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ON THE EXTENDED GRAPH ASSOCIATED WITH THE SET OF ALL NON-ZERO ANNIHILATING IDEALS OF A COMMUTATIVE RING

HIREN D. PATEL*

Abstract. Let R be a commutative ring with non-zero identity which is not an integral domain. An ideal I of a ring R is called an annihilating ideal if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of R and $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$. In this article, we introduce a new graph associated with R denoted by $H(R)$ whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices I, J are adjacent in this graph if and only if $IJ = (0)$ or $I + J \in \mathbb{A}(R)$. The aim of this article is to study the interplay between the ring-theoretic properties of a ring R and the graph-theoretic properties of $H(R)$. For such a ring R, we prove that $H(R)$ is connected and find its diameter. Moreover, we determine girth of $H(R)$. Furthermore, we provide some sufficient conditions under which $H(R)$ is a complete graph.

1. Introduction

The study of algebraic structures with the help of graph theory is an exciting research topic. There are many research articles on assigning a graph to a ring. The rings considered in this article are commutative with non-zero identity which are not integral domains.

Let R be a commutative ring with non-zero identity which is not an integral domain. Let $Z(R)$ denote the set of all zero-divisors of R and $Z(R)^* = Z(R)\backslash\{0\}$. The idea of zero-divisor graph of a commutative ring was given by D. F. Anderson and P. S. Livingston in [2]. Recall from [2] that the *zero-divisor graph* $\Gamma(R)$ of a ring R is an undirected graph whose vertex set is $Z(R)^*$ and two distinct vertices x, y are adjacent in this graph if and only if $xy = 0$. The generalization of zero-divisor graph was introduced by M. Behboodi and Z. Rakeei in [5]. Recall from [5] that an ideal I of R is said to be an *annihilating ideal* if $Ir = (0)$ for some $r \in R\backslash\{0\}$. As in [5], we denote by $\mathbb{A}(R)$, the set of all annihilating ideals of R and by $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$. Recall from [5] that the *annihilating-ideal graph* of a ring R denoted by $\mathbb{AG}(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I, J \in \mathbb{A}(R)^*$ are adjacent in this graph if and only if $IJ = (0)$. There are several research articles in the literature which studied the

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* Corresponding author.
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interplay between the ring-theoretic properties of a commutative ring R and the graph-theoretic properties of $\mathbb{AG}(R)$ (see for example, [5, 6, 15]).

In [13], we introduced the concept of a graph associated with the set of all non-zero annihilating ideals of a commutative ring. Recall from [13] that a graph associated with R denoted by $\Omega(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I, J \in \mathbb{A}(R)^*$ are adjacent in this graph if and only if $I+J \in \mathbb{A}(R)$. The main aim of [13] was to study some properties of $\Omega(R)$. Some further properties of $\Omega(R)$ were proved in [1, 11]. Moreover, A. Cherrabi et al. introduced a new extension of the zero-divisor graph denoted by $\widetilde{\Gamma}(R)$ in [7]. Recall from [7] that $\widetilde{\Gamma}(R)$ is an undirected simple graph whose vertices are the non-zero zero-divisors of R and two distinct vertices x, y are adjacent in this graph if and only if $xy = 0$ or $x + y \in Z(R)$. The authors of [7, 8] explored the relationship of $\widetilde{\Gamma}(R)$ with the zero-divisor graph and with a subgraph of the total graph of a ring. Furthermore, the results proved on the graph considered in the present article is also motivated by the interesting results proved in [14]. Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Recall from [14] that $G(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices I and J are adjacent in G(R) if and only if either $IJ \neq (0)$ or $I + J \notin \mathbb{A}(R)$.

Motivated by the work done on $\Gamma(R), \mathbb{AG}(R), \Omega(R)$,$\widetilde{\Gamma}(R)$, and $G(R)$ mentioned above, in this article, we introduce extended graph associated with R denoted by $H(R)$ whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices I, J are adjacent in this graph if and only if $IJ = (0)$ or $I + J \in \mathbb{A}(R)$. The main aim of this article is to study some properties of $H(R)$. In this article, we focus on sum and product operations on ideals to describe the properties of $H(R)$. Before we give a brief account of results proved in this article, it is useful to recall the following definitions from commutative ring theory.

Let I be an ideal of a ring R with $I \neq R$. Recall from [10] that a prime ideal $\mathfrak{p}$ of R is said to be a *maximal N-prime of* I if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(\frac{R}{I}) = \{r \in R \mid rx \in I \text{ for some } x \in R\backslash I\}$. Hence, $\mathfrak{p}$ is a maximal N-prime of (0) if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. It is clear that $S = R\backslash Z(R)$ is a multiplicatively closed subset of R. Let $x \in Z(R)$. Then $Rx \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [12, Theorem 1] that there exists a maximal N-prime $\mathfrak{p}$ of (0) in R such that $x \in \mathfrak{p}$. Therefore, if $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ is the set of all maximal N-primes of (0) in R, then it follows that $Z(R) = \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$.

