Gulf Journal of Mathematics ISSN: 2309-4966
Vol 13, Issue 1 (2022) 54-87 https://doi.org/10.56947/gjom.v13i1.927

ON EIGENSURFACE OF p-BIHARMONIC OPERATOR AND ASSOCIATED CONCAVE-CONVEX TYPE EQUATION

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Abstract. In this article, we are interested in the simplicity and the existence of the first eigensurface for the third order spectrum of p-biharmonic operator plus potential with weights and the existence of multiple solutions for associated concave-convex type equation under Navier boundary conditions.

1. Introduction

Our aim is to study the following eigensurface problems:

$$\textbf{Problem1}\begin{cases}\text{Find}\quad (u,\beta,\lambda)\in X\setminus\{0\}\times\mathbb{R}^N\times\mathbb{R} \text{ such that}\\ \Delta_p^2 u+H(\beta,u)+V(x)|u|^{p-2}u=\lambda m(x)|u|^{p-2}u \text{ in } \Omega\\ u=\Delta u=0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega\end{cases}\tag{1.1}$$

where $H(\beta,u)=2\beta\cdot\nabla(|\Delta u|^{p-2}\Delta u)+|\beta|^2|\Delta u|^{p-2}\Delta u$, $X:=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$, Ω is a bounded smooth domain of $\mathbb{R}^N$ (with $N\geq 1$) with $\partial\Omega$ its boundary, V, $m\in L^\infty(\Omega)$ are functions possibly indefinite such that $m^+\not\equiv 0$ and Δ_p^2 denotes the p-biharmonic operator defined by $\Delta_p^2 u=\Delta(|\Delta u|^{p-2}\Delta u)$ (with $1<p<\infty$) and associated concave-convex problem

$$\textbf{Problem2}\begin{cases}\text{Find } (u,\beta,\lambda)\in X\setminus\{0\}\times\mathbb{R}^N\times\mathbb{R}_+^* \text{ such that}\\ \Delta_p^2 u+H(\beta,u)+V(x)|u|^{p-2}u=\lambda a(x)|u|^{r-2}u+b(x)|u|^{q-2}u \text{ in } \Omega\\ u=\Delta u=0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega\end{cases}\tag{1.2}$$

where a, $b\in L^\infty(\Omega)$ are functions possibly indefinite, $1<r<p<q<p^*$ with

$$p^*=\begin{cases}\frac{2N}{N-2p} \text{ if } p<\frac{N}{2}\\ +\infty \text{ if } p\geq\frac{N}{2}.\end{cases}$$

The problem (1.1) was considered by P. Drábek and M. Ôtani for $\beta=0_{\mathbb{R}^N}$, $V\equiv 0$ and $m\equiv 1$ [9]. By using a transformation of the problem to a known Poisson problem, they showed that (1.1) has a principal positive eigenvalue which is simple and isolated. In the case $N=1$ they gave a description of all eigenvalues

Date: Received: Sep 14, 2021; Accepted: Jan 26, 2022.

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2010 *Mathematics Subject Classification.* Primary 35J35, 35J40; Secondary 35J60, 35P30.

Key words and phrases. Nonlinear eigenvalue problem, p-biharmonic operator plus a potential, eigensurface, Concave-convex problem, Indefinite weights.

and associated eigenfunctions. A similar problem is considered in [2] in a context of $p(x)$-biharmonic operator where the existence of solutions is established using an approach based on the Minty-Browder theorem applied to compact operators and operators of (S_+) type. There have been increasing interest in the spectrum of p-biharmonic operator under various boundary conditions.

In [20] M. Talbi and N. Tsouli considered the spectrum of the weighted p-biharmonic operator with weights and showed that the following eigenvalue problem

$$\begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u \text{ in } \Omega \\ u = \Delta u = 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

where $\rho \in C(\bar{\Omega})$ and $\rho > 0$, possesses at least one non-decreasing sequence of eigenvalues and studied the one dimensional case. The authors, in the same reference gave the first eigenvalue λ_1 and showed that if $m \geq 0$ a.e., λ_1 is simple and associated to positive eigenfunction. Also they showed that if $m \in C(\bar{\Omega})$, λ_1 is isolated and every positive or negative eigenfunction is associated to λ_1.

The authors in [15] have recently focused on the spectrum of the weighted p-biharmonic operator under mixed Dirichlet-Steklov boundary conditions and similar results are proved applying both variational method and Ljusternick-Schnirelmann principle.

Recently in the particular case of (1.1) where $V \equiv 0$ and $m \geq 0$, the authors in [11], have showed that the spectrum of problem (1.1) contains at least one sequence of positive eigensurfaces. The authors in [12] have showed that the first eigensurface, in the same case, is simple and associated to positive eigenfunction. Also they showed that if $m \in C(\bar{\Omega})$ the first eigensurface is isolated and associated to every positive or negative eigenfunction.

In the case $\beta = 0_{\mathbb{R}^N}$, $V \equiv 0$ (or $V = c \in \mathbb{R}$), problem (1.2) has been studied extensively in the literature (see for example: [7], [14], [18], [22]) and remains a challenge considering more general conditions as we are concerned with in this manuscript.

The space $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endorses with the equivalent norm $\|u\| := \|\Delta u\|_p$ is a reflexive Banach space. Here $\|v\|_p$, for $v \in L^p(\Omega)$, denotes the L^p-norm of v. The weak convergence in X is denoted by $\rightharpoonup$. The positive and negative part of a function u are denoted by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

For all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) \text{ in } \Omega \\ u = 0 \quad\;\; \text{on } \partial\Omega \end{cases}$$

is uniquely solvable in X (see for example [13]). We denote by Λ the inverse operator of $-\Delta : X \longrightarrow L^p(\Omega)$. The following lemma gives us some properties of the operator Λ that will be needed:

Lemma 1.1. *([9, 20]).*

(1) *(Continuity) There exists a constant* $c_p > 0$ *such that*

$$\|\Lambda f\|_{W^{2,p}} \leq c_p \|f\|_p$$

holds for all $p \in (1, \infty)$ *and* $f \in L^p(\Omega)$.

(2) *(Continuity) Given* $k \in \mathbb{N}^*$, *there exists a constant* $c_{p,k} > 0$ *such that*

$$\|\Lambda f\|_{W^{k+2,p}} \leq c_{p,k}\|f\|_{W^{k,p}}$$

holds for all $p \in (1, \infty)$ *and* $f \in W^{k,p}(\Omega)$.

(3) *(Symmetry) The identity*

$$\int_\Omega \Lambda u.v dx = \int_\Omega u.\Lambda v dx$$

holds for $u \in L^p(\Omega)$ *and* $u \in L^{p'}(\Omega)$ *with* $p \in (1, \infty)$ *and* $p' = \frac{p}{p-1}$ *is the conjugate of* p.

(4) *(Regularity) Given* $f \in L^\infty(\Omega)$, *we have* $\Lambda f \in C^{1,\alpha}(\bar{\Omega})$ *for all* $\alpha \in (0, 1)$. *Moreover, there exists* $c_\alpha > 0$ *such that*

$$\|\Lambda f\|_{C^{1,\alpha}(\Omega)} \leq c_\alpha \|f\|_\infty.$$

(5) *(Regularity and Hopf-type maximum principle) Let* $f \in C(\bar{\Omega})$ *and* $f \geq 0$ *then* $w = \Lambda f \in C^{1,\alpha}(\bar{\Omega})$, *for all* $\alpha \in (0, 1)$ *and* w *satisfies:* $w > 0$ *in* Ω, $\frac{\partial w}{\partial n} < 0$ *on* $\partial\Omega$.

(6) *(Order preserving property) Given* f *and* g *in* $L^p(\Omega)$, *if* $f \leq g$ *in* Ω *then* $\Lambda f < \Lambda g$ *in* Ω.

This paper is organized as follows. In section 2 we study an eigencurve associated to Problem1. In section 3 we show that Problem1 has principal eigensurfaces under some suitable conditions. In section 4 we describe the Nehari manifold, the fibering map and the different sign-subsets of the Nehari manifold that will be used to find critical point of energy functional for Problem2. In section 5 we prove existence and multiplicity results for Problem2.

