Gulf Journal of Mathematics
Vol 13, Issue 1 (2022) 41-53

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v13i1.926

ON TRAIN ALGEBRAS OF DEGREE 2 AND EXPONENT 4

WENDINGOUDI ACHILE ZANGRE [1] AND ANDRÉ CONSEIBO[2*]

Abstract. In this paper, we deal with a class of nonassociative algebras called train algebras of degree 2 and exponent 4; Thus we give some results on algebras verifying a train identity of degree 2 and exponent 4. The structure of this class of algebras is studied through Peirce decomposition relative to a non zero idempotent. We give the necessary and sufficient conditions for an algebra verifying a train identity of degree 2 and exponent 4 to be Bernstein or train algebra of rank less than or equal to 3. Finally, we give some necessary conditions that must be verified by a train algebra of degree 2 and exponent 4, mainly in dimension four.

1. Introduction

A commutative algebra A over a field K is called Bernstein algebra if any element x of A verifies the identity $x^2x^2 = \omega(x)^2x^2$ where $\omega : A \to K$ is a non zero morphism of algebras called the weight function of A. Bernstein algebras which have been the subject of many publications ([1],[12],[14],[17]) from the years 1975 are the formulation in terms of nonassociative algebras of stationarity principle or Hardy Weinberg's law in Mendelian genetics. They were introduced by Holgate ([11],[10]). Several authors have subsequently studied algebras which enter into attempts to model genetic phenomena. These include, among others, train algebras. In ([18]) the author establishes the following results: in a Bernstein algebra over a field K of characteristic different from 2, $2x^ix^j = \omega(x)^ix^j + \omega(x)^jx^i$ for any x in A and for all $i, j \geq 2$. In particular, for $i = j = 4$ we obtain the identity $x^4x^4 = \omega(x)^4x^4$ which characterizes the algebras which are the subject of this paper; thus the elements in a Bernstein algebra verifies this latter identity. The train algebras of degree 2 and exponent n (the case $i = j = n$) in general and the train algebras of degree 2 and exponent 3 in particular have been studied in ([4]). Moreover, we find there some results on the structure of these algebras and the link with other classes of algebras. In these classes of algebras, the notion of idempotent is very important. The idempotents constituting equilibrium states in terms of genetic interpretation and are useful for the Peirce decomposition, which facilitates the study of the structure of these algebras. In the following, A will be a commutative algebra not necessarily associative over a field K; the field

Date: Received: Sep 24, 2021; Accepted: Jan 26, 2022.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 17D92,17A05.

Key words and phrases. Peirce decomposition, Train algebra of degree 2 and exponent 4, Bernstein algebra, Principal Train algebra, Idempotent.

K will be commutative and algebraically closed of characteristic different from 2 and 3.

2. Algebras verifying train identity of degree 2 and exponent 4

An algebra A satisfies a train identity of degree 2 and exponent 4 if for any x in A, we have:

$$(x^4)^2 = \omega(x)^4 x^4. \tag{2.1}$$

For an element x of weight 1 in A (i.e. $\omega(x) = 1$), the identity (2.1) gives $(x^4)^2 = x^4$ and thus x^4 is an non zero idempotent of A.

As we pointed out in the introduction, Bernstein algebras satisfies a train identity of degree 2 and exponent 4. We give in the following, results on the structure of algebras satisfying a train identity of degree 2 and exponent 4.

2.1. **Peirce Decomposition.**

This section is devoted to the Peirce decomposition of algebras verifying train identity of degree 2 and exponent 4.

Lemma 2.1. *Let A be an algebra satisfying train identity of degree 2 and exponent 4 then for all x, y and z in A we have:*

(*i*) $4x^4(x(x(xy)))+2x^4(x(yx^2))+2x^4(yx^3) = 2\omega(x)^4x(x(xy))+\omega(x)^4x(yx^2)+\omega(x)^4yx^3 + 4\omega(x)^3\omega(y)x^4$;

(*ii*) $4(x(y(xy))^2+4(y(x(xy))^2+(x(xy^2))^2+8(x(y(xy)))(y(x(xy)))+4(x(y(xy)))(x(xy^2))+4(x(y(xy)))(y(yx^2))+4(y(x(xy)))(x(xy^2))+4(y(x(xy)))(y(yx^2))+2(x(xy^2))(y(yx^2))+8(x(x(xy)))(y(y(xy)))+4(x(yx^2))(y(y(xy)))+4(yx^3)(y(y(xy)))+4(y(xy^2))(x(x(xy)))+2(x(yx^2))(y(xy^2)) + 2(xy^3)(x(yx^2)) + 4(x(x(xy)))(xy^3)4(xy^3)(x(yx^2)) + 2(yx^3)(xy^3)+2x^4y^4 = \omega(x)^4y^4+8\omega(x)^3\omega(y)y(y(xy))+4\omega(x)^3\omega(y)y(xy^2+4\omega(x)^3\omega(y)xy^3+8\omega(x)\omega(y)^3x(x(xy))+4\omega(x)\omega(y)^3x(yx^2)+4\omega(x)\omega(y)^3yx^3+12\omega(x)^2\omega(y)^2x(y(xy))+12\omega(x)^2\omega(y)^2y(x(xy))+6\omega(x)^2\omega(y)^2x(xy^2)+6\omega(x)^2\omega(y)^2y(yx^2))+\omega(y)^4x^4$;

(*iii*) $2(y(x^2z)x^4+4(y(x(xz)))x^4+2(x^3y)(x^3z)+2(x^3y)(x(x^2z))+4(x^3y)(x(x(xz)))+2(z(x^2y))x^4+4(x(y(xz)))x^4+2(x(x^2y))(x^3z)+2(x(x^2y))(x(x^2z))+4(x(x^2y))(x(x(xz)))+4(z(x(xy)))x^4+4(x(z(xy)))x^4+4(x(x(yz)))x^4+4(x(x(xy)))(x^3z)+4(x(x(xy)))(x(x^2z))+8(x(x(xy)))(x(x(xz))) = 12\omega(y)\omega(z)\omega(x)^2x^4+4\omega(x)^3\omega(y)zx^3+4\omega(x)^3\omega(y)x(zx^2)+8\omega(x)^3\omega(y)x(x(xz))+8\omega(x)^3\omega(z)x(x(xy))+2\omega(x)^4z(x(xy))+2\omega(x)^4x(z(xy))+\omega(x)^4x(x(yz))+4\omega(x)^3\omega(z)x(yx^2)+\omega(x)^4z(yx^2)+2\omega(x)^4x(y(xz))+4\omega(x)^3\omega(z)yx^3+\omega(x)^4y(zx^2) + 2\omega(x)^4y(x(xz))$.