Let I be an ideal of a ring R with $I \neq R$. Recall from [9] that a prime ideal $\mathfrak{p}$ of R is said to be an *associated prime of* I *in the sense of Bourbaki*, if $\mathfrak{p} = (I :_R x)$ for some $x \in R$. In this case, we say that $\mathfrak{p}$ is a B-prime of I. Let $\mathfrak{p}$ be a maximal N-prime of (0) in R. It is easy to verify that $\mathfrak{p}$ is a B-prime of (0) in R if and only if $\mathfrak{p} \in \mathbb{A}(R)$.

A ring R is said to be *quasilocal* if R has a unique maximal ideal. By a *local ring*, we mean a Noetherian quasilocal ring. A local ring R with unique maximal ideal $\mathfrak{m}$ is denoted by $(R, \mathfrak{m})$. An element x of a ring R is said to be *nilpotent* if there exists a least positive integer n such that $x^n = 0$. The *nilradical* of a ring R is the set of all nilpotent elements in the ring R. We use $nil(R)$ to denote

the nilradical of a ring R. Clearly, $nil(R)$ is an ideal of R. A ring R is said to be *reduced* if it has no non-zero nilpotent elements. Note that R is reduced if $nil(R) = (0)$. An ideal I of a ring R is said to be a *nilpotent ideal* if there exists $k \in \mathbb{N}$ such that $I^k = (0)$. Recall that a ring R is said to be a *chained ring* if the ideals of R are linearly ordered under the inclusion relation. We denote the set of all minimal prime ideals of R by $Min(R)$. Let $n \in \mathbb{N}$ with $n \geq 2$. We denote the ring of integer modulo n by $\mathbb{Z}_n$. Any unexplained terminology from commutative ring theory can be referred from the standard textbooks like [3, 12].

Next we recall the following definitions from graph theory which are useful in this article. The graphs considered in this article are undirected. Let $G = (V, E)$ be a graph. A graph G is said to be *connected* if for each pair of distinct vertices $a, b \in V$, there exists at least one path in G between a and b. Let $G = (V, E)$ be a connected graph. Recall from [4] that for distinct vertices a, b of G, the *distance* between a and b denoted by $d(a, b)$ is defined as the length of a shortest path between a and b in G. We define $d(a, a) = 0$. We define the *diameter* of G denoted by $diam(G)$ as $diam(G) = sup\{d(a, b) \mid a, b \in V\}$.

Let $G = (V, E)$ be a connected graph. Let $v \in V$. Recall from [4] that the *eccentricity of* v denoted by $E(v)$ is defined as $E(v) = sup\{d(v, w) \mid w \in V\}$. The *radius of* G denoted by $r(G)$ is defined as $r(G) = min\{E(v) \mid v \in V\}$. A vertex $v \in V$ with minimum eccentricity is called a *center of* G. Recall that the *girth* of G denoted by $girth(G)$ or $gr(G)$ is defined as the length of a shortest cycle in G. If G does not contain any cycle, then we set $gr(G) = \infty$. A simple graph G is said to be *complete* if every pair of distinct vertices of G are adjacent in G. Let $n \in \mathbb{N}$. A complete graph on n vertices is denoted by K_n.

Given a graph G, a *subgraph* H of G is a graph whose vertex set and edge set are subsets of those of G. A subgraph H of G is called an *induced subgraph* if all edges of G joining two vertices in H are also edges of H. A *spanning subgraph* H of G is a subgraph containing all the vertices of G. Recall that a vertex a of G is said to be a *Smarandache vertex* or simply an *S-vertex* if there exist three distinct vertices x, y, and b $(\neq a)$ in G such that $x - a, a - b$, and $b - y$ are edges in G but there is no edge between x and y in G. Note that b is also an S-vertex of G. Further concepts of graph theory can be referred from [4].

We now give a brief account of the results proved in this article. This article consists of three sections. Section 1 is on introduction. In Section 2, some basic properties of $H(R)$ are proved with several examples (see Examples 2.3, 2.5, 2.9, and 2.12). It is shown in Corollary 2.4 that $H(R)$ is connected and $diam(H(R)) \leq 2$. It is proved in Theorem 2.7 that if $|\mathbb{A}(R)^*| \geq 3$, then $gr(H(R)) = 3$. Let R be a reduced ring which is not an integral domain. Let $n \in \mathbb{N}$ be such that $|Min(R)| = n$. It is deduced in Proposition 2.10 that $r(H(R)) = 1$ and $H(R)$ has at least n centers. Let R be a ring which is not reduced and let $\mathbb{NI}(R)$ be the set of all nilpotent ideals of R. It is proved in Lemma 2.11 that $r(H(R)) = 1$. Moreover, it is proved that $H(R)$ has at least $|\mathbb{NI}(R)^*|$ centers.