2. An eigenvalue curve associated to Problem1

It is well established that (see for example [5], [6], [8], [17]), in order to prove the existence of principal eigensurfaces of (1.1), one fixes $\lambda \in \mathbb{R}$ and embeds the problem into the new eigenvalue problem of parameter μ:

$$\begin{cases} \text{Find } (u, \beta, \lambda) \in X \setminus \{0\} \times \mathbb{R}^N \times \mathbb{R} \text{ such that} \\ \Delta_p^2 u + H(\beta, u) + V(x)|u|^{p-2}u - \lambda m(x)|u|^{p-2}u = \mu|u|^{p-2}u \text{ in } \Omega \\ u = \Delta u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.1. *The problem* (1.1) *(resp.* (2.1)*) is equivalent to the problem*

$$\begin{cases} \text{Find } (u, \lambda) \in X \setminus \{0\} \times \mathbb{R} \text{ such that} \\ \Delta_p^{2,\beta} u + V(x)e^{\beta\cdot x}|u|^{p-2}u = \lambda m(x)e^{\beta\cdot x}|u|^{p-2}u \text{ in } \Omega \\ u = \Delta u = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{2.2}$$

$$\left(\text{resp.} \begin{cases} \text{Find } (u, \mu) \in X \setminus \{0\} \times \mathbb{R} \text{ such that} \\ \Delta_p^{2,\beta} u + (V - \lambda m)e^{\beta\cdot x}|u|^{p-2}u = \mu e^{\beta\cdot x}|u|^{p-2}u \text{ in } \Omega \\ u = \Delta u = 0 \qquad \text{on } \quad \partial\Omega \end{cases} \right) \tag{2.3}$$

where $\Delta_p^{2,\beta} u = \Delta(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u)$.

Proof. The ideas of the proof follow those of Lemma 3.2 in [11]. □

Definition 2.2. .

(1) The set of couples $(\beta,\lambda) \in \mathbb{R}^N\times\mathbb{R}$ $\big($resp. $(\beta,\mu)\in\mathbb{R}^N\times\mathbb{R}\big)$ such that there exists a solution $(u,\beta,\lambda)\in X\times\mathbb{R}^N\times\mathbb{R}$ $\big($resp. $(u,\beta,\mu)\in X\times\mathbb{R}^N\times\mathbb{R}\big)$ of (1.1) (resp. (2.1)) is called the third-order spectrum of the p-biharmonic operator plus potential. The couple (β,λ) (resp. (β,μ)) is then called a third-order eigenvalue and u is said to be the associated eigenfunction of (1.1) (resp. (2.1)). Moreover, a set of third-order eigenvalues of the form $(\beta, f(\beta))$, for $\beta\in\mathbb{R}^N$ and some function $f:\mathbb{R}^N\longrightarrow\mathbb{R}$, is called an eigensurface.

(2) For $\beta\in\mathbb{R}^N$, $(u,\mu)\in X\times\mathbb{R}$ (resp. $(u,\lambda)\in X\times\mathbb{R}$) is a weak solution to problem (2.3) (resp. (2.2)) if

$$\int_\Omega e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\Delta\varphi dx+\int_\Omega (V-\lambda m)e^{\beta\cdot x}|u|^{p-2}u\varphi dx=\mu\int_\Omega e^{\beta\cdot x}|u|^{p-2}u\varphi dx, \tag{2.4}$$

$$\left(\text{resp. } \int_\Omega e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\Delta\varphi dx+\int_\Omega Ve^{\beta\cdot x}|u|^{p-2}u\varphi dx=\lambda\int_\Omega me^{\beta\cdot x}|u|^{p-2}u\varphi dx\right) \tag{2.5}$$

$\forall\varphi\in X$.

(3) If $(u,\mu(\beta,\lambda))\in X\times\mathbb{R}$ (resp. $(u,\lambda(V,\beta,m))\in X\times\mathbb{R}$) is a weak solution to problem (2.3) (resp. (2.2)), u shall be called an eigenfunction of the problem (2.3) (resp. (2.2)) associated to the eigenvalue $\mu(\beta,\lambda)$ (resp. $\lambda(V,\beta,m)$).

So u shall be called an eigenfunction of the problem (1.1) (resp. (2.1)) associated to the eigensurface $\lambda(V,.,m)$ (resp. $\mu(.,\lambda)$).

(4) By a principal eigensurface (or a principal eigenvalue) we mean an eigensurface (or an eigenvalue) which is associated to a positive eigenfunction or negative eigenfunction.

We are going to consider the smallest eigenvalue $\mu(\beta,\lambda)\in\mathbb{R}$ of problem (2.3). In order to do so, we define the energy functional

$$\begin{array}{rcl} J_{\lambda,\beta}: X & \longrightarrow & \mathbb{R}\\ u & \longmapsto & J_{\lambda,\beta}(u)=E_{V,\beta}(u)-\lambda M_\beta(u)\end{array}$$

where

$$E_{V,\beta}(u) = \int_\Omega e^{\beta\cdot x}|\Delta u|^p dx+\int_\Omega Ve^{\beta\cdot x}|u|^p dx$$

and

$$M_\beta(u) = \int_\Omega me^{\beta\cdot x}|u|^p dx.$$

$M_\beta, E_{V,\beta}$ and $J_{\lambda,\beta}$ are of class C^1 on X and for all $u \in X$

$$\begin{aligned}
\langle E'_{V,\beta}(u), \varphi \rangle &= p\int_\Omega e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u \Delta\varphi dx + p\int_\Omega V e^{\beta\cdot x}|u|^{p-2}u\varphi dx,\\
\langle M'_\beta(u), \varphi \rangle &= p\int_\Omega m e^{\beta\cdot x}|u|^{p-2}u\varphi dx,\\
\langle J'_{\lambda,\beta}(u), \varphi \rangle &= p\int_\Omega e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u \Delta\varphi dx + p\int_\Omega V e^{\beta\cdot x}|u|^{p-2}u\varphi dx\\
&- p\lambda\int_\Omega m e^{\beta\cdot x}|u|^{p-2}u\varphi dx,
\end{aligned}$$

for all $\varphi \in X$.

Lemma 2.3. *Let $w, V \in L^\infty(\Omega)\setminus\{0\}$. If $w > 0$ on Ω and $\beta \in \mathbb{R}^N$ then there exist two positive constants c_1, c_2 such that*

$$\|u\|^p \leq c_1 E_{V,\beta}(u) + c_2\int_\Omega w e^{\beta\cdot x}|u|^p dx, \quad \forall u \in X.$$

Proof. We can easily adapt the proof made in [8]. □

Let us now consider the manifold

$$S = \{u \in X : I_\beta(u) = 1\}$$

with

$$I_\beta(u) := \int_\Omega e^{\beta\cdot x}|u|^p dx, \quad \forall u \in X.$$

In the following, we construct for each $\lambda \in \mathbb{R}$, the smallest eigensurface $\beta \in \mathbb{R}^N \longmapsto \mu_1(\beta,\lambda)$ from which we get the eigenvalue curve $\lambda \in \mathbb{R} \longmapsto \mu_1(\beta,\lambda)$.

Proposition 2.4. *$\mu_1(.,\lambda)$ is the smallest eigensurface of (2.1) with*

$$\mu_1(\beta,\lambda) := \inf\{J_{\lambda,\beta}(u) : u \in S\} \tag{2.6}$$

for $\beta \in \mathbb{R}^N$.

Proof. By Lemma 2.3 we have that $\mu_1(\beta,\lambda) \in \mathbb{R}$ for all $\beta \in \mathbb{R}^N$. Furthermore any minimizing sequence (u_n) is bounded. Thus there exists $u_0 \in X$ such that, up to a subsequence, (u_n) converges weakly to u_0 in X and strongly in $L^p(\Omega)$. Hence

$$J_{\lambda,\beta}(u_0) \leq \lim_{n\longrightarrow\infty} J_{\lambda,\beta}(u_n) = \mu_1(\beta,\lambda), \quad u_0 \in S$$

and consequently $J_{\lambda,\beta}(u_0) = \mu_1(\beta,\lambda)$ for all $\beta \in \mathbb{R}^N$. By the Lagrange multipliers rule, $\mu_1(\beta,\lambda)$ is an eigenvalue for (2.3) and u_0 is an associated eigenfunction. Consequently $\mu_1(.,\lambda)$ is an eigensurface for (2.1) and u_0 is an associated eigenfunction.

