ON THE EXTENDED GRAPH ASSOCIATED WITH THE SET OF ALL NON-ZERO ANNIHILATING IDEALS OF A COMMUTATIVE RING

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Abstract. Let $R$ be a commutative ring with non-zero identity which is not an integral domain. An ideal $I$ of a ring $R$ is called an annihilating ideal if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$. Let $\mathcal{A}(R)$ denote the set of all annihilating ideals of $R$ and $\mathcal{A}(R)^* = \mathcal{A}(R) \setminus \{(0)\}$. In this article, we introduce a new graph associated with $R$ denoted by $H(R)$ whose vertex set is $\mathcal{A}(R)^*$ and two distinct vertices $I, J$ are adjacent in this graph if and only if $IJ = (0)$ or $I + J \in \mathcal{A}(R)$. The aim of this article is to study the interplay between the ring-theoretic properties of a ring $R$ and the graph-theoretic properties of $H(R)$. For such a ring $R$, we prove that $H(R)$ is connected and find its diameter. Moreover, we determine girth of $H(R)$. Furthermore, we provide some sufficient conditions under which $H(R)$ is a complete graph.

1. Introduction

The study of algebraic structures with the help of graph theory is an exciting research topic. There are many research articles on assigning a graph to a ring. The rings considered in this article are commutative with non-zero identity which are not integral domains.

Let $R$ be a commutative ring with non-zero identity which is not an integral domain. Let $Z(R)$ denote the set of all zero-divisors of $R$ and $Z(R)^* = Z(R) \setminus \{0\}$. The idea of zero-divisor graph of a commutative ring was given by D. F. Anderson and P. S. Livingston in [2]. Recall from [2] that the zero-divisor graph $\Gamma(R)$ of a ring $R$ is an undirected graph whose vertex set is $Z(R)^*$ and two distinct vertices $x, y$ are adjacent in this graph if and only if $xy = 0$. The generalization of zero-divisor graph was introduced by M. Behboodi and Z. Rakeei in [5]. Recall from [5] that an ideal $I$ of $R$ is said to be an annihilating ideal if $Ir = (0)$ for some $r \in R \setminus \{0\}$. As in [5], we denote by $\mathcal{A}(R)$, the set of all annihilating ideals of $R$ and by $\mathcal{A}(R)^* = \mathcal{A}(R) \setminus \{(0)\}$. Recall from [5] that the annihilating-ideal graph of a ring $R$ denoted by $\mathcal{A}G(R)$ is an undirected graph whose vertex set is $\mathcal{A}(R)^*$ and two distinct vertices $I, J \in \mathcal{A}(R)^*$ are adjacent in this graph if and only if $IJ = (0)$. There are several research articles in the literature which studied the

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interplay between the ring-theoretic properties of a commutative ring $R$ and the graph-theoretic properties of $\mathbb{A}\mathbb{G}(R)$ (see for example, [5, 6, 15]).

In [13], we introduced the concept of a graph associated with the set of all non-zero annihilating ideals of a commutative ring. Recall from [13] that a graph associated with $R$ denoted by $\Omega(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I, J \in \mathbb{A}(R)^*$ are adjacent in this graph if and only if $I + J \in \mathbb{A}(R)$. The main aim of [13] was to study some properties of $\Omega(R)$. Some further properties of $\Omega(R)$ were proved in [1, 11]. Moreover, A. Cherrabi et al. introduced a new extension of the zero-divisor graph denoted by $\widetilde{\Gamma}(R)$ in [7]. Recall from [7] that $\widetilde{\Gamma}(R)$ is an undirected simple graph whose vertices are the non-zero zero-divisors of $R$ and two distinct vertices $x, y$ are adjacent in this graph if and only if $xy = 0$ or $x + y \in Z(R)$. The authors of [7, 8] explored the relationship of $\widetilde{\Gamma}(R)$ with the zero-divisor graph and with a subgraph of the total graph of a ring. Furthermore, the results proved on the graph considered in the present article is also motivated by the interesting results proved in [14]. Let $R$ be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Recall from [14] that $G(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I$ and $J$ are adjacent in $G(R)$ if and only if either $IJ \neq (0)$ or $I + J \notin \mathbb{A}(R)$.

Motivated by the work done on $\Gamma(R), \mathbb{A}\mathbb{G}(R), \Omega(R), \widetilde{\Gamma}(R)$, and $G(R)$ mentioned above, in this article, we introduce extended graph associated with $R$ denoted by $H(R)$ whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I, J$ are adjacent in this graph if and only if $IJ = (0)$ or $I + J \in \mathbb{A}(R)$. The main aim of this article is to study some properties of $H(R)$. In this article, we focus on sum and product operations on ideals to describe the properties of $H(R)$. Before we give a brief account of results proved in this article, it is useful to recall the following definitions from commutative ring theory.