Proof. The identities $(i), (ii)$ and (iii) are obtained by partially linearizing the identity (2.1). □

Proposition 2.2. *Let A be an algebra satisfying train identity of degree 2 and exponent 4 and e an idempotent of A. Then A has a Peirce decomposition as follows $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$, where $A_\alpha = \{y \in ker(\omega) \mid ey = \alpha y\}$ for $\alpha \in \{0; 1/2; \lambda = -\frac{1+i\sqrt{7}}{4}; \overline{\lambda} = \frac{-1+i\sqrt{7}}{4}\}$.*

Proof. By setting $x = e$ and $y \in Ker(\omega)$ in the identity (i) of Lemma 2.1 we get: $4e(e(e(ey))) + e(ey) - ey = 0$, so $(4\ell_e^4 + \ell_e^2 - \ell_e)(y) = 0$ for any $y \in ker(\omega)$, where $\ell_e : ker\omega \to ker\omega, y \mapsto ey$. So it comes that the minimal polynomial of the operator ℓ_e is : $P(X) = 4X^4 + X^2 - X$ and the roots of this one are $0, \frac{1}{2}, -\frac{1+i\sqrt{7}}{4}$ and $\frac{-1+i\sqrt{7}}{4}$. According to the kernel decomposition lemma : $ker\omega = A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ where $A_\alpha = \{y \in ker(\omega) \mid ey = \alpha y\}$ for $\alpha \in \{0; 1/2; \lambda = -\frac{1+i\sqrt{7}}{4}; \overline{\lambda} = \frac{-1+i\sqrt{7}}{4}\}$ and we get a Peirce decomposition $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$. □

Proposition 2.3. *Let $A = Ke \oplus A_0 \oplus A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be the Peirce decomposition relatively to a non-zero idempotent e of an algebra satisfying train identity of degree 2 and exponent 4. If A is of finite dimension then the dimensions of A_0, $A_{1/2}$, A_λ and $A_{\overline{\lambda}}$ are independent of the chosen idempotent e.*

Proof. This is a particular case ($n = 4$) of the general case treated in [4] (Proposition 2).

□

in view of the previous proposition, we define in what follows the type of an algebra satisfying train identity of degree 2 and exponent 4 of dimension n. We can therefore observe that the type of an algebra verifying a train identity of degree 2 and exponent 4 is an invariant of the algebra.

Definition 2.4. Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be a Peirce decomposition of an algebra satisfying train identity of degree 2 and exponent 4 of dimension $n \geq 1$. We call type of A, the quadruplet $(1 + r, s, t_1, t_2)$ such that $1 + r + s + t_1 + t_2 = n$, with $r = dimA_{1/2}$, $s = dimA_0$, $t_1 = dimA_\lambda$, $t_2 = dimA_{\overline{\lambda}}$.

Theorem 2.5. *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be the Peirce decomposition relative to a non-zero idempotent e of an algebra satisfying a train identity of degree 2 and exponent 4. Then :*

(i) $A_0^2 \subset A_0 \oplus A_{1/2}$;
(ii) $A_{1/2}^2 \subset A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$;
(iii) $A_\lambda^2 \subset A_{1/2}$;
(iv) $A_{\overline{\lambda}}^2 \subset A_{1/2}$;
(v) $A_{1/2}A_0 \subset A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$;
(vi) $A_0A_\lambda \subset A_{1/2}$;
(vii) $A_0A_{\overline{\lambda}} \subset A_{1/2}$;
$(viii)$ $A_{1/2}A_\lambda \subset A_{1/2} \oplus A_0 \oplus A_{\overline{\lambda}}$;
(ix) $A_{1/2}A_{\overline{\lambda}} \subset A_{\frac{1}{2}} \oplus A_0 \oplus A_\lambda$;
(x) $A_\lambda A_{\overline{\lambda}} \subset A_{1/2}$.

Proof. By setting $x = e$, $y \in A_\alpha$ and $z \in A_\beta$ in the identity (iii) of the Lemma 2.1, we obtain the equality:
$(-\alpha - \beta - 2\alpha^2 - 2\beta^2 + 2\alpha\beta + 2\alpha^2\beta + 2\alpha\beta^2 + 4\alpha\beta^3 + 4\alpha^3\beta + 2\alpha^2\beta^2 + 4\alpha^3\beta^2 + 4\alpha^2\beta^3 + 8\alpha^3\beta^3)(yz) + (4\alpha^2 + 4\beta^2)e(yz) + (4\alpha + 4\beta - 2)e(e(yz)) + 4e(e(e(yz))) = 0.$

(i) Suppose that $\alpha = \beta = 0$, we get $(2\ell_e^3 - \ell_e^2)(yz) = 0$, so $\ell_e^2(2\ell_e - id)(yz) = 0$ and $A_0^2 \subset A_{1/2} \oplus A_0$, then (i) is verified.

(ii) For $\alpha = \beta = \frac{1}{2}$, we have $(2\ell_e^3 + \ell_e^2 + \ell_e)(yz) = 0$, so $\ell_e(2\ell_e^2 + \ell_e + id)(yz) = 0$ and $A_{1/2}^2 \subset A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$;

(iii) For $\alpha = \beta = \lambda$, we have $(2\ell_e^3 + (4\lambda - 1)\ell_e^2 - 2(\lambda + 1)\ell_e + id)(yz) = 0$, so $A_\lambda^2 \subset A_{1/2}$;

(iv) For $\alpha = \beta = \overline{\lambda}$, we have $(2\ell_e^3 + (4\overline{\lambda} - 1)\ell_e^2 - 2(\overline{\lambda} + 1)\ell_e + id)(yz) = 0$, so $A_{\overline{\lambda}}^2 \subset A_{1/2}$;

(v) For $\alpha = \frac{1}{2}$ and $\beta = 0$, we have $(4\ell_e^3 + \ell_e - id)(yz) = 0$, so $A_0 A_{1/2} \subset A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$;

(vi) For $\alpha = \lambda$ and $\beta = 0$, we have $(4\ell_e^3 + 2(2\lambda - 1)\ell_e^2 - 2(\lambda + 1)\ell_e + id)(yz) = 0$, so $A_0 A_\lambda \subset A_{1/2}$;

(vii) For $\alpha = \overline{\lambda}$ and $\beta = 0$, we have $(4\ell_e^3 + 2(2\overline{\lambda} - 1)\ell_e^2 - 2(\overline{\lambda} + 1)\ell_e + id)(yz) = 0$, so $A_0 A_{\overline{\lambda}} \subset A_{1/2}$;

$(viii)$ For $\alpha = \frac{1}{2}$ and $\beta = \lambda$, we have $\ell_e(4\ell_e^2 + 4\lambda\ell_e - (2\lambda + 1)id)(yz) = 0$, so $A_{1/2} A_\lambda \subset A_{1/2} \oplus A_0 \oplus A_{\overline{\lambda}}$;

(ix) For $\alpha = \frac{1}{2}$ and $\beta = \overline{\lambda}$, we have $\ell_e(4\ell_e^2 + 4\overline{\lambda}\ell_e - (2\overline{\lambda} + 1)id)(yz) = 0$, so $A_{1/2} A_{\overline{\lambda}} \subset A_{1/2} \oplus A_0 \oplus A_\lambda$;

(x) For $\alpha = \overline{\lambda}$ and $\beta = \lambda$, we have $(4\ell_e^3 - 4\ell_e^2 - 3\ell_e + 2id)(yz) = 0$, so $A_\lambda A_{\overline{\lambda}} \subset A_{1/2}$;

□

2.2. **Link with Bernstein algebras.** The examples below show that there are algebras satisfying train identity of degree 2 and exponent 4 which are not Bernstein algebras.

Example 2.6. Let $A =< e, e_0, e_1, e_2 >$ be a commutative algebra whose multiplication table is given by : $e^2 = e$; $ee_0 = \frac{1}{2}e_0$; $ee_2 = \lambda e_2$ where $\lambda = -\frac{1 + i\sqrt{7}}{4}$; $e_0^2 = e_1$, the other products being zero. We can observe that $\omega : A \to K$ such that $\omega(e) = 1$ and $\omega(e_0) = \omega(e_1) = \omega(e_2) = 0$ is a weight function.