Moreover for any eigensurface $\mu(.,\lambda)$ associated to $u \in X\setminus\{0\}$,

$$J_{\lambda,\beta}(u) = \mu(\beta,\lambda)\int_\Omega e^{\beta\cdot x}|u|^p dx$$

and

$$\begin{aligned}\mu_1(\beta,\lambda) &\leq J_{\lambda,\beta}\left(\frac{u}{\left[\int_\Omega e^{\beta\cdot x}|u|^p dx\right]^{\frac{1}{p}}}\right)\\ &= \frac{J_{\lambda,\beta}(u)}{\int_\Omega e^{\beta\cdot x}|u|^p dx}\\ &= \mu(\beta,\lambda)\end{aligned}$$

for all $\beta \in \mathbb{R}^N$. We finally conclude that $\mu_1(.,\lambda)$ is the smallest eigensurface of (2.1). $\square$

3. Principal eigensurfaces

Our purpose is to find a reasonable assumption on V and m so that there exists at least one $\lambda \in \mathbb{R}$ such that $\mu_1(.,\lambda) = 0$. Indeed, it is straightforward proving the following:

Lemma 3.1. *$\mu_1(.,\lambda(V,.,m)) = 0$ if and only if $\lambda(V,.,m)$ is an eigensurface of (2.2).*

We will denote

$$\begin{aligned}L(\Omega) &:= [L^p(\Omega)\setminus\{0\}]\times\mathbb{R},\\ L_0(\Omega) &:= [L^p(\Omega)\setminus\{0\}]\times\{0\}.\end{aligned}$$

We apply some results proved by P. Drábek and M. Ôtani [9] and some ideas used by M. Talbi and N. Tsouli [20].

Remark 3.2. .

(1) $\forall u \in X$, $\forall v \in L^p(\Omega)$: $v = -\Delta u \Longleftrightarrow u = \Lambda v$.

(2) Let N_p be the Nemytskii operator defined by

$$N_p(u)(x) = \begin{cases}|u(x)|^{p-2}u(x) & if \quad u(x)\neq 0\\ 0 & if \quad u(x) = 0.\end{cases}$$

We have

$$\forall v \in L^p(\Omega), \quad \forall w \in L^{p'}(\Omega): \quad N_p(v) = w \Longleftrightarrow v = N_{p'}(w)$$

with $p' = \frac{p}{p-1}$.

(3) If u is an eigenfunction of (2.3) associated to μ then $v = -\Delta u$ satisfies:

$$e^{\beta\cdot x}N_p(v) + \Lambda(Ve^{\beta\cdot x}N_p(\Lambda v)) - \lambda\Lambda(me^{\beta\cdot x}N_p(\Lambda v)) = \mu\Lambda(e^{\beta\cdot x}N_p(\Lambda v)).$$

We have, $v = N_{p'}(e^{-\beta\cdot x}\Lambda[m_1 N_p(\Lambda v)])$ where $m_1 = (\mu+\lambda m - V)e^{\beta\cdot x}$. Hence

(a) (u,μ) is a solution of (2.3) if and only if (v,μ) is a solution of problem

$$\begin{cases}Find \quad (v,\mu)\in L(\Omega) \quad such \quad that\\ e^{\beta\cdot x}N_p(v) = \Lambda\left([\mu+\lambda m - V]e^{\beta\cdot x}N_p(\Lambda v)\right)\end{cases} \tag{3.1}$$

with $v = -\Delta u$.

(b) $(v,\mu_1(\lambda)) \in L_0(\Omega)$ is a solution of (3.1) if and only if (v,λ) is a solution of problem

$$\begin{cases} Find \quad (v,\lambda) \in L(\Omega) \quad such \quad that \\ e^{\beta\cdot x} N_p(v) = \Lambda\left([\lambda m - V] e^{\beta\cdot x} N_p(\Lambda v)\right). \end{cases} \tag{3.2}$$

(c) $\mu_1(\beta,\lambda) := \inf\left\{ F_{\lambda,\beta}(v) : \quad v \in L^p(\Omega), \int_\Omega e^{\beta\cdot x}|\Lambda v|^p dx = 1 \right\}$ where

$$F_{\lambda,\beta}(v) \;=\; \int_\Omega e^{\beta\cdot x}|v|^p dx + \int_\Omega V e^{\beta\cdot x}|\Lambda v|^p dx - \lambda \int_\Omega m e^{\beta\cdot x}|\Lambda v|^p dx.$$

$F_{\lambda,\beta}$ is of class C^1 in $L^p(\Omega)$ and for all $v \in L^p(\Omega)$,

$$\begin{aligned} \langle F'_{\lambda,\beta}(v), w\rangle &= p\int_\Omega e^{\beta\cdot x}|v|^{p-2} v w dx + p\int_\Omega V e^{\beta\cdot x}|\Lambda v|^{p-2}\Lambda v \Lambda w dx \\ &- p\lambda \int_\Omega m e^{\beta\cdot x}|\Lambda v|^{p-2}\Lambda v\Lambda w dx, \quad \forall w \in L^p(\Omega). \end{aligned}$$

Lemma 3.3. *If (u,μ) is a solution of (2.3), then $u \in C^{1,\alpha}(\bar\Omega)$ for all $\alpha \in (0,1)$ and $\Delta u \in C(\bar\Omega)$.*

Proof. Let $(u,\mu) \in X\times\mathbb{R}$ be a solution of problem (2.3), then (v,μ) is a solution of problem (3.1) with $v = -\Delta u = e^{\frac{-\beta\cdot x}{p-1}} N_{p'}(\Lambda[m_1 N_p(\Lambda v)]) \in L^p(\Omega)$ and $m_1 = (\mu + \lambda m - V)e^{\beta\cdot x} \in L^\infty(\Omega)$. The authors M. Talbi and N. Tsouli have showed in [20] that $\lambda^{\frac{1}{p-1}}\rho^{\frac{-1}{p-1}} N_{p'}(\Lambda[m N_p(\Lambda v)]) \in C(\bar\Omega)$. The proof for $e^{\frac{-\beta\cdot x}{p-1}} N_{p'}(\Lambda[m_1 N_p(\Lambda v)]) \in C(\bar\Omega)$ can be done in the same way. Hence we deduce that $v \in L^\infty(\Omega)$ and from the assertion in Lemma 1.1, we have $u = \Lambda v \in C^{1,\alpha}(\bar\Omega)$ for all $\alpha \in (0,1)$. □

Lemma 3.4. *$(v_1,\mu_1(\beta,\lambda)) \in L(\Omega)$ is a solution of problem (3.1), if and only if*

$$G_{\lambda,\beta}(v_1) = 0 = \min_{v\in L^p(\Omega)\setminus\{0\}} G_{\lambda,\beta}(v)$$

where

$$G_{\lambda,\beta}(v) \;=\; F_{\lambda,\beta}(v) - \mu_1 \int_\Omega e^{\beta\cdot x}|\Lambda v|^p dx.$$

Proof. An easy adaptation of the proof of Lemma 3.2 in [12]. □

Lemma 3.5. *Assume that $\lambda(V,\beta,m)$ is an eigenvalue of problem (3.2). If $V - \lambda(V,\beta,m)m \geq 0$ a.e. in Ω, then $\lambda(V,\beta,m)$ is non principal.*

Proof. Assume by contradiction that $\lambda(V,\beta,m)$ is a principal eigenvalue of problem (3.2) associated to $v > 0$ in Ω such that $V - \lambda(V,\beta,m)m \geq 0$ a.e. in Ω. Then we have:

$$\begin{aligned} v > 0 \;&\Longrightarrow\; \Lambda v > 0, \\ &\Longrightarrow\; [\lambda(V,\beta,m)m - V]e^{\beta\cdot x}N_p(\Lambda v) \leq 0, \\ &\Longrightarrow\; \Lambda\left([\lambda(V,\beta,m)m - V]e^{\beta\cdot x}N_p(\Lambda v)\right) \leq 0, \\ &\Longrightarrow\; e^{\frac{-\beta\cdot x}{p-1}} N_{p'}\left(\Lambda\left[(\lambda(V,\beta,m)m - V)\, e^{\beta\cdot x} N_p(\Lambda v)\right]\right) = v \leq 0, \end{aligned}$$

a contradiction. □

Remark 3.6. We can only look for the principal eigenvalue of the problem (3.2) in the set

$$\{\lambda \in \mathbb{R} : \quad V - \lambda m \leq 0 \quad in \quad \Omega\}. \tag{3.3}$$