Let $I$ be an ideal of a ring $R$ with $I \neq R$. Recall from [10] that a prime ideal $p$ of $R$ is said to be a maximal N-prime of $I$ if $p$ is maximal with respect to the property of being contained in $Z(R/I) = \{ r \in R \mid rx \in I \text{ for some } x \in R\setminus I \}$. Hence, $p$ is a maximal N-prime of $(0)$ if $p$ is maximal with respect to the property of being contained in $Z(R)$. It is clear that $S = R\setminus Z(R)$ is a multiplicatively closed subset of $R$. Let $x \in Z(R)$. Then $Rx \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [12, Theorem 1] that there exists a maximal N-prime $p$ of $(0)$ in $R$ such that $x \in p$. Therefore, if $\{p_\alpha\}_{\alpha \in \Lambda}$ is the set of all maximal N-primes of $(0)$ in $R$, then it follows that $Z(R) = \bigcup_{\alpha \in \Lambda} p_\alpha$.

Let $I$ be an ideal of a ring $R$ with $I \neq R$. Recall from [9] that a prime ideal $p$ of $R$ is said to be an associated prime of $I$ in the sense of Bourbaki, if $p = (I :_R x)$ for some $x \in R$. In this case, we say that $p$ is a B-prime of $I$. Let $p$ be a maximal N-prime of $(0)$ in $R$. It is easy to verify that $p$ is a B-prime of $(0)$ in $R$ if and only if $p \in \mathbb{A}(R)$.

A ring $R$ is said to be quasi-local if $R$ has a unique maximal ideal. By a local ring, we mean a Noetherian quasilocal ring. A local ring $R$ with unique maximal ideal $m$ is denoted by $(R, m)$. An element $x$ of a ring $R$ is said to be nilpotent if there exists a least positive integer $n$ such that $x^n = 0$. The nilradical of a ring $R$ is the set of all nilpotent elements in the ring $R$. We use $\text{nil}(R)$ to denote
the nilradical of a ring $R$. Clearly, $\text{nil}(R)$ is an ideal of $R$. A ring $R$ is said to be reduced if it has no non-zero nilpotent elements. Note that $R$ is reduced if $\text{nil}(R) = (0)$. An ideal $I$ of a ring $R$ is said to be a nilpotent ideal if there exists $k \in \mathbb{N}$ such that $I^k = (0)$. Recall that a ring $R$ is said to be a chained ring if the ideals of $R$ are linearly ordered under the inclusion relation. We denote the set of all minimal prime ideals of $R$ by $\text{Min}(R)$. Let $n \in \mathbb{N}$ with $n \geq 2$. We denote the ring of integer modulo $n$ by $\mathbb{Z}_n$. Any unexplained terminology from commutative ring theory can be referred from the standard textbooks like [3, 12].

Next we recall the following definitions from graph theory which are useful in this article. The graphs considered in this article are undirected. Let $G = (V, E)$ be a graph. A graph $G$ is said to be connected if for each pair of distinct vertices $a, b \in V$, there exists at least one path in $G$ between $a$ and $b$. Let $G = (V, E)$ be a connected graph. Recall from [4] that for distinct vertices $a, b$ of $G$, the distance between $a$ and $b$ denoted by $d(a, b)$ is defined as the length of a shortest path between $a$ and $b$ in $G$. We define $d(a, a) = 0$. We define the diameter of $G$ denoted by $\text{diam}(G)$ as $\text{diam}(G) = \sup \{d(a, b) \mid a, b \in V\}$.

Let $G = (V, E)$ be a connected graph. Let $v \in V$. Recall from [4] that the eccentricity of $v$ denoted by $E(v)$ is defined as $E(v) = \sup \{d(v, w) \mid w \in V\}$. The radius of $G$ denoted by $r(G)$ is defined as $r(G) = \min \{E(v) \mid v \in V\}$. A vertex $v \in V$ with minimum eccentricity is called a center of $G$. Recall that the girth of $G$ denoted by $\text{girth}(G)$ or $\text{gr}(G)$ is defined as the length of a shortest cycle in $G$. If $G$ does not contain any cycle, then we set $\text{gr}(G) = \infty$. A simple graph $G$ is said to be complete if every pair of distinct vertices of $G$ are adjacent in $G$. Let $n \in \mathbb{N}$. A complete graph on $n$ vertices is denoted by $K_n$.

Given a graph $G$, a subgraph $H$ of $G$ is a graph whose vertex set and edge set are subsets of those of $G$. A subgraph $H$ of $G$ is called an induced subgraph if all edges of $G$ joining two vertices in $H$ are also edges of $H$. A spanning subgraph $H$ of $G$ is a subgraph containing all the vertices of $G$. Recall that a vertex $a$ of $G$ is said to be a Smarandache vertex or simply an $S$-vertex if there exist three distinct vertices $x, y,$ and $b$ $(\neq a)$ in $G$ such that $x - a, a - b,$ and $b - y$ are edges in $G$ but there is no edge between $x$ and $y$ in $G$. Note that $b$ is also an $S$-vertex of $G$. Further concepts of graph theory can be referred from [4].