Let $x = \mu e + \alpha e_0 + \beta e_1 + \gamma e_2$ an element in A, then :

$x^2 = \mu^2 e + \alpha^2 e_1 + \mu\alpha e_0 + 2\mu\lambda\gamma e_2$;

$(x^2)^2 = \mu^4 e + \mu^3\alpha e_0 + \mu^2\alpha^2 e_1 + 4\mu^3\lambda^2\gamma e_2 \neq \omega(x)^2 x^2$ if $\gamma\mu \neq 0$;

$x^3 = \mu^3 e + \mu^2\alpha e_0 + \mu\alpha^2 e_1 + \mu^2\gamma(2\lambda^2 + \lambda)e_2$;

$x^4 = \mu^4 e + \mu^3\alpha e_0 + \mu^2\alpha^2 e_1 + \mu^3\lambda\gamma(2\lambda^2 + \lambda + 1)e_2 = \mu^4 e + \mu^3\alpha e_0 + \mu^2\alpha^2 e_1$; and $(x^4)^2 = \mu^8 e + \mu^7\alpha e_0 + \mu^6\alpha^2 e_1 = \mu^4(\mu^4 e + \mu^3\alpha e_0 + \mu^2\alpha^2 e_1) = \omega(x)^4 x^4$.

Thus A is an algebra satisfying a train identity of degree 2 and exponent 4 but it is not Bernstein algebra.

Example 2.7. Let $A =< e, e_0, e_1, e_2 >$ be a commutative algebra such that : $e^2 = e$; $ee_0 = \frac{1}{2}e_0$; $e_0^2 = e_1$; $e_2^2 = e_1$; the other products being zero; As in the previous example, A has a weight morphism $\omega : A \to K$ defined in a similar way.

Let $x = \mu e + \alpha e_0 + \beta e_1 + \gamma e_2$ an element in A. Then :

$x^2 = \mu^2 e + \mu\alpha e_0 + (\alpha^2 + \gamma^2)e_1$;

$(x^2)^2 = \mu^4 e + \mu^3\alpha e_0 + \mu^2\alpha^2 e_1 \neq \omega(x)^2 x^2$ if $\gamma\mu \neq 0$;

$x^3 = \mu^3 e + \mu^2\alpha e_0 + \mu\alpha^2 e_1$; $x^4 = \mu^4 e + \mu^3\alpha e_0 + \mu^2\alpha^2 e_1$
$(x^4)^2 = \mu^8 e + \mu^7\alpha e_0 + \mu^6\alpha^2 e_1 = \omega(x)^4 x^4$.

So A satisfies a train identity of degree 2 and exponent 4 but is not Bernstein algebra.

Lemma 2.8. *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ the Peirce decomposition with respect to a non-zero idempotent e of an algebra satisfying a train identity of degree 2 and exponent 4. Let x_α in A_α with $\alpha \in \{0; 1/2; \lambda; \overline{\lambda}\}$. We have :*

(i) If $A_\lambda = 0$, $A_{\overline{\lambda}} = 0$ and $A_0^2 \subset A_{1/2}$, then :
- *(i1)* $x_{1/2}(x_{1/2}x_0) = 0$;
- *(i2)* $x_{1/2}x_0^2 = 0$;
- *(i3)* $x_{1/2}^3 = 0$;
- *(i4)* $x_{1/2}^2 x_0^2 = 0$;
- *(i5)* $(x_{1/2}x_0)^2 = 0$;
- *(i6)* $(x_0^2)^2 = 0$;
- *(i7)* $(x_{1/2}^2)^2 = 0$.

(ii) If $A_0 = 0$ and $A_\lambda = 0$ then :
- *(ii1)* $x_{1/2}^3 = 0$;
- *(ii2)* $x_{1/2}(x_{1/2}x_{\overline{\lambda}}) = 0$;
- *(ii3)* $x_{1/2}x_{\overline{\lambda}}^2 = 0$;
- *(ii4)* $(x_{1/2}^2)^2 = 0$;
- *(ii5)* $(x_{\overline{\lambda}}^2)^2 = 0$;
- *(ii6)* $(x_{\overline{\lambda}}x_{1/2})^2 = 0$;

(iii) If $A_0 = 0$ and $A_{\overline{\lambda}} = 0$ then :
- *(iii1)* $x_{1/2}^3 = 0$;
- *(iii2)* $x_{1/2}(x_{1/2}x_\lambda) = 0$;
- *(iii3)* $x_{1/2}x_\lambda^2 = 0$;
- *(iii4)* $(x_{1/2}^2)^2 = 0$;
- *(iii5)* $(x_\lambda^2)^2 = 0$;
- *(iii6)* $(x_\lambda x_{1/2})^2 = 0$;

(iv) If $A_\lambda = 0$ and $A_{\overline{\lambda}} = 0$ then :
- *(iv1)* $2e(x_0x_{1/2}^2) + 2x_{1/2}(x_0x_{1/2}) - x_0x_{1/2}^2 = 0$;
- *(iv2)* $x_{1/2}(ex_0^2) - x_{1/2}x_0^2 = 0$;
- *(iv3)* $x_{1/2}^3 = 0$;
- *(iv4)* $(x_{1/2}^2)^2 = 0$;
- *(iv5)* $(e(ex_0^2))^2 + 2ex_0^4 - x_0^4 = 0$.

Proof. (i) Indeed let us suppose that A is an algebra verifying an train identity of degree 2 and exponent 4 such that $A_\lambda = A_{\overline{\lambda}} = 0$ and $A_0^2 \subset A_{1/2}$. let $x = e + \alpha x_{1/2} + \beta x_0 \in A$; then :

$$x^4 = e + \alpha x_{1/2} + \tfrac{\beta^2}{4}x_0^2 + \alpha^2 x_{1/2}^2 + 3\frac{\alpha^3}{2}x_{1/2}^3 + 2\alpha\beta x_0x_{1/2} + \beta^3 x_0^3 + \alpha^2\beta[\tfrac{3}{2}x_0x_{1/2}^2 + 2x_{1/2}(x_0x_{1/2})] + \alpha\beta^2[\tfrac{1}{2}x_0^2x_{1/2} + 3x_0(x_0x_{1/2})] + \alpha^3\beta[x_0x_{1/2}^3 + 2x_{1/2}(x_{1/2}(x_0x_{1/2})) + x_{1/2}(x_0x_{1/2}^2)] + \alpha\beta^3[x_0^3x_{1/2} + 2x_0(x_0(x_0x_{1/2})) + x_0(x_0^2x_{1/2})] + \alpha^2\beta^2[x_{1/2}(x_0^2x_{1/2}) +$$

$2x_{1/2}(x_0(x_0x_{1/2}))+x_0(x_0x_{1/2}^2)+2x_0(x_{1/2}(x_0x_{1/2}))]+\beta^4x_0^4+\alpha^4x_{1/2}^4$.