We may now assume the following conditions:

$$(H_\lambda) : V - \lambda m \not\equiv 0 \quad and \quad (V - \lambda m)^+ \equiv 0. \tag{3.4}$$

Lemma 3.7. *If (H_λ) holds and $(v_1, \mu_1(\beta, \lambda)) \in L_0(\Omega)$ is a solution of problem (3.1) then $(|v_1|, \mu_1(\beta, \lambda))$ is a solution of problem (3.1).*

Proof. Assume that (H_λ) holds and $(v_1, \mu_1(\beta, \lambda)) \in L_0(\Omega)$ is a solution of problem (3.1). Then $G_{\lambda,\beta}(v_1) = 0$, $\mu_1(\beta, \lambda) = 0$, $V - \lambda m \leq 0$ and $G_{\lambda,\beta}(|v_1|) \geq 0$. For every $v \in L^p(\Omega)$ we have $\Lambda|v| \geq |\Lambda v|$. Then

$$\begin{aligned} \Lambda|v_1| \geq |\Lambda v_1| \quad &\Longrightarrow \quad |\Lambda|v_1||^p \geq |\Lambda v_1|^p \\ &\Longrightarrow \quad F_{\lambda,\beta}(|v_1|) \leq F_{\lambda,\beta}(v_1) \\ &\Longrightarrow \quad G_{\lambda,\beta}(|v_1|) \leq G_{\lambda,\beta}(v_1) = 0. \end{aligned}$$

Thus $G_{\lambda,\beta}(|v_1|) = 0$ and $(|v_1|, \mu_1(\beta, \lambda))$ is solution of problem (3.1). □

Proposition 3.8. *If $(H_{\lambda(V,.,m)})$ holds and $\mu_1(., \lambda) = 0$ then $\lambda(V, ., m)$ is a principal eigensurface of problem (1.1).*

Proof. Assume that $(H_{\lambda(V,.,m)})$ holds and $\mu_1(., \lambda) = 0$. Then $\lambda(V, ., m)$ is an eigensurface of problem (1.1) associated with $u \in X \setminus \{0\}$. We deduce that $(v, \mu_1(\beta, \lambda(V, \beta, m))) \in L_0(\Omega)$ and $(|v|, 0)$ are solution of problem (3.2) with $v = -\Delta u$ for $\beta \in \mathbb{R}^N$. Therefore

$$\begin{aligned} |v| \geq 0 \quad &\Longrightarrow \quad \Lambda|v| > 0, \\ &\Longrightarrow \quad N_p(\Lambda|v|) > 0, \\ &\Longrightarrow \quad [\lambda(V, \beta, m)m - V]\, e^{\beta \cdot x} N_p(\Lambda|v|) \geq 0, \\ &\Longrightarrow \quad \Lambda\left([\lambda(V, \beta, m)m - V]e^{\beta \cdot x} N_p(\Lambda|v|)\right) > 0, \\ &\Longrightarrow \quad |v| = e^{\frac{-\beta \cdot x}{p-1}} N_{p'}\left(\Lambda\left[(\lambda m - V)e^{\beta \cdot x} N_p(\Lambda|v|)\right]\right) > 0. \end{aligned}$$

Thus $v < 0$ in Ω or $v > 0$ in Ω. Take $v > 0$ in Ω. The assertion in Lemma 3.3 implies that $v \in C(\bar{\Omega})$. Consequently, $u = \Lambda v > 0$ in Ω from the Lemma 1.1. We conclude that if $(H_{\lambda(V,.,m)})$ holds and $\mu_1(., \lambda) = 0$ then $\lambda(V, ., m)$ is a principal eigensurface of problem (1.1). □

Lemma 3.9. *Suppose that $(H_{\lambda(V,\beta,m)})$ holds. If v and w are positive eigenfunctions of problem (3.1) associated to $\mu_1(\beta, \lambda(V, \beta, m)) = 0$, then functions* max *and* min *defined in Ω by*

$$\max(x) := \max\{v(x), w(x)\} \quad and \quad \min(x) := \min\{v(x), w(x)\}$$

are eigenfunctions of (3.1) associated to $\mu_1(\beta, \lambda(V, \beta, m)) = 0$.

Proof. An easy adaptation of the proof of Lemma 3.4 in [12]. □

Proposition 3.10. *Assume that $(H_{\lambda(V,.,m)})$ holds. If $\mu_1(., \lambda) = 0$, then $\lambda(V, ., m)$ is a simple eigensurface of problem (1.1).*

Proof. Assume that $(H_{\lambda(V,.,m)})$ holds and $\mu_1(.,\lambda)=0$. Then $\lambda(V,.,m)$ is a principal eigensurface of problem (1.1). Let u_1 and u_2 be two positive eigenfunctions of (1.1) associated to $\lambda(V,.,m)$. Then, $(|v|,0)$, $(|w|,0)\in L_0(\Omega)$ are solutions of problem (3.1) with $v=-\Delta u_1$ and $w=-\Delta u_2$ for $\beta\in\mathbb{R}^N$. We can take v, $w>0$ and for $x_0\in\Omega$, set $k=\frac{v(x_0)}{w(x_0)}$ and $\max_k(x)=\max\{v(x),kw(x)\}$, $\forall x\in\Omega$. Lemma 3.9 enables us to claim that $\max_k$ is an eigenfunction of problem (3.1) associated to 0. It follows that v, w, $\max_k\in C(\bar\Omega)$ and Λv, Λw, $\Lambda\max_k\in C^{1,\alpha}(\bar\Omega)$. Hence $N_p(v)$, $N_p(w)$, $N_p(\max_k)\in C^{1,\alpha}(\bar\Omega)$ and $N_p(v)$ and $N_p(w)$ are positive in Ω. Then $\frac{N_p(v)}{N_p(w)}\in C^1(\Omega)$. For any unit vector e we have

$$\begin{cases} N_p(v)(x_0+te)-N_p(v)(x_0)\le N_p(\max_k)(x_0+te)-N_p(\max_k)(x_0)\\ N_p(kw)(x_0+te)-N_p(kw)(x_0)\le N_p(\max_k)(x_0+te)-N_p(\max_k)(x_0).\end{cases}$$

Dividing these inequalities by $t>0$ and $t<0$ and letting t tend to $0^{\pm}$, we get

$$\nabla N_p(v)(x_0)=\nabla N_p(\max_k)(x_0)=\nabla N_p(kw)(x_0)=k^{p-1}\nabla N_p(w)(x_0).$$

Thus,

$$\begin{aligned}\nabla\left(\frac{N_p(v)}{N_p(w)}\right)(x_0) &= \frac{\nabla(N_p(v))(x_0)N_p(w)(x_0)-N_p(v)(x_0)\nabla(N_p(w))(x_0)}{(N_p(w)(x_0))^2}\\ &= \frac{[N_p(w)(x_0)-k^{1-p}N_p(v)(x_0)]\nabla(N_p(v))(x_0)}{(N_p(w)(x_0))^2}\\ &= 0.\end{aligned}$$

Hence,

$$\frac{N_p(v)}{N_p(w)}(x)=const=\frac{N_p(v)}{N_p(w)}(x_0)=\left(\frac{v(x_0)}{w(x_0)}\right)^{p-1}=k^{p-1},\quad \forall x\in\Omega.$$

Consequently $\dfrac{v}{w}=k$ in Ω i.e. $v=kw$. We deduce that $u_1=ku_2$ and the result follows. □

Proposition 3.11. *Suppose that $m\not\equiv 0$ and $\beta\in\mathbb{R}^N$.*

(1) *$\mu_1(\beta,.):\mathbb{R}\longrightarrow\mathbb{R}$ is concave and differentiable with $\mu_1'(\beta,\lambda)=\dfrac{\partial\mu_1(\beta,\lambda)}{\partial\lambda}=-M_\beta(\varphi_0)$ where $\varphi_0\in S$ is some eigenfunction of (2.3) associated to $\mu_1(\beta,\lambda)$ for all $\lambda\in\mathbb{R}$.*

(2) *If $m^+\not\equiv 0$, then $\lim_{\lambda\to+\infty}\mu_1(\beta,\lambda)=-\infty$.*

(3) *If $m^-\not\equiv 0$, then $\lim_{\lambda\to-\infty}\mu_1(\beta,\lambda)=-\infty$.*

(4) *If $m\ge 0$ in Ω (resp. $m\le 0$ in Ω), then $\mu_1(\beta,.)$ is strictly decreasing (resp. strictly increasing).*