Let R be a ring with $|\mathbb{A}(R)^*| \geq 1$. In Section 3, we focus on answering the question of when $H(R)$ is a complete graph. Several sufficient conditions are provided on a ring R in order that $H(R)$ to be a complete graph (see Lemma 3.4, Lemma 3.5, and Corollary 3.6). Let R be a finite ring which is not an

integral domain. It is proved in Theorem 3.7 that $\mathbb{AG}(R) = H(R)$ if and only if $\mathbb{AG}(R)$ is a complete graph. It is shown in Theorem 3.10 that for a finite ring R, $\Omega(R) = H(R)$ if and only if $\Omega(R)$ is complete. Moreover, finite rings R are characterized in Theorem 3.13 in order that $H(R)$ to be a complete graph. Furthermore, several examples are given to illustrate the results proved in this section (see Examples 3.2, 3.3, 3.9, and 3.12).

2. Basic properties of $H(R)$

Let R be a ring such that $|\mathbb{A}(R)^*| \geq 1$. The aim of this section is to state and prove some basic properties of $H(R)$. We prove that $H(R)$ is connected and $diam(H(R)) \leq 2$. If $|\mathbb{A}(R)^*| \geq 3$, then we prove that $gr(H(R)) = 3$. We start this section with the following remark.

Remark 2.1. 1) Let R be a ring. Then $H(R)$ is the union of $\mathbb{AG}(R)$ and $\Omega(R)$. Moreover, $\mathbb{AG}(R)$ and $\Omega(R)$ are spanning subgraphs of $H(R)$.
2) We know that $\mathbb{AG}(R)$ is connected with $diam(\mathbb{AG}(R)) \leq 3$ [5, Theorem 2.1]. As $\mathbb{AG}(R)$ is a spanning subgraph of $H(R)$, it follows that $H(R)$ is connected with $diam(H(R)) \leq 3$. However, with the help of Lemma 2.2, we prove in Corollary 2.4 that $H(R)$ is connected with $diam(H(R)) \leq 2$.

Lemma 2.2. *Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Let $I, J \in \mathbb{A}(R)^*$ with $I \neq J$. Then in $H(R)$, there exists a path of length at most two between I and J.*

Proof. We can assume that I and J are not adjacent in $H(R)$. Then $IJ \neq (0)$ and $I + J \notin \mathbb{A}(R)$. Note that $(0) \neq IJ \subseteq I \cap J$. This gives that $IJ \subseteq I$ and $IJ \subseteq J$. Therefore, $I + IJ = I \in \mathbb{A}(R)$ and $J + IJ = J \in \mathbb{A}(R)$. This proves that $I - IJ - J$ is a path of length two between I and J in $H(R)$. □

Example 2.3. Let $R = \mathbb{Z}_{18}$. Then $\mathbb{A}(\mathbb{Z}_{18})^* = \{(2), (3), (6), (9)\}$. Observe that $(2)(3) \neq (0)$ and $(2) + (3) \notin \mathbb{A}(R)$. It is obvious that $(2) + (6) = (2) \in \mathbb{A}(R)$ and $(3) + (6) = (3) \in \mathbb{A}(R)$. This concludes that (2) and (3) are not adjacent in $H(R)$ but $(2) - (2)(3) - (3)$ is a path of length two between (2) and (3) in $H(R)$.

Corollary 2.4. *Let R be a ring which is not an integral domain. Then $H(R)$ is connected and $diam(H(R)) \leq 2$.*

Proof. Let $I, J \in \mathbb{A}(R)^*$ be such that $I \neq J$. We know from Lemma 2.2 that in $H(R)$, there exists a path of length at most two between I and J. Therefore, we obtain that $H(R)$ is connected and $diam(H(R)) \leq 2$. □

We now present an example to illustrate Corollary 2.4.

Example 2.5. 1) Let $R \in \{\mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}\}$. Then (2) (respectively, $(x + (x^2))$) is the only non-zero annihilating ideal of R. Hence, $H(R)$ is a graph with single vertex and so, $H(R)$ is connected and $diam(H(R)) = 0$.
2) Let $R = \mathbb{Z}_6$. Then (2) and (3) are the only non-zero annihilating ideals of $\mathbb{Z}_6$ and $(2)(3) = (0)$. Hence, $H(\mathbb{Z}_6)$ is connected and $diam(H(\mathbb{Z}_6)) = 1$.
3) Let $R = \mathbb{Z}_{12}$. Then $\mathbb{A}(\mathbb{Z}_{12})^* = \{(2), (3), (4), (6)\}$. It is easy to verify that $(2)(6) = (3)(4) = (4)(6) = (0), (2) + (4) = (2) \in \mathbb{A}(\mathbb{Z}_{12})$, and $(3) + (6) = (3) \in \mathbb{A}(\mathbb{Z}_{12})$. Note that $(2)(3) \neq (0)$ and $(2) + (3) \notin \mathbb{A}(\mathbb{Z}_{12})$. Hence, (2)

and (3) are not adjacent in $H(\mathbb{Z}_{12})$. Clearly, $(2)+(2)(3)=(2)\in\mathbb{A}(\mathbb{Z}_{12})$ and $(3)+(2)(3)=(3)\in\mathbb{A}(\mathbb{Z}_{12})$. Therefore, $(2)-(2)(3)-(3)$ is a path of length two between (2) and (3). Thus, $H(\mathbb{Z}_{12})$ is a connected graph and $diam(H(\mathbb{Z}_{12}))=2$.