We now give a brief account of the results proved in this article. This article consists of three sections. Section 1 is on introduction. In Section 2, some basic properties of $H(R)$ are proved with several examples (see Examples 2.3, 2.5, 2.9, and 2.12). It is shown in Corollary 2.4 that $H(R)$ is connected and $\text{diam}(H(R)) \leq 2$. It is proved in Theorem 2.7 that if $|A(R)^*| \geq 3$, then $\text{gr}(H(R)) = 3$. Let $R$ be a reduced ring which is not an integral domain. Let $n \in \mathbb{N}$ be such that $|\text{Min}(R)| = n$. It is deduced in Proposition 2.10 that $r(H(R)) = 1$ and $H(R)$ has at least $n$ centers. Let $R$ be a ring which is not reduced and let $\mathbb{N}(R)$ be the set of all nilpotent ideals of $R$. It is proved in Lemma 2.11 that $r(H(R)) = 1$. Moreover, it is proved that $H(R)$ has at least $|\mathbb{N}(R)^*|$ centers.

Let $R$ be a ring with $|A(R)^*| \geq 1$. In Section 3, we focus on answering the question of when $H(R)$ is a complete graph. Several sufficient conditions are provided on a ring $R$ in order that $H(R)$ to be a complete graph (see Lemma 3.4, Lemma 3.5, and Corollary 3.6). Let $R$ be a finite ring which is not an
integral domain. It is proved in Theorem 3.7 that \( \mathbb{A}G(R) = H(R) \) if and only if \( \mathbb{A}G(R) \) is a complete graph. It is shown in Theorem 3.10 that for a finite ring \( R \), \( \Omega(R) = H(R) \) if and only if \( \Omega(R) \) is complete. Moreover, finite rings \( R \) are characterized in Theorem 3.13 in order that \( H(R) \) to be a complete graph. Furthermore, several examples are given to illustrate the results proved in this section (see Examples 3.2, 3.3, 3.9, and 3.12).

2. Basic properties of \( H(R) \)

Let \( R \) be a ring such that \( |\mathbb{A}(R)^*| \geq 1 \). The aim of this section is to state and prove some basic properties of \( H(R) \). We prove that \( H(R) \) is connected and \( diam(H(R)) \leq 2 \). If \( |\mathbb{A}(R)^*| \geq 3 \), then we prove that \( gr(H(R)) = 3 \). We start this section with the following remark.

Remark 2.1. 1) Let \( R \) be a ring. Then \( H(R) \) is the union of \( \mathbb{A}G(R) \) and \( \Omega(R) \). Moreover, \( \mathbb{A}G(R) \) and \( \Omega(R) \) are spanning subgraphs of \( H(R) \).

2) We know that \( \mathbb{A}G(R) \) is connected with \( diam(\mathbb{A}G(R)) \leq 3 \) [5, Theorem 2.1]. As \( \mathbb{A}G(R) \) is a spanning subgraph of \( H(R) \), it follows that \( H(R) \) is connected with \( diam(H(R)) \leq 3 \). However, with the help of Lemma 2.2, we prove in Corollary 2.4 that \( H(R) \) is connected with \( diam(H(R)) \leq 2 \).

Lemma 2.2. Let \( R \) be a ring such that \( \mathbb{A}(R)^* \neq \emptyset \). Let \( I, J \in \mathbb{A}(R)^* \) with \( I \neq J \). Then in \( H(R) \), there exists a path of length at most two between \( I \) and \( J \).

Proof. We can assume that \( I \) and \( J \) are not adjacent in \( H(R) \). Then \( IJ \neq (0) \) and \( I + J \notin \mathbb{A}(R) \). Note that \( (0) \neq IJ \subseteq I \cap J \). This gives that \( IJ \subseteq I \) and \( IJ \subseteq J \). Therefore, \( I + IJ = I \in \mathbb{A}(R) \) and \( J + IJ = J \in \mathbb{A}(R) \). This proves that \( I - IJ - J \) is a path of length two between \( I \) and \( J \) in \( H(R) \).

Example 2.3. Let \( R = \mathbb{Z}_{18} \). Then \( \mathbb{A}(\mathbb{Z}_{18})^* = \{(2), (3), (6), (9)\} \). Observe that \( (2)(3) \neq (0) \) and \( (2) + (3) \notin \mathbb{A}(R) \). It is obvious that \( (2) + (6) = (2) \in \mathbb{A}(R) \) and \( (3) + (6) = (3) \in \mathbb{A}(R) \). This concludes that \( (2) \) and \( (3) \) are not adjacent in \( H(R) \) but \( (2) - (2)(3) - (3) \) is a path of length two between \( (2) \) and \( (3) \) in \( H(R) \).

Corollary 2.4. Let \( R \) be a ring which is not an integral domain. Then \( H(R) \) is connected and \( diam(H(R)) \leq 2 \).

Proof. Let \( I, J \in \mathbb{A}(R)^* \) be such that \( I \neq J \). We know from Lemma 2.2 that in \( H(R) \), there exists a path of length at most two between \( I \) and \( J \). Therefore, we obtain that \( H(R) \) is connected and \( diam(H(R)) \leq 2 \).

We now present an example to illustrate Corollary 2.4.

Example 2.5. 1) Let \( R \in \{ \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2) \} \). Then \( (2) \) (respectively, \( (x + (x^2)) \)) is the only non-zero annihilating ideal of \( R \). Hence, \( H(R) \) is a graph with single vertex and so, \( H(R) \) is connected and \( diam(H(R)) = 0 \).