(5) *$\sup_{\lambda\in\mathbb{R}}\mu_1(\beta,\lambda)=\alpha(V,\beta,m)$ where*

$$\alpha(V,\beta,m)=\inf\{E_{V,\beta}(u):\quad u\in S;\quad M_\beta(u)=0\}. \tag{3.5}$$

Proof. (1) The concavity of $\mu_1(\beta,.)$ follows from the concavity of the mapping $\lambda\mapsto J_{\lambda,\beta}(u)$, for a fixed $u\in X$. In particular $\mu_1(\beta,.)$ is continuous. Now let $\lambda_n \longrightarrow \lambda$ and φ_n, φ_λ be the $I_\beta-$normalized (i.e. $I_\beta(\varphi_n) = I_\beta(\varphi_\lambda) = 1$) eigenfunctions related to $\mu_1(\beta, \lambda_n)$, $\mu_1(\beta, \lambda)$ respectively. We apply Lemma 2.3 with $w = 1$ to get

$$\|\varphi_n\|^p \leq c_1 E_{V-\lambda_n m,\beta}(\varphi_n) + c_2 = c_1\mu_1(\beta, \lambda_n) + c_2.$$

So we conclude that φ_n is a bounded sequence in X. Hence there exists φ_0 such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi_0$ in X, strongly in $L^p(\Omega)$. Then $I_\beta(\varphi_0) = 1$ and from

$$\mu_1(\beta, \lambda) = \lim_{n\longrightarrow\infty} J_{\lambda,\beta}(\varphi_n) \geq J_{\lambda,\beta}(\varphi_0) \geq \mu_1(\beta, \lambda),$$

we infer that φ_0 is an eigenfunction associated to $\mu_1(\beta, \lambda)$. Furthermore

$$\begin{cases} \mu_1(\beta, \lambda_n) \leq E_{V-\lambda_n m,\beta}(\varphi_0) \\ \mu_1(\beta, \lambda) \leq E_{V-\lambda m,\beta}(\varphi_n). \end{cases}$$

Hence

$$\begin{cases} -M_\beta(\varphi_n) \leq \dfrac{\mu_1(\beta, \lambda_n) - \mu_1(\beta, \lambda)}{\lambda_n - \lambda} \leq -M_\beta(\varphi_0), \quad if \quad \lambda_n > \lambda \\ -M_\beta(\varphi_0) \leq \dfrac{\mu_1(\beta, \lambda_n) - \mu_1(\beta, \lambda)}{\lambda_n - \lambda} \leq -M_\beta(\varphi_n), \quad if \quad \lambda_n < \lambda. \end{cases}$$

Passing to the limit we get $\mu_1^{'}(\beta, \lambda) = -M_\beta(\varphi_0)$.

(2) Since $m^+ \not\equiv 0$, there exists a function $v \in X$ such that $M_\beta(v) > 0$ and $I_\beta(v) = 1$. Then, for all $\lambda > 0$

$$\mu_1(\beta, \lambda) \leq J_{\lambda,\beta}(v) \longrightarrow -\infty \quad as \quad \lambda \longrightarrow +\infty.$$

(3) A similar proof holds if $m^- \not\equiv 0$.

(4) If $m \geq 0$ in Ω the result is clear from the fact that $M_\beta(\varphi_\lambda) > 0$ for any $\lambda \in \mathbb{R}$. Indeed, if $\lambda_1 < \lambda_2$ then

$$\begin{aligned} \mu_1(\beta, \lambda_1) &= E_{V,\beta}(\varphi_{\lambda_1}) - \lambda_1 M_\beta(\varphi_{\lambda_1}) \\ &\geq E_{V,\beta}(\varphi_{\lambda_1}) - \lambda_2 M_\beta(\varphi_{\lambda_1}) \\ &\geq \mu_1(\beta, \lambda_2). \end{aligned}$$

If $m \leq 0$ in Ω, similar argument to the previous case.

(5) Let us prove that

$$\sup_{\lambda\in\mathbb{R}} \mu_1(\beta, \lambda) = \alpha(V, \beta, m).$$

Notice that trivially $\sup\limits_{\lambda\in\mathbb{R}} \mu_1(\beta, \lambda) \leq \alpha(V, \beta, m)$ for all $\lambda \in \mathbb{R}$. We distinguish the following three complementary cases:

(a) $m \geq 0$.

In this case we know that $\mu_1(\beta, .)$ is strictly decreasing so that

$$\sup_{\lambda\in\mathbb{R}} \mu_1(\beta, \lambda) = \lim_{\lambda\longrightarrow-\infty} \mu_1(\beta, \lambda).$$

Let $\lambda_n \longrightarrow -\infty$ when $n \longrightarrow +\infty$ and let φ_n be the associated I_β−normalized eigenfunction. We have

$$\begin{aligned} \mu_1(\beta, \lambda_n) &= E_{V,\beta}(\varphi_n) - \lambda_n M_\beta(\varphi_n) \\ &\geq E_{V,\beta}(\varphi_n), \quad \forall \lambda_n \leq 0 \end{aligned}$$

and from Lemma 2.3 with $w = 1$:

$$\|\varphi_n\|^p \leq c_1 E_{V,\beta}(\varphi_n) + c_2.$$

Thus the sequence (φ_n) is bounded in X. Then there exists $\varphi_0 \in X$ such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi_0 \in X$ and $\varphi_n \longrightarrow \varphi_0 \in L^p(\Omega)$. Thus

$$\begin{aligned} E_{V,\beta}(\varphi_0) &\leq \lim_{n \longrightarrow +\infty} \inf E_{V,\beta}(\varphi_n) \leq \lim_{n \to +\infty} \mu_1(\beta, \lambda_n), \\ I_\beta(\varphi_0) &= \lim_{n \longrightarrow +\infty} I_\beta(\varphi_n) = 1, \\ \lim_{n \longrightarrow +\infty} M_\beta(\varphi_n) &= M_\beta(\varphi_0) \geq 0. \end{aligned}$$

If $M_\beta(\varphi_0) = 0$, then:

$$\alpha \leq E_{V,\beta}(\varphi_0) \leq \lim_{n \to +\infty} \mu_1(\beta, \lambda_n) \leq \alpha(V, \beta, m)$$

and

$$\sup_{\lambda \in \mathbb{R}} \mu_1(\beta, \lambda) = \alpha(V, \beta, m).$$

If $M_\beta(\varphi_0) > 0$, then:

$$\begin{aligned} \lim_{n \to +\infty} \mu_1(\beta, \lambda_n) &= \lim_{n \to +\infty} E_{V,\beta}(\varphi_n) - \lambda_n M_\beta(\varphi_n) \\ &\geq E_{V,\beta}(\varphi_0) - \lim_{n \to +\infty} \lambda_n M_\beta(\varphi_n) \\ &= +\infty \end{aligned}$$

for $\lim_{n \longrightarrow +\infty} \lambda_n = -\infty$ and $\lim_{n \longrightarrow +\infty} M_\beta(\varphi_n) = M_\beta(\varphi_0) > 0$. Thus

$$\sup_{\lambda \in \mathbb{R}} \mu_1(\beta, \lambda) = \alpha(V, \beta, m).$$

(b) $m \leq 0$.
We argue similarly.

(c) $m^+ \not\equiv 0$ and $m^- \not\equiv 0$.
In this case it follows from hereinbefore that $\mu_1(\beta, .)$ is bounded from above. Then $\sup_{\lambda \in \mathbb{R}} \mu_1(\beta, \lambda)$ is achieved at some λ_0 that satisfies $\mu_1^{'}(\beta, \lambda_0) = -M_\beta(\varphi_0)$ for some φ_0 eigenfunction associated to $0 = \mu_1(\beta, \lambda_0)$. We conclude that

$$\alpha(V, \beta, m) \leq E_{V,\beta}(\varphi_0) = \mu_1(\beta, \lambda_0) = \sup_{\lambda \in \mathbb{R}} \mu_1(\beta, \lambda)$$

and

$$\sup_{\lambda \in \mathbb{R}} \mu_1(\beta, \lambda) = \alpha(V, \beta, m).$$

□

The main result of this section reads as follows:

Theorem 3.12. *Suppose that $m \not\equiv 0$.*

(1) (a) *If $m \geq 0$ in Ω, then there exists an eigensurface of problem (1.1) if and only if $\alpha(V,.,m) > 0$. In this case the eigensurface $\lambda_1(V,.,m)$ is unique and it is characterized by:*

$$\lambda_1(V,\beta,m) = \min_{u\in\mathcal{M}_\beta^+} E_{V,\beta}(u) \tag{3.6}$$

for $\beta \in \mathbb{R}^N$ where

$$\mathcal{M}_\beta^+ = \{u \in X : M_\beta(u) = 1\}.$$

Moreover if $(H_{\lambda_1(V,.,m)})$ holds, then $\lambda_1(V,.,m)$ is a principal eigensurface and simple.