$(x^4)^2=e+\alpha x_{1/2}+\frac{\beta^2}{16}x_0^2+\alpha^2x_{1/2}^2+2\alpha\beta x_{1/2}x_0+7\frac{\alpha^3}{2}x_{1/2}^3+\beta^3x_0^3+\alpha^2\beta[\frac{3}{2}x_0x_{1/2}^2+4x_{1/2}(x_0x_{1/2})+\alpha\beta^2[3x_0(x_0x_{1/2})+\frac{1}{2}x_0^2x_{1/2}]+\alpha^3\beta[x_0x_{1/2}^3+6x_{1/2}(x_{1/2}(x_{1/2}x_0))+3x_{1/2}(x_0x_{1/2}^2)+4x_{1/2}^2(x_{1/2}x_0)]+\alpha\beta^2[3x_0(x_0x_{1/2})+\frac{1}{2}x_0^2x_{1/2}]+\alpha^3\beta[x_0x_{1/2}^3+6x_{1/2}(x_{1/2}(x_{1/2}x_0))+3x_{1/2}(x_0x_{1/2}^2)+4x_{1/2}^2(x_{1/2}x_0)]+\alpha\beta^3[2x_0(x_0(x_0x_{1/2}))+x_0(x_{1/2}x_0^2)+2x_0^3x_{1/2}+x_0^2(x_0x_{1/2})]+\alpha^2\beta^2[4(x_0^2x_{1/2})+2x_{1/2}(x_0^2x_{1/2})+x_0(x_0x_{1/2}^2)+2x_0(x_{1/2}(x_{1/2}x_0))+x_0(x_0x_{1/2}^2)+2x_0(x_{1/2}(x_0x_{1/2}))+x_{1/2}(x_{1/2}x_0^2)+6x_{1/2}(x_0(x_0x_{1/2}))+\frac{1}{2}x_0^2x_{1/2}^2]+\alpha^4[3x_{1/2}^4+(x_{1/2}^2)^2]+\beta^4[x_0^4+\frac{1}{16}(x_0^2)^2]+\dots$

Theses identities are obtained by identifying the coefficients of $\alpha^i\beta^j$ in equality $(x^4)^2=x^4$ for $1\leq j+i\leq 4$.

(ii), (iii) and (iv). Theses identities are obtained using a method similar to that of (i). □

The theorem below gives the necessary and sufficient conditions for a train algebra of degree 2 and exponent 4 to be a Bernstein algebra.

Theorem 2.9. *Let $A=Ke\oplus A_{1/2}\oplus A_0\oplus A_\lambda\oplus A_{\overline{\lambda}}$ be the Peirce decomposition relative to a non-zero idempotent e of a train algebra of degree 2 and exponent 4. Then A is a Bernstein algebra if and only if $A_\lambda=A_{\overline{\lambda}}=0$ and $A_0^2\subset A_{1/2}$.*

Proof. Indeed, suppose that A satisfies these conditions and establish that this one is Bernstein. Let $x=e+x_{1/2}+x_0$ an element of weight 1 in A then :

$x^2=e+x_{1/2}+2x_0x_{1/2}+x_0^2+x_{1/2}^2$;

$(x^2)^2=e+x_{1/2}+2x_0x_{1/2}+x_0^2+x_{1/2}^2+4x_{1/2}(x_0x_{1/2})+2x_{1/2}x_0^2+4x_0^2(x_0x_{1/2})+4x_{1/2}^2(x_0x_{1/2})+2x_0^2x_{1/2}^2+4(x_0x_{1/2})^2+(x_0^2)^2$.

By using identities of the Lemma 2.8, we shown that $x^2=(x^2)^2$.Since the set of elements of weight 1 is dense in A according to the Zariski topology, then for any x in A, $(x^2)^2=\omega(x)^2x^2$ and A is a Bernstein algebra.

Conversely, suppose that A is a Bernstein algebra then the relations $A_\lambda=A_{\overline{\lambda}}=0$ and $A_0^2\subset A_{1/2}$ are immediate (see [1]) .

□

2.3. **Link with principal train algebras.** Let A be a commutative nonassociative algebra and $\omega:A\to K$ his weight function. We say that A is principal train algebra of rank r if there exist $\gamma_1,\dots,\gamma_r$ in K such that $x^r+\gamma_1\omega(x)x^{r-1}+\dots+\gamma_{r-1}\omega(x)^{r-1}x=0$, $\forall x\in A$, where $r\geq 2$ is the smallest integer for which such an equation holds. The Equation is called the train equation of A.

An algebra satisfying train identity of degree 2 and exponent 4 is principal train algebra under some conditions.

Proposition 2.10. *Let $A=Ke\oplus A_{1/2}\oplus A_0\oplus A_\lambda\oplus A_{\overline{\lambda}}$ be the Peirce decomposition relative to an idempotent e of an algebra satisfying train identity of degree 2 and exponent 4. If $A_{1/2}=0$ then A check the train equation :*

$$x^7-\frac{1}{2}\omega(x)x^6-\frac{1}{2}\omega(x)^3x^4=0.$$

Proof. : Indeed, let A be an algebra verifying an train identity of degree 2 and exponent 4 such that $A_{1/2} = 0$. Then $A_0^2 \subset A_0$, $A_\lambda^2 = A_{\overline{\lambda}}^2 = A_\lambda A_{\overline{\lambda}} = A_0 A_\lambda = A_0 A_{\overline{\lambda}} = 0$. Let $x = e + \alpha x_0 + \gamma x_\lambda + \mu x_{\overline{\lambda}}$ any element of weight 1 in A.

We have: $x^2 = e + \alpha^2 x_0^2 + 2\lambda\gamma x_\lambda + 2\overline{\lambda}\mu x_{\overline{\lambda}}$; $x^3 = e + \gamma(2\lambda^2 + \lambda)x_\lambda + \mu(2\overline{\lambda}^2 + \overline{\lambda})x_{\overline{\lambda}} + \alpha^3 x_0^3 = e - \gamma x_\lambda - \mu x_{\overline{\lambda}} + \alpha^3 x_0^3$; $x^4 = e + \alpha^4 x_0^4$ and $(x^4)^2 = e + \alpha^8 (x_0^4)^2$; or $(x_0^4)^2 = 0$, hence $x_0^4 = 0$ due to identity $(x^4)^2 = x^4$, so $x^4 = e$. We have $x^3 + x = 2e + \alpha x_0 + \alpha^3 x_0^3 = 2x^4 + \alpha^3 x_0^3$ so $2x^4 - x^3 - x = -\alpha x_0 - \alpha x_0^3$. The multiplication of the last equality three successive times by x give $x^7 - \frac{1}{2}x^6 - \frac{1}{2}x^4 = 0$. The elements of weight 1 being dense in A by Zariski topology, then for any x in A $x^7 - \frac{1}{2}\omega(x)x^6 - \frac{1}{2}\omega(x)^3 x^4 = 0$. □

Theorem 2.11. *Let A be an algebra satisfying train identity of degree 2 and exponent 4. If A is train of rank 3, then the train equation of A is*

$$x^3 - (1+\alpha)\omega(x)x^2 + \alpha\omega(x)^2 x = 0 = 0;$$

where $\alpha \in \{0; \lambda = -\frac{1+i\sqrt{7}}{4}; \overline{\lambda} = \frac{-1+i\sqrt{7}}{4}\}$.

Proof. Let A be an algebra satisfying train identity of degree 2 and exponent 4. Suppose that A is principal train of rank 3, then for any x in A, the following equations hold :

$$x^3 - (1+\alpha)\omega(x)x^2 + \alpha\omega(x)^2 x = 0 \quad (1).$$

A partial linearization of (1) gives

$$x^2 y + 2x(xy) = (1+\alpha)\omega(y)x^2 + 2(1+\alpha)\omega(x)xy - 2\alpha\omega(xy)x - \alpha\omega(x)^2 y \quad (2)$$

.