2) Let \( R = \mathbb{Z}_6 \). Then \( (2) \) and \( (3) \) are the only non-zero annihilating ideals of \( \mathbb{Z}_6 \) and \( (2)(3) = (0) \). Hence, \( H(\mathbb{Z}_6) \) is connected and \( diam(H(\mathbb{Z}_6)) = 1 \).

3) Let \( R = \mathbb{Z}_{12} \). Then \( \mathbb{A}(\mathbb{Z}_{12})^* = \{(2), (3), (4), (6)\} \). It is easy to verify that \( (2)(6) = (3)(4) = (4)(6) = (0), (2) + (4) = (2) \in \mathbb{A}(\mathbb{Z}_{12}) \), and \( (3) + (6) = (3) \in \mathbb{A}(\mathbb{Z}_{12}) \). Note that \( (2)(3) \neq (0) \) and \( (2) + (3) \notin \mathbb{A}(\mathbb{Z}_{12}) \). Hence, \( (2) \)
and (3) are not adjacent in \(H(\mathbb{Z}_{12})\). Clearly, \((2) + (2)(3) = (2) \in \mathbb{A}(\mathbb{Z}_{12})\) and \((3) + (2)(3) = (3) \in \mathbb{A}(\mathbb{Z}_{12})\). Therefore, \((2) - (2)(3) - (3)\) is a path of length two between \((2)\) and \((3)\). Thus, \(H(\mathbb{Z}_{12})\) is a connected graph and \(\text{diam}(H(\mathbb{Z}_{12})) = 2\).

**Lemma 2.6.** Let \(R\) be a reduced ring which is not an integral domain. If \(\text{Min}(R) \notin \mathbb{A}(R)\), then there exist \(I, J \in \mathbb{A}(R)^*\) with \(I \neq J\) such that \(I\) and \(J\) are adjacent in \(H(R)\). Moreover, \(I, J, \text{ and } I + J\) forms a triangle in \(H(R)\).

**Proof.** Since \(R\) is not an integral domain, there exist \(x, y \in R\setminus \{0\}\) such that \(xy = 0\). We are assuming that \(\text{Min}(R) \notin \mathbb{A}(R)\). Let \(p \in \text{Min}(R)\) be such that \(p \notin \mathbb{A}(R)\). Then \(px \neq (0)\) and \(py \neq (0)\). Hence, \(p \notin ((0): x) \cup ((0): y)\). Therefore, there exists \(p \in \mathfrak{p}\) such that \(px \neq 0\) and \(py \neq 0\). Let us denote \(Rpx\) by \(I\) and \(Rpy\) by \(J\). From \(xy = 0\), it follows that \(IJ = (0)\) and so, \(I, J \in \mathbb{A}(R)^*\).

As \(R\) is reduced, \(I^2 \neq (0)\) and \(J^2 \neq (0)\). Therefore, from \(IJ = (0)\), we get that \(I \neq J\). We know from \[12, \text{Theorem 84}\] that \(\mathfrak{p} \subseteq Z(R)\). Hence, \(p \in Z(R)\) and so \(Rp \in \mathbb{A}(R)\). Observe that \(I + J \subseteq Rp\). Hence, \(I + J \in \mathbb{A}(R)\). From \(IJ = (0)\) and \(I + J \in \mathbb{A}(R)\), we obtain that \(I\) and \(J\) are adjacent in \(H(R)\). From \(IJ = (0)\), \(I^2 \neq (0)\), and \(J^2 \neq (0)\), we get that \(I + J \notin \{I, J\}\). It is easy to note that \(I + (I + J) = J + (I + J) = I + J \in \mathbb{A}(R)\). This implies that \(I, J, \text{ and } I + J\) forms a triangle in \(H(R)\).

**Theorem 2.7.** Let \(R\) be a ring. If \(|\mathbb{A}(R)^*| \geq 3\), then \(\text{gr}(H(R)) = 3\).

**Proof.** If \(H(R)\) is complete, then \(\text{gr}(H(R)) = 3\). Suppose that there exist two distinct vertices \(I_1\) and \(I_2\) which are not adjacent in \(H(R)\). This gives that \(I_1I_2 \neq (0)\) and \(I_1 + I_2 \notin \mathbb{A}(R)\). As \(I_1, I_2 \in \mathbb{A}(R)^*\), there exist \(J_1, J_2 \in \mathbb{A}(R)^*\) such that \(I_1J_1 = I_2J_2 = (0)\). We claim that \(J_i \notin \{I_1, I_2\}\) for each \(i \in \{1, 2\}\).

First, we verify that \(J_i \notin \{I_1, I_2\}\). If \(J_1 = I_1\), then from \(I_1J_1 = (0)\), it follows that \(I_1^2 = (0)\). In such a case, we obtain from \[6, \text{Lemma 1.5}\] that \(I_1 + I_2 \in \mathbb{A}(R)\) and this is in contradiction to the assumption that \(I_1 + I_2 \notin \mathbb{A}(R)\). If \(J_1 = I_2\), then from \(I_1J_1 = (0)\), we get that \(I_1I_2 = (0)\) and this contradicts the assumption that \(I_1I_2 \neq (0)\). Therefore, \(J_1 \notin \{I_1, I_2\}\). Similarly, it can be shown that \(J_2 \notin \{I_1, I_2\}\).