(b) *If $m \leq 0$ in Ω, then there exists an eigensurface of problem (1.1) if and only if $\alpha(V,.,m) > 0$. In this case the eigensurface $\lambda_{-1}(V,.,m)$ is unique and it is characterized by:*

$$\lambda_{-1}(V,\beta,m) = -\min_{u\in\mathcal{M}_\beta^-} E_{V,\beta}(u) \tag{3.7}$$

for $\beta \in \mathbb{R}^N$ where

$$\mathcal{M}_\beta^- = \{u \in X : M_\beta(u) = -1\}.$$

Moreover if $(H_{\lambda_{-1}(V,.,m)})$ holds, then $\lambda_{-1}(V,.,m)$ is a principal eigensurface and simple.

(2) *If $m^+ \not\equiv 0$ and $m^- \not\equiv 0$, then there exists an eigensurface of problem (1.1) if and only if $\alpha(V,.,m) \geq 0$. More precisely:*

(a) *if $\alpha(V,.,m) > 0$, then problem (1.1) admits exactly two eigensurfaces $\lambda_{-1}(V,.,m)$ and $\lambda_1(V,.,m)$ with $\lambda_{-1}(V,.,m) < \lambda_1(V,.,m)$. Moreover if $(H_{\lambda_{\pm 1}(V,.,m)})$ hold, then $\lambda_{\pm 1}(V,.,m)$ are principal eigensurfaces and simple.*

(b) *if $\alpha(V,.,m) = 0$, then problem (1.1) has an unique eigensurface $\lambda_0(V,.,m)$ given by*

$$\lambda_0(V,\beta,m) = \inf_{u\in\mathcal{M}_\beta^+} E_{V,\beta}(u) = -\inf_{u\in\mathcal{M}_\beta^-} E_{V,\beta}(u) \tag{3.8}$$

for $\beta \in \mathbb{R}^N$. These infima are not achieved. Moreover any function $u \not\equiv 0$ in X satisfying $E_{V,\beta}(u) = M_\beta(u) = 0$ is an eigenfunction associated to $\lambda_0(V,.,m)$ and if $(H_{\lambda_0(V,.,m)})$ holds, then $\lambda_0(V,.,m)$ is a principal eigensurface and simple.

Proof. (1) (a) Assume that $m \geq 0$ in Ω. Then from Proposition 3.11, $\mu_1(\beta,.)$ is strictly decreasing and $\alpha(V,\beta,m) = \sup_{\lambda\in\mathbb{R}} \mu_1(\beta,\lambda)$ for $\beta \in \mathbb{R}^N$. Likewise, from Lemma 3.1, $\mu_1(.,\lambda(V,.,m)) = 0$ if and only if $\lambda(V,.,m)$ is eigensurface of problem (1.1). It is clear that $\alpha(V,\beta,m) > 0$ is a necessary and sufficient condition for the curve $\mu_1(\beta,.)$ to vanish at some unique value $\lambda_1(V,\beta,m)$.

From Proposition 3.11, we know that for $\beta \in \mathbb{R}^N$: $\mu_1'(\beta, \lambda_1(V,\beta,m)) = -M_\beta(\varphi_0)$ and

$$\begin{aligned}\mu_1(\beta, \lambda_1(V,\beta,m)) &= E_{V,\beta}(\varphi_{\lambda_1(V,\beta,m)}) - \lambda_1 M_\beta(\varphi_{\lambda_1(V,\beta,m)}) \\ &= E_{V,\beta}(\varphi_0) - \lambda_1 M_\beta(\varphi_0)\end{aligned}$$

with $\varphi_{\lambda_1(V,\beta,m)}$, $\varphi_0 \in S$. Then $E_{V,\beta}(\varphi_0) = \lambda_1(V,\beta,m) M_\beta(\varphi_0)$ and $M_\beta(\varphi_0) \geq 0$.
If $M_\beta(\varphi_0) = 0$, then $\alpha(V,\beta,m) = E_{V-\lambda_1(V,\beta,m)m,\beta}(\varphi_0) = 0$, a contradiction. Thus, $M_\beta(\varphi_0) > 0$ and consequently (3.6) holds.
Moreover if $(H_{\lambda_1(V,.,m)})$ holds, from Proposition 3.10, $\lambda_1(V,.,m)$ is a principal eigensurface and simple.

(b) If $m \leq 0$ in Ω, similar to the previous case.

(2) Assume that $m^+ \not\equiv 0$ and $m^- \not\equiv 0$. By Proposition 3.11, $\mu_1(\beta, .)$ is a concave differentiable function going to $-\infty$ when $\lambda \longrightarrow \pm\infty$ for $\beta \in \mathbb{R}^N$. Likewise, from Lemma 3.1, $\mu_1(., \lambda(V,.,m)) = 0$ if and only if $\lambda(V,.,m)$ is eigensurface of problem (1.1). It is clear that $\alpha(V,\beta,m) \geq 0$ is a necessary and sufficient condition for the curve $\mu_1(\beta, .)$ to vanish at some value $\lambda(V,\beta,m)$.

(a) If $\alpha(V,\beta,m) > 0$, then there exists $\lambda_{-1}(V,\beta,m) < \lambda_1(V,\beta,m)$ such that

$$\mu_1(\beta, \lambda_{-1}(V,\beta,m)) \;=\; \mu_1(\beta, \lambda_1(V,\beta,m)) = 0$$

and

$$\mu_1'(\beta, \lambda_1(V,\beta,m)) < 0 < \mu_1'(\beta, \lambda_{-1}(V,\beta,m)).$$

Thus there exist $\varphi_{\lambda_{-1}(V,\beta,m)}, \varphi_{\lambda_1(V,\beta,m)} \in S$ such that:

$$\begin{cases}\mu_1(\beta, \lambda_{-1}(V,\beta,m)) = E_{V,\beta}(\varphi_{\lambda_{-1}(V,\beta,m)}) - \lambda_{-1}(V,\beta,m) M_\beta(\varphi_{\lambda_{-1}(V,\beta,m)}) \\ \mu_1'(\beta, \lambda_{-1}(V,\beta,m)) = -M_\beta(\varphi_{\lambda_{-1}(V,\beta,m)})\end{cases}$$

and

$$\begin{cases}\mu_1(\beta, \lambda_1(V,\beta,m)) = E_{V,\beta}(\varphi_{\lambda_1(V,\beta,m)}) - \lambda_1 M_\beta(\varphi_{\lambda_1(V,\beta,m)}) \\ \mu_1'(\beta, \lambda_1(V,\beta,m)) = -M_\beta(\varphi_{\lambda_1(V,\beta,m)}).\end{cases}$$

Consequently, $\begin{cases}M_\beta(\varphi_{\lambda_1(V,\beta,m)}) > 0 \\ M_\beta(\varphi_{\lambda_{-1}(V,\beta,m)}) < 0\end{cases}$.

Take now

$$\bar{\varphi}_{\lambda_\varepsilon(V,\beta,m)} = \frac{\varphi_{\lambda_\varepsilon(V,\beta,m)}}{\left[\varepsilon M_\beta(\varphi_{\lambda_\varepsilon(V,\beta,m)})\right]^{\frac{1}{p}}},$$

with $\varepsilon = \mp 1$. It is easy to see that

$$\begin{aligned}M_\beta(\bar{\varphi}_{\lambda_\varepsilon(V,\beta,m)}) &= \varepsilon, \\ \varepsilon\lambda_\varepsilon(V,\beta,m) &= E_{V,\beta}(\bar{\varphi}_{\lambda_\varepsilon(V,\beta,m)})\end{aligned}$$

and

$$\varepsilon\lambda_\varepsilon(V,\beta,m) \leq E_{V,\beta}(u), \forall u \in \mathcal{M}_\beta^\varepsilon.$$

Therefore $\varepsilon\lambda_\varepsilon(V,\beta,m) = \min_{u\in\mathcal{M}_\beta^\varepsilon} E_{V,\beta}(u)$. Thus (3.6) and (3.7) hold.