Lemma 2.6. *Let R be a reduced ring which is not an integral domain. If $Min(R)\nsubseteq\mathbb{A}(R)$, then there exist $I,J\in\mathbb{A}(R)^*$ with $I\neq J$ such that I and J are adjacent in $H(R)$. Moreover, I,J, and $I+J$ forms a triangle in $H(R)$.*

Proof. Since R is not an integral domain, there exist $x,y\in R\backslash\{0\}$ such that $xy=0$. We are assuming that $Min(R)\nsubseteq\mathbb{A}(R)$. Let $\mathfrak{p}\in Min(R)$ be such that $\mathfrak{p}\notin\mathbb{A}(R)$. Then $\mathfrak{p}x\neq(0)$ and $\mathfrak{p}y\neq(0)$. Hence, $\mathfrak{p}\nsubseteq((0):x)\cup((0):y)$. Therefore, there exists $p\in\mathfrak{p}$ such that $px\neq 0$ and $py\neq 0$. Let us denote Rpx by I and Rpy by J. From $xy=0$, it follows that $IJ=(0)$ and so, $I,J\in\mathbb{A}(R)^*$. As R is reduced, $I^2\neq(0)$ and $J^2\neq(0)$. Therefore, from $IJ=(0)$, we get that $I\neq J$. We know from [12, Theorem 84] that $\mathfrak{p}\subseteq Z(R)$. Hence, $p\in Z(R)$ and so $Rp\in\mathbb{A}(R)$. Observe that $I+J\subseteq Rp$. Hence, $I+J\in\mathbb{A}(R)$. From $IJ=(0)$ and $I+J\in\mathbb{A}(R)$, we obtain that I and J are adjacent in $H(R)$. From $IJ=(0)$, $I^2\neq(0)$, and $J^2\neq(0)$, we get that $I+J\notin\{I,J\}$. It is easy to note that $I+(I+J)=J+(I+J)=I+J\in\mathbb{A}(R)$. This implies that I,J, and $I+J$ forms a triangle in $H(R)$. □

Theorem 2.7. *Let R be a ring. If $|\mathbb{A}(R)^*|\geq 3$, then $gr(H(R))=3$.*

Proof. If $H(R)$ is complete, then $gr(H(R))=3$. Suppose that there exist two distinct vertices I_1 and I_2 which are not adjacent in $H(R)$. This gives that $I_1I_2\neq(0)$ and $I_1+I_2\notin\mathbb{A}(R)$. As $I_1,I_2\in\mathbb{A}(R)^*$, there exist $J_1,J_2\in\mathbb{A}(R)^*$ such that $I_1J_1=I_2J_2=(0)$. We claim that $J_i\notin\{I_1,I_2\}$ for each $i\in\{1,2\}$. First, we verify that $J_1\notin\{I_1,I_2\}$. If $J_1=I_1$, then from $I_1J_1=(0)$, it follows that $I_1^2=(0)$. In such a case, we obtain from [6, Lemma 1.5] that $I_1+I_2\in\mathbb{A}(R)$ and this is in contradiction to the assumption that $I_1+I_2\notin\mathbb{A}(R)$. If $J_1=I_2$, then from $I_1J_1=(0)$, we get that $I_1I_2=(0)$ and this contradicts the assumption that $I_1I_2\neq(0)$. Therefore, $J_1\notin\{I_1,I_2\}$. Similarly, it can be shown that $J_2\notin\{I_1,I_2\}$.

Clearly $J_1\neq J_2$, since if $J_1=J_2=J$ (say), then $I_1J=I_2J=(0)$. This implies that $(I_1+I_2)J=(0)$, a contradiction with the assumption that $I_1+I_2\notin\mathbb{A}(R)$. In this case, $J_1J_2=(0)$, since if $J_1J_2\neq(0)$, then $J_1J_2(I_1+I_2)=(0)$, again a contradiction. Note that $J_1(J_2+I_1)=(0)$. This implies that $J_2+I_1\in\mathbb{A}(R)$ and hence $d(I_1,J_2)=1$ in $H(R)$. Thus, we obtain a cycle $I_1-J_1-J_2-I_1$ of length 3 in $H(R)$. This proves that $gr(H(R))=3$. □