Clearly \(J_1 \neq J_2\), since if \(J_1 = J_2 = J\) (say), then \(I_1J_1 = I_2J_2 = (0)\). This implies that \((I_1 + I_2)J = (0)\), a contradiction with the assumption that \(I_1 + I_2 \notin \mathbb{A}(R)\).

In this case, \(J_1J_2 = (0)\), since if \(J_1J_2 \neq (0)\), then \(J_1J_2(I_1 + I_2) = (0)\), again a contradiction. Note that \(J_1(J_2 + I_1) = (0)\). This implies that \(J_2 + I_1 \in \mathbb{A}(R)\) and hence \(d(I_1, J_2) = 1\) in \(H(R)\). Thus, we obtain a cycle \(I_1 - J_1 - J_2 - I_1\) of length \(3\) in \(H(R)\). This proves that \(\text{gr}(H(R)) = 3\).

**Corollary 2.8.** Let \(R\) be a ring which is not an integral domain. Let \(I_1, I_2 \in \mathbb{A}(R)^*\) be such that \(I_1\) and \(I_2\) are not adjacent in \(H(R)\). Then \(H(R)\) contains \(S\)-vertices.

**Proof.** By hypothesis, \(I_1, I_2 \in \mathbb{A}(R)^*\) are such that \(I_1\) and \(I_2\) are not adjacent in \(H(R)\). We know from the proof of Theorem 2.7 that there exist distinct \(J_1, J_2 \in \mathbb{A}(R)^* \setminus \{I_1, I_2\}\) such that \(I_iJ_i = (0)\) for each \(i \in \{1, 2\}\). Observe that \((J_1 + J_2)I_1I_2 = (0)\). From \(I_1I_2 \neq (0)\), it follows that \(J_1 + J_2 \in \mathbb{A}(R)\) and so, \(J_1\) and \(J_2\) are adjacent in \(H(R)\). From the above discussion, it is clear that \(I_1 - J_1\),
Let $J_1 - J_2$, $J_2 - I_2$ be edges of $H(R)$ but $I_1$ and $I_2$ are not adjacent in $H(R)$. This proves that $J_1$ and $J_2$ are $S$-vertices in $H(R)$. □

**Example 2.9.** Let $R = \mathbb{Z}/12$. Then, $\mathcal{A}(R)^* = \{(2), (3), (4), (6)\}$. Note that (2) and (3) are the only vertices in $H(R)$ for which $d((2), (3)) = 2$. It is clear that $(2)(6) = (0), (6)(4) = (0), (4) + (2) = (2)$. Hence, $(2) - (6) - (4) - (2)$ is a cycle of length 3 in $H(R)$. Therefore, $gr(H(R)) = 3$. Observe that (2) and (3) are not adjacent in $H(R)$ but $(2) - (6) - (4) - (3)$ is a path from (2) to (3) in $H(R)$. Hence, (4) and (6) are $S$-vertices in $H(R)$.

**Proposition 2.10.** Let $R$ be a reduced ring which is not an integral domain. Let $n \in \mathbb{N}$ be such that $|\text{Min}(R)| = n$. Then $r(H(R)) = 1$ and $H(R)$ has at least $n$ centers.

**Proof.** By hypothesis, $R$ is reduced and $|\text{Min}(R)| = n$ for some $n \in \mathbb{N}$. Let $\text{Min}(R) = \{p_i \mid i \in \{1, 2, \ldots, n\}\}$. As $\text{nil}(R) = (0)$, it follows from [3, Proposition 1.8] and [12, Theorem 10] that $\bigcap_{i=1}^n p_i = (0)$. It is easy to verify that $Z(R) = \bigcup_{i=1}^n p_i$. As $R$ is not an integral domain, we get that $n \geq 2$. Let $i \in \{1, 2, \ldots, n\}$. We claim that $\bigcap_{j \in \{1, 2, \ldots, n\}\setminus\{i\}} p_j = (0)$. Suppose that $\bigcap_{j \in \{1, 2, \ldots, n\}\setminus\{i\}} p_j = (0)$. This implies that $p_i \ni (0) = \bigcap_{j \in \{1, 2, \ldots, n\}\setminus\{i\}} p_j$. Hence, we obtain from [3, Proposition 1.11(ii)] that $p_i \ni p_j$ for some $j \in \{1, 2, \ldots, n\}$ with $j \neq i$. This is impossible, since distinct minimal prime ideals of a ring are not comparable under the inclusion relation. Therefore, $\bigcap_{j \in \{1, 2, \ldots, n\}\setminus\{i\}} p_j \neq (0)$. Let $a_i \in \bigcap_{j \in \{1, 2, \ldots, n\}\setminus\{i\}} p_j \setminus \{0\}$. Then $p_i a_i = (0)$. Hence, $p_i \ni (0) :_R a_i$. Since $a_i \notin p_i$, it follows that $((0) :_R a_i) \subseteq p_i$ and so, $p_i = ((0) :_R a_i)$. Let $i \in \{1, 2, \ldots, n\}$. Let us denote $Ra_i$ by $I_i$. It is clear that $I_i \in \mathcal{A}(R)^*$. Let $J \in \mathcal{A}(R)^*$ be such that $J \neq I_i$. Note that $J \subseteq Z(R) = \bigcup_{k=1}^n p_k$. Hence, we obtain from [3, Proposition 1.11(i)] that $J \subseteq p_k$ for some $k \in \{1, 2, \ldots, n\}$. If $k = i$, then $JI_i = J(Ra_i) = (0)$. Suppose that $k \neq i$. Observe that $Ja_k = (0)$. As $I_i \subseteq p_k$, we get that $I_i a_k = (0)$. Therefore, $(I_i + J)a_k = (0)$ and so, $I_i + J \in \mathcal{A}(R)$. This shows that $I_i$ and $J$ are adjacent in $H(R)$ for each $J \in \mathcal{A}(R)^*$ with $J \neq I_i$. Hence, $E(I_i) = 1$ in $H(R)$ and so, $r(H(R)) = 1$. It is shown that $I_i = Ra_i$ is a center of $H(R)$ for each $i \in \{1, 2, \ldots, n\}$. Let $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Observe that $p_i = ((0) :_R a_i)$ and $p_j = ((0) :_R a_j)$. As $p_i \neq p_j$, we get that $Ra_i \neq Ra_j$. This proves that $H(R)$ has at least $n$ centers. □