For $y = x^2$ in (2), we get

$$\begin{aligned} x^2x^2 &= -2x^4 + (1+\alpha)\omega(x)^2x^2 + 2(1+\alpha)\omega(x)x^3 - 2\alpha\omega(x)^3x - \alpha\omega(x)^2x^2 \\ &= (1+2\alpha)\omega(x)^2x^2 - 2\alpha\omega(x)^3x. \end{aligned}$$

By multiplying the equation (1) by x we obtain :

$$\begin{aligned} x^4 &= (1+\alpha)\omega(x)x^3 - \alpha\omega(x)^2x^2 \\ &= (1+\alpha)\omega(x)[(1+\alpha\omega(x)x^2 - \alpha\omega(x)^2x] - \alpha\omega(x)^2x^2 \\ &= (1+\alpha)^2\omega(x)^2x^2 - \alpha(1+\alpha)\omega(x)^3x - \alpha\omega(x)^2x^2. \end{aligned}$$

So it comes that :

$$x^4 = (1+\alpha+\alpha^2)\omega(x)^2x^2 - \alpha(1+\alpha)\omega(x)^3x \qquad (a)$$

Relation (a) gives us:

$$\begin{aligned} x^4x^4 = (1+\alpha+\alpha^2)^2\omega(x)^4x^2x^2 - 2\alpha(1+\alpha)(1+\alpha+\alpha^2)\omega(x)^5x^3 \\ + \alpha^2(1+\alpha)^2\omega(x)^6x^2 \end{aligned}$$

and if we replace x^2x^2 and x^3 by their expressions, we obtain:

$$\begin{aligned}x^4x^4 &= (1+\alpha+\alpha^2)^2\omega(x)^4[(1+2\alpha)\omega(x)^2x^2 - 2\alpha\omega(x)^3x]\\ &- 2\alpha(1+\alpha)(1+\alpha+\alpha^2)\omega(x)^5[(1+\alpha)\omega(x)x^2 - \alpha\omega(x)^2x] + \alpha 2(1+\alpha)^2\omega(x)^6x^2; \qquad = (1+\\ &- 2\alpha(1+\alpha)^2(1+\alpha+\alpha^2)\omega(x)^6x^2 + 2\alpha^2(1+\alpha)(1+\alpha+\alpha^2)\omega(x)^7x + \alpha^2(1+\alpha)^2\omega(x)^6x^2\\ &= [(1+2\alpha)(1+\alpha+\alpha^2)^2 - 2\alpha(1+\alpha)^2(1+\alpha+\alpha^2) + \alpha^2(1+\alpha)^2]\omega(x)^6x^2\\ &+ 2\alpha(1+\alpha+\alpha^2)[\alpha(1+\alpha) - (1+\alpha+\alpha^2)]\omega(x)^7x\\ &= [(2\alpha^5+6\alpha^4+10\alpha^3+8\alpha^2+4\alpha+1) - (2\alpha^5+6\alpha^4+8\alpha^3+6\alpha^2+2\alpha)]\omega(x)^6x^2\\ &- 2\alpha(1+\alpha+\alpha^2)\omega(x)^7x;\end{aligned}$$

so :

$$x^4x^4 = (2\alpha^3+2\alpha^2+2\alpha+1)\omega(x)^6x^2 - 2\alpha(1+\alpha+\alpha^2)\omega(x)^7x \qquad (b)$$

The multiplication of (a) by $\omega(x)^4$ gives :

$$\omega(x)^4x^4 = (1+\alpha+\alpha^2)\omega(x)^6x^2 - \alpha(1+\alpha)\omega(x)^7x \qquad (c)$$

The difference of $(b) and (c)$ gives :

$$\begin{aligned}x^4x^4 - \omega(x)^4x^4 &= [(2\alpha^3+2\alpha^2+2\alpha+1) - (1+\alpha+\alpha^2)]\omega(x)^6x^2\\ &+ [\alpha(1+\alpha) - 2\alpha(1+\alpha+\alpha^2)]\omega(x)^7x\\ &= \omega(x)^6[\alpha(2\alpha^2+\alpha+1)x^2 - \alpha(2\alpha^2+\alpha+1)\omega(x)x]\end{aligned}$$

Since $x^4x^4 - \omega(x)^4x^4 = 0$, then $\alpha(2\alpha^2+\alpha+1)\omega(x)^6[x^2 - \omega(x)x] = 0$;
so $\alpha = 0$ or $(2\alpha^2+\alpha+1) = 0$ because $x^2 - \omega(x)x \neq 0$. Therefore $\alpha \in \{0; \lambda = -\frac{1+i\sqrt{7}}{4}; \overline{\lambda} = -\frac{1-i\sqrt{7}}{4}\}$ and the train equation of A is one of the forms given in the theorem. □

3. Train algebras of degree 2 and exponent 4

An algebra A is called a train algebra of degree 2 and exponent 4 if it satisfies an train identity of degree 2 and exponent 4 and does not verify a polynomial identity of degree less than 8.

3.1. **General results.**

Proposition 3.1. *Let $A = Ke \oplus A_{1/2} \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be the Peirce decomposition of a train algebra of degree 2 and exponent 4 relatively to an idempotent e. The following assertions are verified* :

(*i*) $A_{1/2} \neq 0$;
(*ii*) *the dimension of A is greater than or equal to 4.*

Proof. (*i*) Indeed if $A_{1/2} = 0$, according to Proposition 2.10, A est principal train algebra of rank less or equal to 7, which is absurd because A is train algebra of degree 2 and exponent 4. Therefore $A_{1/2} \neq 0$.

(*ii*) Since $A_{1/2} \neq 0$, the dimension of A is greater than or equal to 2. If the dimension of A is 2, then $A_0 = A_\lambda = A_{\overline{\lambda}} = 0$ and for any $x \in A$, $x^2 = \omega(x)x$, which is impossible. Suppose dimension of A is 3. Thus the possible types of A is $(2,1,0,0)$, $(2,0,1,0)$, $(2,0,0,1)$, the type $(3,0,0,0)$ being impossible because otherwise A checked the train equation $x^2 = \omega(x)x$. It remains to show that the type of A cannot be one of the other previous types to conclude.

Suppose the type of A is $(2,0,1,0)$ and setting $A_{1/2} =< e_0 >$ and $A_\lambda =< e_1 >$. Then $A =< e, e_0, e_1 >$, $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $e_0^2 = \alpha_0 e_1$, $e_1^2 = \alpha_1 e_0$ and $e_0e_1 = \beta_0 e_0$. We have $0 = e_0^3 = \alpha_0\beta_0 e_0$ and $0 = (e_0^2)^2 = \alpha_0^2\alpha_1 e_0$, so $\alpha_0\beta_0 = 0$ and $\alpha_0\alpha_1 = 0$.

We distinguish two cases :

Case 1 : $\alpha_0 \neq 0$ so $\beta_0 = \alpha_1 = 0$.

Let $x = e + \alpha e_0 + \beta e_1$, an element of A such that $\omega(x) = 1$; so $x^2 = e + \alpha e_0 + (\alpha^2\alpha_0 + 2\lambda\beta)e_1$ and we have $(x^2 - x)^2 = (\alpha^2\alpha_0 + \beta(2\lambda - 1)e_1)^2 = 0$; so $x^2x^2 - 2x^3 + x^2 = 0$ and A satisfies a polynomial identity of degree 4.

Case 2 : $\alpha_0 = 0$.

In this case, we have $x^2 = e + (\alpha + \beta^2\alpha_1 + 2\alpha\beta\beta_0)e_0 + 2\lambda\beta e_1$; and $(x^2 - 2\lambda x)^2 = ((1 - 2\lambda)e + (\alpha(1 - 2\lambda) + \beta^2\alpha_1 + 2\alpha\beta\beta_0)e_0)^2 = (1 - 2\lambda)x^2 - 2(1 + 2\lambda)x$; so $x^2x^2 - 4\lambda x^3 - 3x^2 + 2(1 + 2\lambda)x = 0$. The algebra A satisfies a polynomial equation of degree 4.

We conclude that the type of a train algebra of degree 2 and exponent 4 cannot be $(2,0,1,0)$. By reasoning as previously, we show that the type of A can't be either $(2,0,0,1)$.