Corollary 2.8. *Let R be a ring which is not an integral domain. Let $I_1,I_2\in\mathbb{A}(R)^*$ be such that I_1 and I_2 are not adjacent in $H(R)$. Then $H(R)$ contains S-vertices.*

Proof. By hypothesis, $I_1,I_2\in\mathbb{A}(R)^*$ are such that I_1 and I_2 are not adjacent in $H(R)$. We know from the proof of Theorem 2.7 that there exist distinct $J_1,J_2\in\mathbb{A}(R)^*\backslash\{I_1,I_2\}$ such that $I_iJ_i=(0)$ for each $i\in\{1,2\}$. Observe that $(J_1+J_2)I_1I_2=(0)$. From $I_1I_2\neq(0)$, it follows that $J_1+J_2\in\mathbb{A}(R)$ and so, J_1 and J_2 are adjacent in $H(R)$. From the above discussion, it is clear that I_1-J_1,

$J_1 - J_2$, $J_2 - I_2$ are edges of $H(R)$ but I_1 and I_2 are not adjacent in $H(R)$. This proves that J_1 and J_2 are S-vertices in $H(R)$. □

Example 2.9. Let $R = \mathbb{Z}_{12}$. Then, $\mathbb{A}(R)^* = \{(2), (3), (4), (6)\}$. Note that (2) and (3) are the only vertices in $H(R)$ for which $d((2), (3)) = 2$. It is clear that $(2)(6) = (0), (6)(4) = (0), (4) + (2) = (2)$. Hence, $(2) - (6) - (4) - (2)$ is a cycle of length 3 in $H(R)$. Therefore, $gr(H(R)) = 3$. Observe that (2) and (3) are not adjacent in $H(R)$ but $(2) - (6) - (4) - (3)$ is a path from (2) to (3) in $H(R)$. Hence, (4) and (6) are S-vertices in $H(R)$.

Proposition 2.10. *Let R be a reduced ring which is not an integral domain. Let $n \in \mathbb{N}$ be such that $|Min(R)| = n$. Then $r(H(R)) = 1$ and $H(R)$ has at least n centers.*

Proof. By hypothesis, R is reduced and $|Min(R)| = n$ for some $n \in \mathbb{N}$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. As $nil(R) = (0)$, it follows from [3, Proposition 1.8] and [12, Theorem 10] that $\bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$. It is easy to verify that $Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. As R is not an integral domain, we get that $n \geq 2$. Let $i \in \{1, 2, \ldots, n\}$. We claim that $\bigcap_{j \in \{1,2,\ldots,n\}\setminus\{i\}} \mathfrak{p}_j \neq (0)$. Suppose that $\bigcap_{j \in \{1,2,\ldots,n\}\setminus\{i\}} \mathfrak{p}_j = (0)$. This implies that $\mathfrak{p}_i \supseteq (0) = \bigcap_{j \in \{1,2,\ldots,n\}\setminus\{i\}} \mathfrak{p}_j$. Hence, we obtain from [3, Proposition 1.11(ii)] that $\mathfrak{p}_i \supseteq \mathfrak{p}_j$ for some $j \in \{1, 2, \ldots, n\}$ with $j \neq i$. This is impossible, since distinct minimal prime ideals of a ring are not comparable under the inclusion relation. Therefore, $\bigcap_{j \in \{1,2,\ldots,n\}\setminus\{i\}} \mathfrak{p}_j \neq (0)$. Let $a_i \in (\bigcap_{j \in \{1,2,\ldots,n\}\setminus\{i\}} \mathfrak{p}_j)\setminus\{0\}$. Then $\mathfrak{p}_i a_i = (0)$. Hence, $\mathfrak{p}_i \subseteq ((0) :_R a_i)$. Since $a_i \notin \mathfrak{p}_i$, it follows that $((0) :_R a_i) \subseteq \mathfrak{p}_i$ and so, $\mathfrak{p}_i = ((0) :_R a_i)$. Let $i \in \{1, 2, \ldots, n\}$. Let us denote Ra_i by I_i. It is clear that $I_i \in \mathbb{A}(R)^*$. Let $J \in \mathbb{A}(R)^*$ be such that $J \neq I_i$. Note that $J \subseteq Z(R) = \bigcup_{k=1}^{n} \mathfrak{p}_k$. Hence, we obtain from [3, Proposition 1.11(i)] that $J \subseteq \mathfrak{p}_k$ for some $k \in \{1, 2, \ldots, n\}$. If $k = i$, then $JI_i = J(Ra_i) = (0)$. Suppose that $k \neq i$. Observe that $Ja_k = (0)$. As $I_i \subseteq \mathfrak{p}_k$, we get that $I_i a_k = (0)$. Therefore, $(I_i + J)a_k = (0)$ and so, $I_i + J \in \mathbb{A}(R)$. This shows that I_i and J are adjacent in $H(R)$ for each $J \in \mathbb{A}(R)^*$ with $J \neq I_i$. Hence, $E(I_i) = 1$ in $H(R)$ and so, $r(H(R)) = 1$. It is shown that $I_i = Ra_i$ is a center of $H(R)$ for each $i \in \{1, 2, \ldots, n\}$. Let $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Observe that $\mathfrak{p}_i = ((0) :_R a_i)$ and $\mathfrak{p}_j = ((0) :_R a_j)$. As $\mathfrak{p}_i \neq \mathfrak{p}_j$, we get that $Ra_i \neq Ra_j$. This proves that $H(R)$ has at least n centers. □