**Lemma 2.11.** Let $R$ be a ring which is not reduced and let $\mathbb{N}(R)$ be the set of all nilpotent ideals of $R$. Then $r(H(R)) = 1$. Moreover, $H(R)$ has at least $|\mathbb{N}(R)^*| - 1$ centers.

**Proof.** By hypothesis, $R$ is not reduced. Hence, it is possible to find $x \in R \setminus \{0\}$ such that $x^2 = 0$. It is clear that $Rx \in \mathbb{N}(R)^*$. Hence, $\mathbb{N}(R)^* \neq \emptyset$. Let $I \in \mathbb{N}(R)^*$. Let $J \in \mathcal{A}(R)^* \setminus \{I\}$. We know from [6, Lemma 1.5] that $I + J \in \mathcal{A}(R)$ and so, $I$ and $J$ are adjacent in $H(R)$. This shows that $E(I) = 1$ in $H(R)$ and so, $r(H(R)) = 1$. It is shown that $I$ is a center of $H(R)$ for each $I \in \mathbb{N}(R)^*$. Therefore, $H(R)$ has at least $|\mathbb{N}(R)^*| - 1$ centers. □
Example 2.12. Let $R = \mathbb{Z}_{12}$. We know that $A(\mathbb{Z}_{12})^* = \{(2), (3), (4), (6)\}$. Note that (6) is the only non-zero nilpotent ideal of $\mathbb{Z}_{12}$. It is clear from Lemma 2.11 that (6) is adjacent with (2), (3), and (4) in $H(R)$. Hence, $r(H(R)) = 1$. Observe that $E((6)) = E((4)) = 1$. Therefore, $H(R)$ has exactly 2 centers.

3. When is $H(R)$ a complete graph?

Let $R$ be a ring which is not an integral domain. The purpose of this section is to provide some sufficient conditions under which $H(R)$ is a complete graph. We prove several results with the assumption that $R$ is finite. We start this section with the following remark.

Remark 3.1. Let $R$ be a ring with $|A(R)^*| \geq 1$. As $A_G(R)$ and $\Omega(R)$ are spanning subgraphs of $H(R)$, it follows that $H(R)$ is complete if either $A_G(R)$ or $\Omega(R)$ is complete. In Example 3.2, we mention a ring $R$ such that $H(R)$ is complete but $A_G(R)$ is not complete. In Example 3.3, we provide a ring $R$ such that $H(R)$ is complete but $\Omega(R)$ is not complete.

Example 3.2. Let $R = \mathbb{Z}_{16}$. It is clear that each proper ideal of $R$ is an annihilating ideal of $R$ and $A(R)^* = \{(2), (4), (8)\}$. As $\Omega(R)$ is complete, we get that $H(R)$ is complete. From $(2)(4) \neq (0)$, it follows that $A_G(R)$ is not complete.

Example 3.3. Let $R = F_1 \times F_2$, where $F_i$ is a field for each $i \in \{1, 2\}$. Note that $A(R)^* = \{I = F_1 \times (0), J = (0) \times F_2\}$. From $IJ = (0) \times (0)$, it follows that $A_G(R)$ is complete and so, $H(R)$ is complete. As $I + J = R \notin A(R)$, we get that $\Omega(R)$ is not complete.

Lemma 3.4. Let $R$ be a chained ring which is not an integral domain. Then $\Omega(R)$ is complete and so, $H(R)$ is complete.

Proof. Since $R$ is a chained ring which is not an integral domain, we obtain from [13, Lemma 2.6] that $\Omega(R)$ is complete and so, $H(R)$ is complete.

Lemma 3.5. Let $R$ be a ring which has exactly one maximal $N$-prime $p$ of $(0)$ such that $p$ is a $B$-prime of $(0)$. Then $\Omega(R)$ is complete and so, $H(R)$ is complete.