Suppose the type of A is $(2,1,0,0)$, and setting $A_{1/2} =< e_0 >$, $A_0 =< e_1 >$ then $A =< e, e_0, e_1 >$ and $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = 0$, $e_0^2 = \alpha_0 e_1$, $e_1^2 = \alpha_1 e_0 + \alpha_2 e_1$, $e_0e_1 = \beta_0 e_0$.

We have $0 = e_0^3 = \alpha_0\beta_0 e_0$ and $0 = (e_0^2)^2 = \alpha_0^2\alpha_1 e_0 + \alpha_0^2\alpha_2 e_1 = 0$, then we obtain the following equalities: $\alpha_0\beta_0 = 0$, $\alpha_0\alpha_1 = 0$ and $\alpha_0\alpha_2 = 0$.

Let $x = e + \alpha e_0 + \beta e_1$ an element of weight one of A Let's distinguish the cases :

Case 1 : $\alpha_0 \neq 0$ so $\beta_0 = \alpha_1 = \alpha_2 = 0$.

We have $x^2 = e + \alpha e_0 + \alpha^2\alpha_0 e_1$, so $x^2x^2 = x^2$. Therefore, A verifies Bernstein algebras identity.

Case 2 : $\alpha_0 = 0$.

We have $0 = (e(ee_1^2))^2 + 2ee_1^4 - e_1^4 = -\alpha_2^3 e_1$, so $\alpha_2 = 0$. Since $\alpha_2 = 0$, according to Theorem2.9, A is Bernstein algebra. Thus the type of A can not be $(2,1,0,0)$. □

3.2. **train algebra of degree** 2 **and exponent** 4 **of dimension** 4.

Proposition 3.2. *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_\lambda \oplus A_{\overline{\lambda}}$ be the Peirce decomposition relatively to a nonzero idempotent e of a train algebra of degree* 2 *and exponent* 4. *If A is of dimension* 4, *then the type of A is different from* $(2,0,2,0), (2,0,0,2), (3,0,0,1), (3,0,1,0)$ *and* $(3,1,0,0)$

Proof. Suppose the type of A is $(2,0,2,0)$, and setting $A_{1/2} =< e_0 >$, $A_\lambda =< e_1, e_2 >$. Then $A =< e, e_0, e_1, e_2 >$ with $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \lambda e_1$, $ee_2 = \lambda e_2$, $e_0^2 = \alpha_0 e_1 + \alpha_1 e_2$, $e_1^2 = \beta_0 e_0$ and $e_2^2 = \beta_1 e_0$, $e_1e_2 = \beta_2 e_0$, $e_0e_1 = \gamma_0 e_0$, $e_0e_2 = \gamma_1 e_0$.

By using identities of Lemma 2.8 We have:

$0 = e_0^3 = (\alpha_0\gamma_0 + \alpha_1\gamma_1)e_0$, $0 = (e_0^2)^2 = (\alpha_0^2\beta_0 + \alpha_1^2\beta_1 + 2\alpha_0\alpha_1\beta_2)e_0$, $0 = (e_1^2)^2 = \alpha_0\beta_0^2 e_1 + \alpha_1\beta_0^2 e_2$, $0 = (e_2^2)^2 = \alpha_0\beta_1^2 e_1 + \alpha_1\beta_1^2 e_2$, $0 = (e_1e_2)^2 = \alpha_0\beta_2^2 e_1 + \alpha_1\beta_2^2 e_2$, $0 = (e_0e_1)^2 = \gamma_0^2\alpha_0 e_1 + \gamma_0^2\alpha_1 e_0$ and $0 = (e_0e_2)^2 = \gamma_1^2\alpha_0 e_1 + \gamma_1^2\alpha_1 e_0$.

Therefore, we have the following systems :

$$(\mathcal{S}_1) : \begin{cases} \alpha_0\beta_0 &= 0 \\ \alpha_0\beta_1 &= 0 \\ \alpha_0\beta_2 &= 0 \\ \alpha_0\gamma_0 &= 0 \\ \alpha_0\gamma_1 &= 0 \end{cases} \qquad (\mathcal{S}_2) : \begin{cases} \alpha_1\beta_0 &= 0 \\ \alpha_1\beta_1 &= 0 \\ \alpha_1\beta_2 &= 0 \\ \alpha_1\gamma_0 &= 0 \\ \alpha_1\gamma_1 &= 0 \end{cases}$$

We can thus distinguish the following cases :

Case 1 : $\alpha_0 \neq 0$ so $\beta_0 = \beta_1 = \beta_2 = \gamma_0 = \gamma_1 = 0$.

Let $x = e + \alpha e_0 + \beta e_1 + \gamma e_2$ an element of A such that $\omega(x) = 1$; then $x^2 = e + \alpha e_0 + (\alpha^2\alpha_0 + 2\lambda\beta)e_1 + (\alpha^2\alpha_1 + 2\lambda\gamma)e_2$, so $(x^2 - x)^2 = ((\alpha^2\alpha_0 + \beta(2\lambda - 1))e_1 + (\alpha^2\alpha_1 + \gamma(2\lambda - 1))e_2)^2 = 0$ and for any x in A, $x^2x^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$. We obtain an algebra satisfying a polynomial equation of degree 4.

Case 2 : $\alpha_0 = 0$ and $\alpha_1 \neq 0$.

We show that any element x in A satisfies the equation $x^2x^2 - 2\omega(x)x^3 + \omega(x)^2x^2 = 0$.

Case 3 : $\alpha_0 = \alpha_1 = 0$.

Let $x = e + \alpha e_0 + \beta e_1 + \gamma e_2$ an element of A such that $\omega(x) = 1$. We have $x^2 = e + (\alpha + \beta^2\beta_0 + \gamma^2\beta_1 + 2\alpha\beta\gamma_0 + 2\alpha\gamma\gamma_1 + 2\beta\beta_2\gamma)e_0 + 2\lambda\beta e_1 + 2\lambda\gamma e_2$; $(x^2 - 2\lambda x)^2 = (e(1-2\lambda) + (\alpha(1-2\lambda) + \beta^2\beta_0 + \gamma^2\beta_1 + 2\alpha\beta\gamma_0 + 2\alpha\gamma\gamma_1 + 2\beta\beta_2\gamma)e_0)^2 = (1-2\lambda)x^2 - 2\lambda(1-2\lambda)x$; so $x^2x^2 - 4\lambda x^3 - 3x^2 + 2(1+2\lambda)x = 0$. Thus A satisfies a polynomial equation of degree 4. The type of A can't therefore be $(2,0,2,0)$ or $(2,0,0,2)$.

Suppose the type of A is $(3,0,1,0)$, and setting $A_{1/2} =< e_0, e_1 >$, $A_\lambda =< e_2 >$, so $A =< e, e_0, e_1, e_2 >$, $e^2 = e$; $ee_0 = \frac{1}{2}e_0$; $ee_1 = \frac{1}{2}e_1$; $ee_2 = \lambda e_2$; $e_0^2 = \alpha_0 e_2$; $e_1^2 = \alpha_1 e_2$; $e_0e_1 = \alpha_2 e_2$; $e_2^2 = \beta_0 e_0 + \beta_1 e_1$; $e_0e_2 = \gamma_0 e_0 + \gamma_1 e_1$; $e_1e_2 = \mu_0 e_0 + \mu_1 e_1$.

Using equality $x_{1/2}^3 = 0$, we have : $e_0^3 = \alpha_0\gamma_0 e_0 + \alpha_0\gamma_1 e_1 = 0$; and $e_1^3 = \alpha_1\mu_0 e_0 + \alpha_1\mu_1 e_1 = 0$.