Lemma 2.11. *Let R be a ring which is not reduced and let $\mathbb{NI}(R)$ be the set of all nilpotent ideals of R. Then $r(H(R)) = 1$. Moreover, $H(R)$ has at least $|\mathbb{NI}(R)^*|$ centers.*

Proof. By hypothesis, R is not reduced. Hence, it is possible to find $x \in R\setminus\{0\}$ such that $x^2 = 0$. It is clear that $Rx \in \mathbb{NI}(R)^*$. Hence, $\mathbb{NI}(R)^* \neq \emptyset$. Let $I \in \mathbb{NI}(R)^*$. Let $J \in \mathbb{A}(R)^*\setminus\{I\}$. We know from [6, Lemma 1.5] that $I+J \in \mathbb{A}(R)$ and so, I and J are adjacent in $H(R)$. This shows that $E(I) = 1$ in $H(R)$ and so, $r(H(R)) = 1$. It is shown that I is a center of $H(R)$ for each $I \in \mathbb{NI}(R)^*$. Therefore, $H(R)$ has at least $|\mathbb{NI}(R)^*|$ centers. □

Example 2.12. Let $R = \mathbb{Z}_{12}$. We know that $\mathbb{A}(\mathbb{Z}_{12})^* = \{(2), (3), (4), (6)\}$. Note that (6) is the only non-zero nilpotent ideal of $\mathbb{Z}_{12}$. It is clear from Lemma 2.11 that (6) is adjacent with (2), (3), and (4) in $H(R)$. Hence, $r(H(R)) = 1$. Observe that $E((6)) = E((4)) = 1$. Therefore, $H(R)$ has exactly 2 centers.

3. When is $H(R)$ a complete graph?

Let R be a ring which is not an integral domain. The purpose of this section is to provide some sufficient conditions under which $H(R)$ is a complete graph. We prove several results with the assumption that R is finite. We start this section with the following remark.

Remark 3.1. Let R be a ring with $|\mathbb{A}(R)^*| \geq 1$. As $\mathbb{AG}(R)$ and $\Omega(R)$ are spanning subgraphs of $H(R)$, it follows that $H(R)$ is complete if either $\mathbb{AG}(R)$ or $\Omega(R)$ is complete. In Example 3.2, we mention a ring R such that $H(R)$ is complete but $\mathbb{AG}(R)$ is not complete. In Example 3.3, we provide a ring R such that $H(R)$ is complete but $\Omega(R)$ is not complete.

Example 3.2. Let $R = \mathbb{Z}_{16}$. It is clear that each proper ideal of R is an annihilating ideal of R and $\mathbb{A}(R)^* = \{(2), (4), (8)\}$. As $\Omega(R)$ is complete, we get that $H(R)$ is complete. From $(2)(4) \neq (0)$, it follows that $\mathbb{AG}(R)$ is not complete.

Example 3.3. Let $R = F_1 \times F_2$, where F_i is a field for each $i \in \{1, 2\}$. Note that $\mathbb{A}(R)^* = \{I = F_1 \times (0), J = (0) \times F_2\}$. From $IJ = (0) \times (0)$, it follows that $\mathbb{AG}(R)$ is complete and so, $H(R)$ is complete. As $I + J = R \notin \mathbb{A}(R)$, we get that $\Omega(R)$ is not complete.

Lemma 3.4. *Let R be a chained ring which is not an integral domain. Then $\Omega(R)$ is complete and so, $H(R)$ is complete.*

Proof. Since R is a chained ring which is not an integral domain, we obtain from [13, Lemma 2.6] that $\Omega(R)$ is complete and so, $H(R)$ is complete. □

Lemma 3.5. *Let R be a ring which has exactly one maximal N-prime $\mathfrak{p}$ of (0) such that $\mathfrak{p}$ is a B-prime of (0). Then $\Omega(R)$ is complete and so, $H(R)$ is complete.*

Proof. We know from [13, Lemma 2.3, Case 1] that $\Omega(R)$ is complete. As $\Omega(R)$ is a spanning subgraph of $H(R)$, it follows that $H(R)$ is complete. □

Corollary 3.6. *Let R be a local Artinian ring which is not a field. Then $\Omega(R)$ is complete and so, $H(R)$ is complete. In particular, if R is a finite local ring which is not a field, then $\Omega(R)$ is complete and so, $H(R)$ is complete.*

Proof. Let $\mathfrak{m}$ denote the unique maximal ideal of the local Artinian ring R. Since R is not a field, it follows that $\mathfrak{m} \neq (0)$. We know from [3, Corollary 8.2 and Proposition 8.4] that $\mathfrak{m}$ is nilpotent. Therefore, we obtain from [13, Example 2.4] that $\Omega(R)$ is complete. Therefore, $H(R)$ is complete.