Proof. We know from [13, Lemma 2.3, Case 1] that $\Omega(R)$ is complete. As $\Omega(R)$ is a spanning subgraph of $H(R)$, it follows that $H(R)$ is complete.

Corollary 3.6. Let $R$ be a local Artinian ring which is not a field. Then $\Omega(R)$ is complete and so, $H(R)$ is complete. In particular, if $R$ is a finite local ring which is not a field, then $\Omega(R)$ is complete and so, $H(R)$ is complete.

Proof. Let $m$ denote the unique maximal ideal of the local Artinian ring $R$. Since $R$ is not a field, it follows that $m \neq (0)$. We know from [3, Corollary 8.2 and Proposition 8.4] that $m$ is nilpotent. Therefore, we obtain from [13, Example 2.4] that $\Omega(R)$ is complete. Therefore, $H(R)$ is complete.

Let $R$ be a finite local ring which is not a field. Since any finite ring is Artinian, it follows from the previous paragraph that $\Omega(R)$ is complete and so, $H(R)$ is complete.
Theorem 3.7. Let $R$ be a finite ring which is not an integral domain. Then $\mathcal{A}(R) = H(R)$ if and only if $\mathcal{A}(R)$ is a complete graph.

Proof. Let $R$ be a ring (finite or not). Suppose that $\mathcal{A}(R)$ is complete. Since $\mathcal{A}(R)$ is a spanning subgraph of $H(R)$, we see that if $\mathcal{A}(R)$ is complete, then so is $H(R)$. Hence, $\mathcal{A}(R) = H(R)$.

Conversely, suppose that $\mathcal{A}(R) = H(R)$, where $R$ is a finite ring. Since $R$ is a finite ring, $R \cong R_1 \times \cdots \times R_n$, where $R_1, \ldots, R_n$ are finite local rings. It is convenient to denote $R_1 \times \cdots \times R_n$ by $T$. Since $R \cong T$ as rings and $\mathcal{A}(R) = H(R)$ by assumption, we get that $\mathcal{A}(T) = H(T)$. We claim that $n \leq 2$. Suppose that $n > 2$. Let $i \in \{1, 2, 3, \ldots, n\}$. Let $e_i$ denote the element of $T$ whose $i$-th coordinate equals 1 and $j$-th coordinate equals 0 for all $j \in \{1, 2, 3, \ldots, n\}\backslash\{i\}$. It is clear that $I = Te_2, J = Te_2 + Te_3 \in \mathcal{A}(T)\star$ are such that $I + J = J \in \mathcal{A}(T)$. Hence, $I$ and $J$ are adjacent in $H(T)$. Note that $IJ = I \neq (0) \times (0) \times \cdots \times (0)$. Therefore, $I$ and $J$ are not adjacent in $\mathcal{A}(T)$. This is in contradiction to the fact that $\mathcal{A}(T) = H(T)$. Hence, $n \leq 2$. If $n = 1$, then $R$ is a finite local ring and so, we obtain from Corollary 3.6 that $\Omega(R)$ is complete. Therefore, $H(R)$ is complete. From $\mathcal{A}(R) = H(R)$, it follows that $\mathcal{A}(R)$ is complete. Suppose that $n = 2$. Now, $T = R_1 \times R_2$. We claim that $R_i$ is a field for each $i \in \{1, 2\}$. Let $m_1$ denote the unique maximal ideal of $R_1$. If $R_1$ is not a field, then $m_1 \neq (0)$ and as $R_1$ is finite, $m_1 \in \mathcal{A}(R_1)\star$. Let $A = m_1 \times R_2$ and let $B = (0) \times R_2$. It is clear that $A, B \in \mathcal{A}(T)\star$ and $A + B = A \in \mathcal{A}(T)$. Hence, $A$ and $B$ are adjacent in $H(T)$. Observe that $AB = B \neq (0) \times (0)$. Therefore, $A$ and $B$ are not adjacent in $\mathcal{A}(T)$. This contradicts the fact that $\mathcal{A}(T) = H(T)$. Therefore, $R_1$ is a field. Similarly, it can be shown that $R_2$ is a field. Hence, $\mathcal{A}(T)$ is complete and from $R \cong T$ as rings, we obtain that $\mathcal{A}(R)$ is complete. □

Corollary 3.8. Let $R$ be a finite ring which is not an integral domain. Then $\mathcal{A}(R) = H(R)$ if and only if $R$ is one of the following three types of rings:
1) $R \cong F_1 \times F_2$, where $F_1, F_2$ are fields.
2) $Z(R)$ is an ideal of $R$ with $Z(R)^2 = (0)$.
3) $R$ is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$.

Proof. The proof of this corollary follows from Theorem 3.7 and [5, Theorem 2.7]. □

Example 3.9. Let $R = \mathbb{Z}_p^3$, where $p$ is a prime number. Note that $\mathcal{A}(R)\star = \{(p), (p^2)\}$ and $(p)(p^2) = (0)$. Therefore, $\mathcal{A}(R)$ is complete. Hence, $\mathcal{A}(R) = H(R)$.