Moreover if $(H_{\lambda_\pm(V,.,m)})$ hold, from Proposition 3.10, $\lambda_{\pm1}(V,.,m)$ are principal eigensurfaces and simple.

(b) If $\alpha(V,.,m) = 0$, then $\mu_1(.,\lambda_0(V,.,m)) = \mu_1^{'}(.,\lambda_0(V,.,m)) = 0$ at some $\lambda_0(V,.,m)$ an eigensurface of problem (1.1). Hence, $E_{V,\beta}(\varphi_0) = M_\beta(\varphi_0) = 0$, for $\beta \in \mathbb{R}^N$ and for some $\varphi_0 \in S$ an eigenfunction of problem (3.2) associated to $\mu_1(\beta,\lambda_0(V,\beta,m))$. We know that, if $u \in S$, then $\mu_1(\beta,\lambda_0(V,\beta,m)) \le J_{\lambda_0(V,\beta,m),\beta}(u)$. Thus we have: $\lambda_0(V,\beta,m)M_\beta(u) \le E_{V,\beta}(u)$, $\forall u \in S$. If $u \in \mathcal{M}_\beta^+$, then $\lambda_0(V,\beta,m) \le E_{V,\beta}(u)$, $\forall u \in \mathcal{M}_\beta^+$. Consequently:

$$\lambda_0(V,\beta,m) \le \inf_{u\in\mathcal{M}_\beta^+} E_{V,\beta}(u).$$

Consider the sequence $u_n \in \mathcal{M}_\beta^+$ given by $u_n = \dfrac{\varphi_0 + \frac{\psi}{n}}{[M_\beta(\varphi_0+\frac{\psi}{n})]^{\frac{1}{p}}}$ for some fixed $0 < \psi \in C_0^\infty(\Omega)$ and $\beta \in \mathbb{R}^N$ such that

$$M_\beta(\psi) > 0 \quad and \quad \int_\Omega m e^{\beta\cdot x}|\varphi_0|^{p-2}\varphi_0\psi dx > 0.$$

One can easily prove that

$$\int_\Omega m e^{\beta\cdot x}|\varphi_0 + \frac{\psi}{n}|^p dx > 0$$

for n large enough. Moreover, for such n we can find $0 < t_n,\ s_n < 1$ such that

$$\begin{aligned} E_{V,\beta}\left(\varphi_0 + \frac{\psi}{n}\right) &= \frac{p}{n}\int_\Omega e^{\beta\cdot x}|\Delta\varphi_0 + t_n\Delta\psi|^{p-2}\left(\Delta\varphi_0 + t_n\Delta\psi\right)\Delta\psi dx \\ &+ \frac{p}{n}\int_\Omega V e^{\beta\cdot x}|\varphi_0 + t_n\psi|^{p-2}\left(\varphi_0 + t_n\psi\right)\psi dx \end{aligned}$$

and

$$M_\beta\left(\varphi_0 + \frac{\psi}{n}\right) = \frac{p}{n}\int_\Omega m e^{\beta\cdot x}|\varphi_0 + s_n\psi|^{p-2}\left(\varphi_0 + s_n\psi\right)\psi dx.$$

Hence $E_{V,\beta}(u_n) \longrightarrow \lambda_0(V,\beta,m)$. Since $M_\beta(\varphi_0) = 0$ it is clear that $\inf_{u\in\mathcal{M}_\beta^+} E_{V,\beta}(u)$ is not achieved.

By repeating the above argument for $u \in \mathcal{M}_\beta^-$ and $\psi < 0$ such that

$$M_\beta(\psi) < 0 \quad and \quad \int_\Omega m e^{\beta\cdot x}|\varphi_0|^{p-2}\varphi_0\psi dx < 0.$$

One can easily prove that

$$E_{V,\beta}(u_n) \longrightarrow \lambda_0(V,\beta,m) = -\inf_{u\in\mathcal{M}_\beta^-} E_{V,\beta}(u).$$

Finally for any $I_\beta-$normalized function u satisfying $E_{V,\beta}(u) = M_\beta(u) = 0$, one has

$$0 = \max_{\lambda\in\mathbb{R}} \mu_1(\beta,\lambda) = E_{V,\beta}(u) = E_{V-\lambda m,\beta}(u) \geq \mu_1(\beta,\lambda_0(V,\beta,m)) = 0.$$

So u achieves $\mu_1(\beta,\lambda_0(V,\beta,m))$ for $\beta \in \mathbb{R}^N$. Consequently u is eigenfunction of problem (1.1) associated by $\lambda_0(V,.,m)$. Moreover if $(H_{\lambda_0(V,.,m)})$ holds, from Proposition 3.10, $\lambda_0(V,.,m)$ is a principal eigensurface and simple.

□

4. The Nehari manifold and the fibering maps for concave-convex problem

First and foremost, we establish as in Lemma 2.1 that:

Lemma 4.1. *The problem* (1.2) *is equivalent to the problem*

$$\begin{cases} Find \quad (u,\lambda) \in X\setminus\{0\}\times\mathbb{R}_+^* \quad such \quad that \\ \Delta_p^{2,\beta} u + V(x)e^{\beta\cdot x}|u|^{p-2}u = \lambda a(x)e^{\beta\cdot x}|u|^{r-2}u + \lambda b(x)e^{\beta\cdot x}|u|^{q-2}u, in \quad \Omega \\ u = \Delta u = 0 \qquad\qquad\qquad\qquad\qquad\qquad on \quad \partial\Omega. \end{cases} \tag{4.1}$$

The couple $(u,\lambda) \in X\times\mathbb{R}$ is a (weak) solution to problem (4.1) if

$$\begin{aligned}\int_\Omega e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u\Delta\varphi dx \quad &+ \quad \int_\Omega Ve^{\beta\cdot x}|u|^{p-2}u\varphi dx = \lambda\int_\Omega ae^{\beta\cdot x}|u|^{r-2}u\varphi dx \\ &+ \quad \int_\Omega be^{\beta\cdot x}|u|^{q-2}u\varphi dx, \quad \forall\varphi\in X.\end{aligned}$$

The search of solutions for problem (1.2) can be stated in an abstract form as the search of critical points of an energy functional $I_{\lambda,\beta}$ defined by

$$\begin{aligned} I_{\lambda,\beta} : X &\longrightarrow \mathbb{R} \\ u &\longmapsto \frac{1}{p}E_{V,\beta}(u) - \frac{\lambda}{r}A_\beta(u) - \frac{1}{q}B_\beta(u) \end{aligned}$$

where

$$\begin{aligned} A_\beta(u) &= \int_\Omega ae^{\beta\cdot x}|u|^r dx, \\ B_\beta(u) &= \int_\Omega be^{\beta\cdot x}|u|^q dx. \end{aligned}$$

$I_{\lambda,\beta}$ may be unbounded from below on the set $\{u \in X : B_\beta(u) > 0\}$ then it is useful to consider the functional $I_{\lambda,\beta}$ restricted to the so called Nehari set

$$\begin{aligned} \mathcal{N}_{\lambda,\beta} &= \left\{u \in X\setminus\{0\} : \langle I'_{\lambda,\beta}(u), u\rangle = 0\right\} \\ &= \{u \in X\setminus\{0\} : E_{V,\beta}(u) = \lambda A_\beta(u) + B_\beta(u)\} \end{aligned}$$

where $\langle .,.\rangle$ is the usual duality map defined on $X'\times X$.