Let R be a finite local ring which is not a field. Since any finite ring is Artinian, it follows from the previous paragraph that $\Omega(R)$ is complete and so, $H(R)$ is complete. □

Theorem 3.7. *Let R be a finite ring which is not an integral domain. Then $\mathbb{AG}(R) = H(R)$ if and only if $\mathbb{AG}(R)$ is a complete graph.*

Proof. Let R be a ring (finite or not). Suppose that $\mathbb{AG}(R)$ is complete. Since $\mathbb{AG}(R)$ is a spanning subgraph of $H(R)$, we see that if $\mathbb{AG}(R)$ is complete, then so is $H(R)$. Hence, $\mathbb{AG}(R) = H(R)$.

Conversely, suppose that $\mathbb{AG}(R) = H(R)$, where R is a finite ring. Since R is a finite ring, $R \cong R_1 \times \cdots \times R_n$, where $R_1, \ldots, R_n$ are finite local rings. It is convenient to denote $R_1 \times \cdots \times R_n$ by T. Since $R \cong T$ as rings and $\mathbb{AG}(R) = H(R)$ by assumption, we get that $\mathbb{AG}(T) = H(T)$. We claim that $n \leq 2$. Suppose that $n > 2$. Let $i \in \{1, 2, 3, \ldots, n\}$. Let e_i denote the element of T whose i-th coordinate equals 1 and j-th coordinate equals 0 for all $j \in \{1, 2, 3, \ldots, n\} \backslash \{i\}$. It is clear that $I = Te_2, J = Te_2 + Te_3 \in \mathbb{A}(T)^*$ are such that $I + J = J \in \mathbb{A}(T)$. Hence, I and J are adjacent in $H(T)$. Note that $IJ = I \neq (0) \times (0) \times (0) \times \cdots \times (0)$. Therefore, I and J are not adjacent in $\mathbb{AG}(T)$. This is in contradiction to the fact that $\mathbb{AG}(T) = H(T)$. Hence, $n \leq 2$. If $n = 1$, then R is a finite local ring and so, we obtain from Corollary 3.6 that $\Omega(R)$ is complete. Therefore, $H(R)$ is complete. From $\mathbb{AG}(R) = H(R)$, it follows that $\mathbb{AG}(R)$ is complete. Suppose that $n = 2$. Now, $T = R_1 \times R_2$. We claim that R_i is a field for each $i \in \{1, 2\}$. Let $\mathfrak{m}_1$ denote the unique maximal ideal of R_1. If R_1 is not a field, then $\mathfrak{m}_1 \neq (0)$ and as R_1 is finite, $\mathfrak{m}_1 \in \mathbb{A}(R_1)^*$. Let $A = \mathfrak{m}_1 \times R_2$ and let $B = (0) \times R_2$. It is clear that $A, B \in \mathbb{A}(T)^*$ and $A + B = A \in \mathbb{A}(T)$. Hence, A and B are adjacent in $H(T)$. Observe that $AB = B \neq (0) \times (0)$. Therefore, A and B are not adjacent in $\mathbb{AG}(T)$. This contradicts the fact that $\mathbb{AG}(T) = H(T)$. Therefore, R_1 is a field. Similarly, it can be shown that R_2 is a field. Hence, $\mathbb{AG}(T)$ is complete and from $R \cong T$ as rings, we obtain that $\mathbb{AG}(R)$ is complete. □

Corollary 3.8. *Let R be a finite ring which is not an integral domain. Then $\mathbb{AG}(R) = H(R)$ if and only if R is one of the following three types of rings:*
1) $R \cong F_1 \times F_2$, where F_1, F_2 are fields.
2) $Z(R)$ is an ideal of R with $Z(R)^2 = (0)$.
3) R is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$.

Proof. The proof of this corollary follows from Theorem 3.7 and [5, Theorem 2.7]. □

Example 3.9. Let $R = \mathbb{Z}_{p^3}$, where p is a prime number. Note that $\mathbb{A}(R)^* = \{(p), (p^2)\}$ and $(p)(p^2) = (0)$. Therefore, $\mathbb{AG}(R)$ is complete. Hence, $\mathbb{AG}(R) = H(R)$.

Theorem 3.10. *Let R be a finite ring which is not an integral domain. Then $\Omega(R) = H(R)$ if and only if $\Omega(R)$ is complete.*

Proof. Let R be a ring (finite or not). Suppose that $\Omega(R)$ is complete. As $\Omega(R)$ is a spanning subgraph of $H(R)$, it follows that $H(R)$ is complete. Hence, $\Omega(R) = H(R)$.

