times$$

$$\times \sum_{\ell=0}^{2\ell \leq i+n^\star+n} (-i-n^\star-n)_{2\ell} \frac{\left(\frac{\frac{y}{2}-\frac{1}{4}}{x^2}\right)^\ell}{\ell!} \sum_{j=0}^{2j \leq i+n^\star} (-i-n^\star)_{2j} \frac{\left(\frac{-\frac{y}{2}-\frac{1}{4}}{x^2}\right)^j}{j!}\ t^n$$

$$= e^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} \frac{x^{i+n^\star+n}(-x)^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!} \times$$

$${}_2F_0\left[\begin{matrix} \frac{-(i+n^\star+n)}{2}, \frac{-(i+n^\star+n)+1}{2} & ; \\ \text{———} & ; \end{matrix} \ \frac{\frac{y}{2}-\frac{1}{4}}{x^2}\right] {}_2F_0\left[\begin{matrix} \frac{-(i+n^\star)}{2}, \frac{-(i+n^\star)+1}{2} & ; \\ \text{———} & ; \end{matrix} \ \frac{-\frac{y}{2}-\frac{1}{4}}{x^2}\right] t^n$$

$$= e^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} \frac{x^{i+n^\star+n}(-x)^{i+n^\star}}{(i+n^\star+n)!\ (i+n^\star)!} \times$$

$$\times \left(\frac{\sqrt{\frac{1}{4}-\frac{y}{2}}}{2x}\right)^{i+n^\star+n} H_{i+n^\star+n}\left(\frac{x}{\sqrt{\frac{1}{4}-\frac{y}{2}}}\right) \times$$

$$\times \left(\frac{\sqrt{\frac{1}{4}+\frac{y}{2}}}{2x}\right)^{i+n^\star} H_{i+n^\star}\left(\frac{x}{\sqrt{\frac{1}{4}+\frac{y}{2}}}\right) t^n. \tag{4.4}$$

On comparison of equations (4.4) and (1.3), Result 4.2 follows.

Result 4.3. *The generalized Bessel functions $J_n^{(m)}(x,y)$ have the following explicit representation in terms of Gould-Hopper polynomials:*

$$J_n^{(m)}(x,y) = \sum_{i=0}^{\infty} \frac{g^m_{i+n^\star+n}\left(\frac{x}{2},\frac{y}{2}\right)\ g^m_{i+n^\star}\left(-\frac{x}{2},-\frac{y}{2}\right)}{(i+n^\star+n)!\ (i+n^\star)!}. \tag{4.5}$$

Interpretation: Taking $\beta = \frac{x}{2}$, $\gamma = \frac{y}{2}$, $\theta = \nu = m$, $\lambda = -\frac{x}{2}$, $\sigma = -\frac{y}{2}$ in Corollary 3.5, we get

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^m - \frac{1}{t^m}\right)\right] = \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} \frac{g^m_{i+n^\star+n}\left(\frac{x}{2},\frac{y}{2}\right)\ g^m_{i+n^\star}\left(-\frac{x}{2},-\frac{y}{2}\right)}{(i+n^\star+n)!\ (i+n^\star)!}\ t^n. \tag{4.6}$$

On comparison of equations (4.6) and (1.2), Result 4.3 follows.

Result 4.4. *The Hermite-Bessel functions associated with the Bell-type polynomials $H_n^{(3,2)}(x,w,y)$ have the following explicit representation:*

$$
\begin{aligned}
H_n^{(3,2)}(x,w,y)) &= e^{-\frac{w}{2}} \sum_{i=0}^{\infty} \frac{\left(\frac{x}{2}-\frac{3y}{8}\right)^{i+n^\star+n}\left(-\frac{x}{2}+\frac{3y}{8}\right)^{i+n^\star}}{(i+n^\star+n)!\,(i+n^\star)!} \times \\
&\times F^{1:0;0}_{0:0;0}\left(\begin{array}{ccc} [-i-n^\star-n:2,3] & : \text{---} ; & \text{---} ; \\ \text{---} & : \text{---} ; & \text{---} ; \end{array} \frac{w}{4\left(\frac{x}{2}-\frac{3y}{8}\right)^2}, \frac{-y}{8\left(\frac{x}{2}-\frac{3y}{8}\right)^3}\right) \times \\
&\times F^{1:0;0}_{0:0;0}\left(\begin{array}{ccc} [-i-n^\star:2,3] & : \text{---} ; & \text{---} ; \\ \text{---} & : \text{---} ; & \text{---} ; \end{array} \frac{-w}{4\left(-\frac{x}{2}+\frac{3y}{8}\right)^2}, \frac{y}{8\left(-\frac{x}{2}+\frac{3y}{8}\right)^3}\right).
\end{aligned}
\tag{4.7}
$$

Interpretation: Multiplying by $e^{-\frac{w}{2}}$ on both sides of equation (3.3) and then taking $\beta = \frac{x}{2} - \frac{3y}{8}$, $\gamma = \frac{w}{4}$, $\theta = 2$, $\zeta = \frac{y}{8}$, $\phi = 3$, $\lambda = -\frac{x}{2} + \frac{3y}{8}$, $\mu = -\frac{w}{4}$, $\psi = 2$, $\sigma = -\frac{y}{8}$, $\nu = 3$, we get

$$
\begin{aligned}
&\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right) + \frac{w}{4}\left(t-\frac{1}{t}\right)^2 + \frac{y}{8}\left(t-\frac{1}{t}\right)^3\right] \\
&= e^{-\frac{w}{2}} \sum_{n=-\infty}^{\infty} \sum_{i=0}^{\infty} \frac{\left(\frac{x}{2}-\frac{3y}{8}\right)^{i+n^\star+n}\left(-\frac{x}{2}+\frac{3y}{8}\right)^{i+n^\star}}{(i+n^\star+n)!\,(i+n^\star)!} \times \\
&\times F^{1:0;0}_{0:0;0}\left(\begin{array}{ccc} [-i-n^\star-n:2,3] & : \text{---} ; & \text{---} ; \\ \text{---} & : \text{---} ; & \text{---} ; \end{array} \frac{w}{4\left(\frac{x}{2}-\frac{3y}{8}\right)^2}, \frac{-y}{8\left(\frac{x}{2}-\frac{3y}{8}\right)^3}\right) \times \\
&\times F^{1:0;0}_{0:0;0}\left(\begin{array}{ccc} [-i-n^\star:2,3] & : \text{---} ; & \text{---} ; \\ \text{---} & : \text{---} ; & \text{---} ; \end{array} \frac{-w}{4\left(-\frac{x}{2}+\frac{3y}{8}\right)^2}, \frac{y}{8\left(-\frac{x}{2}+\frac{3y}{8}\right)^3}\right) t^n
\end{aligned}
\tag{4.8}
$$

On comparison of equations (4.8) and (1.4), Result 4.4 follows.

5. Conclusion

The present article deals with finding explicit representations of the generalized and mixed Bessel functions. The relatives of the Bessel functions considered in this work have exponential generating functions and thus we get their explicit forms directly from Corollaries 3.3 and 3.5. The explicit forms of other special functions which have generating functions other than exponential type can be obtained by using Theorem 3.1.

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