Also, the equality $(x_{1/2}^2)^2 = 0$ leads to $(e_0^2)^2 = \alpha_0^2\beta_0 e_0 + \alpha_0^2\beta_1 e_1 = 0$, and $(e_1^2)^2 = \alpha_1^2\beta_0 e_0 + \alpha_1^2\beta_1 e_1 = 0$.

The equality $y_{1/2}x_{1/2}^2 + 2x_{1/2}(x_{1/2}y_{1/2}) = 0$ (this identity is obtain by partial linearization of the identity $x_{1/2}^3 = 0$)leads to $e_1e_0^2 + 2e_0(e_1e_0) = (\alpha_0\mu_0 + 2\alpha_2\gamma_0)e_0 + (\alpha_0\mu_1 + 2\alpha_2\gamma_1)e_1 = 0$ and $e_0e_1^2 + 2e_1(e_0e_1) = (\alpha_1\gamma_0 + 2\alpha_2\mu_0)e_0 + (\alpha_1\gamma_1 + 2\alpha_2\mu_1)e_1 = 0$.

Finally, the equality $y_{1/2}^2 x_{1/2}^2 + 2(x_{1/2}y_{1/2})^2 = 0$ (obtain by linearizing partially the identity $(x_{1/2}^2)^2 = 0$) allows us to obtain $e_0^2 e_1^2 + 2(e_0 e_1)^2 = 2\alpha_2^2\beta_0 e_0 + 2\alpha_2^2\beta_1 e_1 = 0$.

From these previous equalities derive the following systems :

$$(\mathcal{S}_1) : \begin{cases} \alpha_0\gamma_0 = 0 & (1) \\ \alpha_0\gamma_1 = 0 & (2) \\ \alpha_1\mu_0 = 0 & (3) \\ \alpha_1\mu_1 = 0 & (4) \end{cases} \quad (\mathcal{S}_2) : \begin{cases} \alpha_0\beta_0 = 0 & (1) \\ \alpha_0\beta_1 = 0 & (2) \\ \alpha_1\beta_0 = 0 & (3) \\ \alpha_1\beta_1 = 0 & (4) \end{cases} \quad (\mathcal{S}_3) : \begin{cases} \alpha_0\mu_0 + 2\alpha_2\gamma_0 = 0 & (1) \\ \alpha_0\mu_1 + 2\alpha_2\gamma_1 = 0 & (2) \\ \alpha_1\gamma_0 + 2\alpha_2\mu_0 = 0 & (3) \\ \alpha_1\gamma_1 + 2\alpha_2\mu_1 = 0 & (4) \end{cases}$$

By multiplying (1) of $(\mathcal{S}_3)$ with γ_0 and using (1) of $(\mathcal{S}_1)$ we obtain $\alpha_2\gamma_0 = 0$ and $\alpha_0\mu_0 = 0$.

By multiplying (2) of $(\mathcal{S}_3)$ with γ_1 then let's use (2) of $(\mathcal{S}_1)$ we obtain $\alpha_2\gamma_1 = 0$ and $\alpha_0\mu_1 = 0$.

If we multiply (3) of $(\mathcal{S}_3)$ by μ_0 then we use (3) of $(\mathcal{S}_1)$ we have $\alpha_2\mu_0 = 0$ and $\alpha_1\gamma_0 = 0$.

Finally, the product of (4) of the system $(\mathcal{S}_3)$ par μ_1 and using (4) de $(\mathcal{S}_1)$ we obtain $\alpha_2\mu_1 = 0$ and $\alpha_1\gamma_1 = 0$.

This allows us to obtain the three new systems :

$$(\mathcal{S}'_1) : \begin{cases} \alpha_0\beta_0 = 0 & (1) \\ \alpha_0\beta_1 = 0 & (2) \\ \alpha_0\gamma_0 = 0 & (3) \\ \alpha_0\gamma_1 = 0 & (4) \\ \alpha_0\mu_0 = 0 & (5) \\ \alpha_0\mu_1 = 0 & (6) \end{cases} \quad (\mathcal{S}'_2) : \begin{cases} \alpha_1\beta_0 = 0 & (1) \\ \alpha_1\beta_1 = 0 & (2) \\ \alpha_1\gamma_0 = 0 & (3) \\ \alpha_1\gamma_1 = 0 & (4) \\ \alpha_1\mu_0 = 0 & (5) \\ \alpha_1\mu_1 = 0 & (6) \end{cases} \quad (\mathcal{S}'_3) : \begin{cases} \alpha_2\beta_0 = 0 & (1) \\ \alpha_2\beta_1 = 0 & (2) \\ \alpha_2\gamma_0 = 0 & (3) \\ \alpha_2\gamma_1 = 0 & (4) \\ \alpha_2\mu_0 = 0 & (5) \\ \alpha_2\mu_1 = 0 & (6) \end{cases}$$

We distinguish two cases :

Case 1 : $\alpha_0 \neq 0$ so $\beta_0 = \beta_1 = \beta_2 = \gamma_0 = \gamma_1 = \mu_0 = \mu_1 = 0$.

Let $x = e + \alpha e_0 + \beta e_1 + \gamma e_2$, an element of A such that $\omega(x) = 1$. So $x^2 = e + \alpha e_0 + \beta e_1 + (\alpha^2\alpha_0 + \beta^2\alpha_1 + 2\alpha\beta\rho_0)e_2 = (x^2)^2$. In this case, A verifies the Bernstein algebras identity.

Case 2 : $\alpha_0 = \alpha_1 = \alpha_1 = 0$.

Let $x = e + \alpha e_0 + \beta e_1 + \gamma e_2$, an element of A such that $\omega(x) = 1$. We have $(x^2 - 2\lambda x)^2 = (1 - 2\lambda)(x^2 - 2\lambda x)$ and A satisfies a train polynomial of degree 4.

Case 3 : $\alpha_0 = 0$ and $\alpha_1 \neq 0$. Then $\beta_0 = \beta_1 = \gamma_0 = \gamma_1 = \mu_0 = \mu_1$.

Let $x = e + \alpha e_0 + \beta e_1 + \gamma e_2$, an element of A such that $\omega(x) = 1$. We have $x^2 = e + \alpha e_0 + \beta e_1 + (\beta^2\alpha_1 + 2\alpha\beta\alpha_2 + 2\lambda\gamma)e_2$, so $(x^2 - x)^2 = 0$ and A satisfies a train polynomial of degree 4.

Case 4 : $\alpha_0 = 0$ and $\alpha_2 \neq 0$. Similar to previous case.

To conclude, the type of a train algebra of degree 2 and exponent 4 can't be $(3, 0, 1, 0)$ or $(3, 0, 0, 1)$.

Suppose the type of A is $(3, 1, 0, 0)$, and setting $A_{1/2} =< e_0, e_1 >$, $A_0 =< e_2 >$ so $A =< e, e_0, e_1, e_2 >$, $e^2 = e$, $ee_0 = \frac{1}{2}e_0$, $ee_1 = \frac{1}{2}e_1$, $ee_2 = 0$, $e_0^2 = \alpha_0 e_2$,

$e_1^2 = \alpha_1 e_2$, $e_2^2 = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$, $e_0 e_2 = \gamma_0 e_0 + \gamma_1 e_1$, $e_1 e_2 = \mu_0 e_0 + \mu_1 e_1$, $e_0 e_1 = \rho_0 e_2$.

Using the equality $x_{1/2}^3 = 0$, we have $e_0^3 = \alpha_0\gamma_0 e_0 + \alpha_0\gamma_1 e_1 = 0$ and $e_1^3 = \alpha_1\mu_0 e_0 + \alpha_1\mu_1 e_1 = 0$.