Theorem 3.10. Let $R$ be a finite ring which is not an integral domain. Then $\Omega(R) = H(R)$ if and only if $\Omega(R)$ is complete.

Proof. Let $R$ be a ring (finite or not). Suppose that $\Omega(R)$ is complete. As $\Omega(R)$ is a spanning subgraph of $H(R)$, it follows that $H(R)$ is complete. Hence, $\Omega(R) = H(R)$.

Conversely, suppose that $\Omega(R) = H(R)$, where $R$ is a finite ring. Since $R$ is a finite ring, $R \cong R_1 \times \cdots \times R_n$, where $R_1, \ldots, R_n$ are finite local rings. We claim that $n = 1$. Suppose that $n \geq 2$. Let us denote the ring $R_1 \times R_2 \times \cdots \times R_n$ by...
$T$. Since $R \cong T$ as rings, we get that $\Omega(T) = H(T)$. Let $I = Te_2$ and $J = Tf_2$, where $e_2 = (0,1,0,\ldots,0)$ and $f_2 = (1,1,\ldots,1) - e_2$. It is clear that $I, J \in A(T)^*$ with $IJ = (0) \times (0) \times \cdots \times (0)$ and so, $I$ and $J$ are adjacent in $H(T)$. From $I + J = T \notin A(T)$, we get that $I$ and $J$ are not adjacent in $\Omega(T)$.

This contradicts the fact that $\Omega(T) = H(T)$. Therefore, $n = 1$ and so, $R$ is a finite local ring and in such a case, we obtain from Corollary 3.6 that $\Omega(R)$ is complete. \hfill $\Box$

**Corollary 3.11.** Let $R$ be a finite ring which is not an integral domain. Then $\Omega(R) = H(R)$ if and only if $R$ is a local ring.

**Proof.** The proof of this corollary follows from the proof of Theorem 3.10 and Corollary 3.6. \hfill $\Box$

**Example 3.12.** Let $R = \mathbb{Z}_{16}$. Observe that $R$ is a finite local ring with maximal ideal $(2)$ and $A(R)^* = \{(2), (4), (8)\}$. It is clear that $(2) + (4) = (2) \in A(R)$, $(2) + (8) = (2) \in A(R)$, $(4) + (8) = (4) \in A(R)$. This implies that $\Omega(R)$ is complete. Hence, $\Omega(R) = H(R)$.

**Theorem 3.13.** Let $R$ be a finite ring. Then $H(R)$ is complete if and only if one of the following holds:
1) $R$ is a local ring.
2) $R \cong F_1 \times F_2$, where $F_1, F_2$ are fields.

**Proof.** Suppose that $H(R)$ is complete. If $R$ is a local ring, then there is nothing to prove. So, suppose that $R$ is not a local ring. Then our claim is to prove that $R \cong F_1 \times F_2$, where $F_1, F_2$ are fields.

Since $R$ is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$ for some $n \in \mathbb{N}, n \geq 2$. Observe that $R_1, R_2, \ldots, R_n$ are finite local rings. Let us denote the ring $R_1 \times R_2 \times \cdots \times R_n$ by $T$. Since $R \cong T$ as rings, and $H(R)$ is complete by assumption, we obtain that $H(T)$ is complete. First, we verify that $n = 2$. Suppose that $n \geq 3$. Let $I = R_1 \times R_2 \times (0) \times \cdots \times (0)$ and let $J = (0) \times R_2 \times R_3 \times \cdots \times R_n$. It is clear that $I, J \in A(T)^*$, $IJ \neq (0) \times (0) \times (0) \times \cdots \times (0)$ and $I + J = T \notin A(T)$. This implies that $I$ and $J$ are not adjacent in $H(T)$. This contradicts the fact that $H(T)$ is complete. Therefore, $n = 2$. Thus, $T = R_1 \times R_2$. We claim that $R_i$ is a field for each $i \in \{1, 2\}$. Suppose that $R_1$ is not a field. Let $m_1$ denote the unique maximal ideal of $R_1$. As $R_1$ is not a field by assumption, it follows that $m_1 \neq (0)$. Note that $m_1$ is nilpotent. Let $A = R_1 \times (0)$ and let $B = m_1 \times R_2$. It is clear that $A, B \in A(T)^*$ with $A \neq B$. Observe that $AB \neq (0) \times (0)$ and $A + B = T \notin A(T)$. Therefore, $A$ and $B$ are not adjacent in $H(T)$. This contradicts the fact that $H(T)$ is complete. Therefore, $R_1$ is a field. Similarly, it can be shown that $R_2$ is a field. Let $i \in \{1, 2\}$. Thus, with $F_i = R_i$, it follows that $F_i$ is a field and $R \cong F_1 \times F_2$ as rings. This shows that if $H(R)$ is complete, then either $R$ is a local ring or $R \cong F_1 \times F_2$ as rings, where $F_i$ is a field for each $i \in \{1, 2\}$.

Conversely, assume that 1) or 2) holds. If 1) holds, then $\Omega(R)$ is complete by Corollary 3.6 and so, $H(R)$ is complete. If 2) holds, then $A_G(R)$ is complete by [5, Theorem 2.7] and so, $H(R)$ is complete. \hfill $\Box$
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