Conversely, suppose that $\Omega(R) = H(R)$, where R is a finite ring. Since R is a finite ring, $R \cong R_1 \times \cdots \times R_n$, where $R_1, \ldots, R_n$ are finite local rings. We claim that $n = 1$. Suppose that $n \geq 2$. Let us denote the ring $R_1 \times R_2 \times \cdots \times R_n$ by

T. Since $R \cong T$ as rings, we get that $\Omega(T) = H(T)$. Let $I = Te_2$ and $J = Tf_2$, where $e_2 = (0, 1, 0, \ldots, 0)$ and $f_2 = (1, 1, \ldots, 1) - e_2$. It is clear that $I, J \in \mathbb{A}(T)^*$ with $IJ = (0) \times (0) \times \cdots \times (0)$ and so, I and J are adjacent in $H(T)$. From $I + J = T \notin \mathbb{A}(T)$, we get that I and J are not adjacent in $\Omega(T)$. This contradicts the fact that $\Omega(T) = H(T)$. Therefore, $n = 1$ and so, R is a finite local ring and in such a case, we obtain from Corollary 3.6 that $\Omega(R)$ is complete. □

Corollary 3.11. *Let R be a finite ring which is not an integral domain. Then $\Omega(R) = H(R)$ if and only if R is a local ring.*

Proof. The proof of this corollary follows from the proof of Theorem 3.10 and Corollary 3.6. □

Example 3.12. Let $R = \mathbb{Z}_{16}$. Observe that R is a finite local ring with maximal ideal (2) and $\mathbb{A}(R)^* = \{(2), (4), (8)\}$. It is clear that $(2) + (4) = (2) \in \mathbb{A}(R)$, $(2) + (8) = (2) \in \mathbb{A}(R)$, $(4) + (8) = (4) \in \mathbb{A}(R)$. This implies that $\Omega(R)$ is complete. Hence, $\Omega(R) = H(R)$.

Theorem 3.13. *Let R be a finite ring. Then $H(R)$ is complete if and only if one of the following holds:*
1) *R is a local ring.*
2) *$R \cong F_1 \times F_2$, where F_1, F_2 are fields.*

Proof. Suppose that $H(R)$ is complete. If R is a local ring, then there is nothing to prove. So, suppose that R is not a local ring. Then our claim is to prove that $R \cong F_1 \times F_2$, where F_1, F_2 are fields.

Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$ for some $n \in \mathbb{N}, n \geq 2$. Observe that $R_1, R_2, \ldots, R_n$ are finite local rings. Let us denote the ring $R_1 \times R_2 \times \cdots \times R_n$ by T. Since $R \cong T$ as rings, and $H(R)$ is complete by assumption, we obtain that $H(T)$ is complete. First, we verify that $n = 2$. Suppose that $n \geq 3$. Let $I = R_1 \times R_2 \times (0) \times \cdots \times (0)$ and let $J = (0) \times R_2 \times R_3 \times \cdots \times R_n$. It is clear that $I, J \in \mathbb{A}(T)^*$, $IJ \neq (0) \times (0) \times (0) \times \cdots \times (0)$ and $I + J = T \notin \mathbb{A}(T)$. This implies that I and J are not adjacent in $H(T)$. This contradicts the fact that $H(T)$ is complete. Therefore, $n = 2$. Thus, $T = R_1 \times R_2$. We claim that R_i is a field for each $i \in \{1, 2\}$. Suppose that R_1 is not a field. Let $\mathfrak{m}_1$ denote the unique maximal ideal of R_1. As R_1 is not a field by assumption, it follows that $\mathfrak{m}_1 \neq (0)$. Note that $\mathfrak{m}_1$ is nilpotent. Let $A = R_1 \times (0)$ and let $B = \mathfrak{m}_1 \times R_2$. It is clear that $A, B \in \mathbb{A}(T)^*$ with $A \neq B$. Observe that $AB \neq (0) \times (0)$ and $A + B = T \notin \mathbb{A}(T)$. Therefore, A and B are not adjacent in $H(T)$. This contradicts the fact that $H(T)$ is complete. Therefore, R_1 is a field. Similarly, it can be shown that R_2 is a field. Let $i \in \{1, 2\}$. Thus, with $F_i = R_i$, it follows that F_i is a field and $R \cong F_1 \times F_2$ as rings. This shows that if $H(R)$ is complete, then either R is a local ring or $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.

Conversely, assume that 1) or 2) holds. If 1) holds, then $\Omega(R)$ is complete by Corollary 3.6 and so, $H(R)$ is complete. If 2) holds, then $\mathbb{AG}(R)$ is complete by [5, Theorem 2.7] and so, $H(R)$ is complete. □

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Faculty, Government Polytechnic, Bhuj, 370001, India.
Email address: hdp12376@gmail.com