The equality $(x^2)^2 = 0$ give $(e_1^2)^2 = \alpha_1^2\beta_0 e_0 + \alpha_1^2\beta_1 e_1 + \alpha_1^2\beta_2 e_2 = 0$ and $(e_0^2)^2 = \alpha_0^2\beta_0 e_0 + \alpha_0^2\beta_1 e_1 + \alpha_0^2\beta_2 e_2 = 0$.

The equality $y_{1/2}x_{1/2}^2 + 2x_{1/2}(x_{1/2}y_{1/2}) = 0$ give $e_1 e_0^2 + 2e_0(e_0 e_1) = (\alpha_0\mu_0 + 2\rho_0\gamma_0)e_0 + (\alpha_0\mu_1 + 2\rho_0\gamma_1)e_1 = 0$ and $e_0 e_1^2 + 2e_1(e_0 e_1) = (\alpha_1\gamma_0 + 2\rho_0\mu_0)e_0 + (\alpha_1\gamma_1 + 2\rho_0\mu_1)e_1 = 0$.

And the equality $x_{1/2}^2 y_{1/2}^2 + 2(x_{1/2}y_{1/2})^2 = 0$ give $e_0^2 e_1^2 + 2(e_0 e_1)^2 = 2(e_0 e_1)^2 = 2\rho_0^2\beta_0 e_0 + 2\rho_0^2\beta_1 e_1 + 2\rho_0^2\beta_2 e_2 = 0$.

It follows the systems :

$$(\mathcal{S}_1): \begin{cases} \alpha_0\beta_0 = 0 & (1)\\ \alpha_0\beta_1 = 0 & (2)\\ \alpha_0\beta_2 = 0 & (3)\\ \alpha_0\gamma_0 = 0 & (4)\\ \alpha_0\gamma_1 = 0 & (5)\\ \alpha_0\mu_0 = 0 & (6)\\ \alpha_0\mu_1 = 0 & (7) \end{cases} \quad (\mathcal{S}_2): \begin{cases} \alpha_1\beta_0 = 0 & (1)\\ \alpha_1\beta_1 = 0 & (2)\\ \alpha_1\beta_2 = 0 & (3)\\ \alpha_1\gamma_0 = 0 & (4)\\ \alpha_1\gamma_1 = 0 & (5)\\ \alpha_1\mu_0 = 0 & (6)\\ \alpha_1\mu_1 = 0 & (7) \end{cases} \quad (\mathcal{S}_3): \begin{cases} \rho_0\beta_0 = 0 & (1)\\ \rho_0\beta_1 = 0 & (2)\\ \rho_0\beta_2 = 0 & (3)\\ \rho_0\gamma_0 = 0 & (4)\\ \rho_0\gamma_1 = 0 & (5)\\ \rho_0\mu_0 = 0 & (6)\\ \rho_0\mu_1 = 0 & (7) \end{cases}$$

Consider then the following cases: case 1: $\alpha_0 \neq 0$ so $\beta_0 = \beta_1 = \beta_2 = \gamma_0 = \gamma_1 = \mu_0 = \mu_1 = 0$; case 2: $\alpha_0 = \alpha_1 = \rho_0 = 0$; case 3: $\alpha_0 = 0$ and $\alpha_1 \neq 0$; case 4: $\alpha_0 = 0$ and $\rho_0 \neq 0$. As for the previous type, we show in each of these cases that A satisfies a polynomial equation of degree 4. □

From the Propositions 3.1 and 3.2 we deduce that the possible types of A in dimension 4 are : $(2, 2, 0, 0)$, $(2, 1, 0, 1)$, $(2, 1, 1, 0)$ and $(2, 0, 1, 1)$.

Acknowledgement. The authors thank the referee for the relevant suggestions which have improved the document.

References

1. M. T. Alcalde, C. Burgueño, A. Labra and A. Micali, *Sur les algèbres de Bernstein. (On Bernstein algebras)*, Proc. Lond. Math. Soc. (3), **58**(1), (1989), 51–68. https://doi.org/10.1112/plms/s3-58.1.51
2. I. Basso, R. Costa, J. Carlos Gutiérrez, and H. Guzzo Jr., *Cubic algebras of exponent 2 : basic properties*, Int. J. Math. Game Theory and Algebra **9**(4), (1999) 245–258.
3. J. Bayara, A. Conseibo, M. Ouattara and F. Zitan *Power associative algebra that are train algebra*, Journal of Algebra **324** (2010). https://doi.org/10.1016/j.jalgebra.2010.06.012
4. J. Bayara, A. Conseibo, M. Ouattara, and A. Micali *Train algebras of degree 2 and exponent 3*, Discrete and continous dynamical systems series, **4**(6), (2011). https://doi.org/10.3934/dcdss.2011.4.1371

5. S. Bernstein, *Démonstration mathématique de la loi d'hérédité de Mendel*, Compt. Rend. Acad. Sci. Paris 177, (1923), 528–531.
6. S. Bernstein, *Principe de stationarit6 et généralisation de la loi de Mendel*, Compt. Rend. Acad. Sci. Paris 177, (1923), 581–584.
7. A. Dembega, A. Konkobo and M. Ouattara, *Derivations and dimensionally nilpotent derivations in Lie triple algebras*, Gulf Journal of Mathematics, **7**(2), (2019), 71–84. https://doi.org/10.56947/gjom.v7i2.192
8. I.M.H. Etherington, *Commutative train algebras of ranks* 2 *and* 3, J.London. Math. soc, **15**, (1940), 136–149. Corrigendum. ibid, **20**, (1945), 238.
9. H. Guzzo Jr., P. Vicente *Train algebras of rank n which are Bernstein or power-associative algebras*, Nova J. Math. Game Theory Algebra **6**(2-3), (1997), 103–112.
10. P. Holgate, *Selfing in genetic algebras*, J. Math. Biol, **6**, (1978), 197–206. https://doi.org/10.1007/BF02450789
11. P. Holgate, *Genetic algebras satisfying Bernstein's stationarity principle*, J. Lond. Math. Soc., II. Ser., **9**, (1975), 613–623.
12. A. Micali, M. Ouattara *Structure des algèbres de Bernstein*, Linear Algebra Appl., **218**, (1995), 77–88. https://doi.org/10.1016/0024-3795(93)00159-W
13. C. Malol, A. Micali et M. Ouattara *sur les algèbres de Bernstein IV*, Linear Algebra Appl., **158**, (1991), 1–26. https://doi.org/10.1016/0024-3795(91)90048-2
14. M. Ouattara *Progrès récents dans la théorie des algèbres de Bernstein*, Matemática Comtemporânea, **6**, (1994), 197–213.
15. M. Ouattara, S. Savadogo, *Evolution train algebras*, Gulf Journal of Mathematics, **8**(1), (2020), 37–51. https://doi.org/10.56947/gjom.v8i1.299
16. R. D. Schafer, *An introduction to nonassociative algebras*, Academic Press, New York, 1966.
17. A. Wörz-Busekros, *Bernstein Algebras*, Arch. Math., **48**, (1987), 388–398. https://doi.org/10.1007/BF01189631
18. A. Wörz-Busekros, *Algebras in Genetics*, Lecture Notes in Biomathematics, **36**, Springer-Verlag, Berlin-New York, (1980).

[1] Département de Mathématiques et Informatique, Université Norbert ZONGO, BP 376 Koudougou, Burkina Faso.
Email address: achilzangre@gmail.com

[2] Département de Mathématiques et Informatique, Université Norbert ZONGO, BP 376 Koudougou, Burkina Faso.
Email address: andreconsebo@yahoo.fr