Clearly $\mathcal{N}_{\lambda,\beta}$ is a much smaller set than X and, as we shall show, $I_{\lambda,\beta}$ is much better behaved on $\mathcal{N}_{\lambda,\beta}$. In particular, on $\mathcal{N}_{\lambda,\beta}$ we have that

$$\begin{aligned} I_{\lambda,\beta}(u) &= \lambda\left(\frac{1}{p}-\frac{1}{r}\right)A_\beta(u)+\left(\frac{1}{p}-\frac{1}{q}\right)B_\beta(u) \\ &= \left(\frac{1}{p}-\frac{1}{r}\right)E_{V,\beta}(u)+\left(\frac{1}{r}-\frac{1}{q}\right)B_\beta(u) \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)E_{V,\beta}(u)-\lambda\left(\frac{1}{r}-\frac{1}{q}\right)A_\beta(u). \end{aligned}$$

Let us introduce the fibering maps associated to $I_{\lambda,\beta}$. For $u\in X\setminus\{0\}$, we define the function

$$\begin{aligned} \Phi_u:(0,\infty) &\longrightarrow \mathbb{R} \\ t &\longmapsto I_{\lambda,\beta}(tu)=\frac{t^p}{p}E_{V,\beta}(u)-\frac{\lambda t^r}{r}A_\beta(u)-\frac{t^q}{q}B_\beta(u). \end{aligned}$$

We have (with $t>0$):

$$\begin{aligned} \Phi_u'(t) &= t^{p-1}E_{V,\beta}(u)-\lambda t^{r-1}A_\beta(u)-t^{q-1}B_\beta(u), \\ \Phi_u''(t) &= (p-1)t^{p-2}E_{V,\beta}(u)-(r-1)\lambda t^{r-2}A_\beta(u)-(q-1)t^{q-2}B_\beta(u). \end{aligned}$$

It is easy to see that:

$$\begin{aligned} u\in\mathcal{N}_{\lambda,\beta} &\iff \Phi_u'(1)=0 \\ t\in(0,\infty), tu\in\mathcal{N}_{\lambda,\beta} &\iff \Phi_u'(t)=0 \\ u\in\mathcal{N}_{\lambda,\beta}\setminus\{0\}\Longrightarrow\Phi_u''(1) &= (p-q)E_{V,\beta}(u)-\lambda(r-q)A_\beta(u), \\ &= (p-r)E_{V,\beta}(u)+(q-r)B_\beta(u), \\ &= (p-q)B_\beta(u)-\lambda(r-p)A_\beta(u). \end{aligned}$$

Thus it is natural to subdivide $\mathcal{N}_{\lambda,\beta}$ into sets corresponding to local minima, local maxima and inflexion points of Φ_u and so we define

$$\begin{aligned} \mathcal{N}_{\lambda,\beta}^+ &= \{u\in\mathcal{N}_{\lambda,\beta}:\Phi_u''(1)>0\}, \\ \mathcal{N}_{\lambda,\beta}^- &= \{u\in\mathcal{N}_{\lambda,\beta}:\Phi_u''(1)<0\}, \\ \mathcal{N}_{\lambda,\beta}^0 &= \{u\in\mathcal{N}_{\lambda,\beta}:\Phi_u''(1)=0\}. \end{aligned}$$

As the behavior of the maps Φ_u is strongly related to the sign of $A_\beta(u)$, $B_\beta(u)$ and $E_{V,\beta}(u)$, we introduce the notation

$$\begin{aligned} \mathcal{A}_\beta^+ &= \{u\in X:A_\beta(u)>0\},\quad \mathcal{A}_\beta^-=\{u\in X:A_\beta(u)<0\}, \\ \mathcal{B}_\beta^+ &= \{u\in X:B_\beta(u)>0\},\quad \mathcal{B}_\beta^-=\{u\in X:A_\beta(u)<0\}, \\ \mathcal{E}_\beta^+ &= \{u\in X:E_{V,\beta}(u)>0\},\quad \mathcal{E}_\beta^-=\{u\in X:E_{V,\beta}(u)<0\}, \\ \mathcal{A}_{0,\beta} &= \{u\in X:A_\beta(u)=0\},\quad \mathcal{B}_{0,\beta}=\{u\in X:B_\beta(u)=0\}, \\ \mathcal{E}_{0,\beta} &= \{u\in X:E_{V,\beta}(u)=0\},\quad \mathcal{A}_{0,\beta}^-=\mathcal{A}_\beta^-\cup\mathcal{A}_{0,\beta},\quad \mathcal{A}_{0,\beta}^+=\mathcal{A}_\beta^+\cup\mathcal{A}_{0,\beta}; \\ \mathcal{B}_{0,\beta}^- &= \mathcal{B}_\beta^-\cup\mathcal{B}_{0,\beta},\quad \mathcal{B}_{0,\beta}^+=\mathcal{B}_\beta^+\cup\mathcal{B}_{0,\beta},\quad \mathcal{E}_{0,\beta}^-=\mathcal{E}_\beta^-\cup\mathcal{E}_{0,\beta},\quad \mathcal{E}_{0,\beta}^+=\mathcal{E}_\beta^+\cup\mathcal{E}_{0,\beta}. \end{aligned}$$

Let us write $\Phi_u'(t)=t^{r-1}[h_u(t)-\lambda A_\beta(u)]$ where $h_u(t)=t^{p-r}E_{V,\beta}(u)-t^{q-r}B_\beta(u)$.

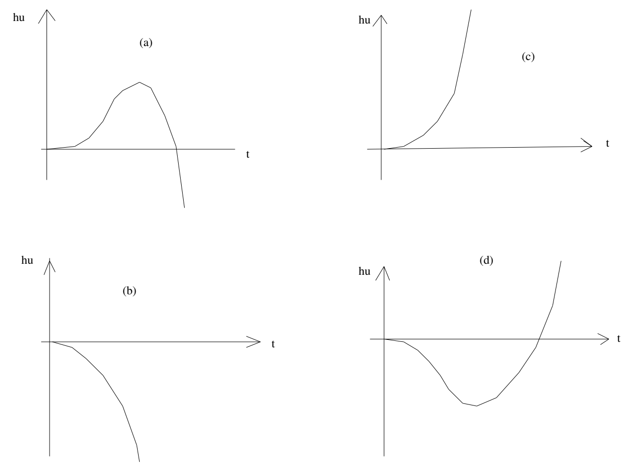

FIGURE 1. Possible forms of h_u

Clearly, for $t>0$, $tu \in \mathcal{N}_{\lambda,\beta}$ if and only if t is a solution of the equation

$$h_u(t) - \lambda A_\beta(u) = 0. \tag{4.2}$$

For $t>0$, if $tu \in \mathcal{N}_{\lambda,\beta}$ then $\Phi''_{tu}(1) = t^{r+1} h'_u(t)$. In particular, $tu \in \mathcal{N}^+_\lambda$ (resp. $\mathcal{N}^-_{\lambda,\beta}$) if and only if $h'_u(t) > 0$ (resp. $h'_u(t) < 0$).

In order to study the solvability of (4.2) let us describe the variation of function $t \longmapsto h_u(t)$ for any $u \notin \mathcal{E}_{0,\beta} \cap \mathcal{B}_{0,\beta}$. Four possible pictures of the graph of h_u can be drawn:

Case (1): $u \in \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$ in Fig. 1(a);
Case (2): $u \in \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^-_{0,\beta}$ in Fig. 1(b);
Case (3): $u \in \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_{0,\beta}$ in Fig. 1(c);
Case (4): $u \in \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ in Fig. 1(d).

Now we describe the behavior of Φ_u according to the sign of $A_\beta(u)$, $B_\beta(u)$ and $E_{V,\beta}(u)$. Assume that $u \notin \mathcal{E}_{0,\beta} \cup \mathcal{B}_{0,\beta}$.

Case (5): $u \in \mathcal{A}^+_\beta \cap \mathcal{E}^+_\beta \cap \mathcal{B}^+_\beta$.

In this case the equation (4.2) has two solutions if and only if

$$\lambda \;<\; \left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-r}} \frac{[E_{V,\beta}(u)]^{\frac{q-r}{q-p}}}{[B_\beta(u)]^{\frac{p-r}{q-p}}}.$$

Take:

$$\begin{aligned}
\sigma_{11} &= \left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-r}};\\
\sigma_{12} &= \inf_{u \in A^+_\beta \cap E^+_\beta \cap B^+_\beta} \frac{[E_{V,\beta}(u)]^{\frac{q-r}{q-p}}}{A_\beta(u)\,[B_\beta(u)]^{\frac{p-r}{q-p}}};\\
\lambda^+_1 &= \sigma_{11}\sigma_{12}.
\end{aligned}$$

Thus, if $\lambda_1^+ > 0$ then, $\forall \lambda \in (0;\lambda_1^+)$ there exists $0 < t_1(u) < t_2(u)$ such that $\Phi_u'(t_1(u)) = \Phi_u'(t_2(u)) = 0$. Consequently, $t_1(u)$ is a local minimum point and $t_2(u)$ is a local maximum point. Therefore $t_1(u)u \in \mathcal{N}_{\lambda,\beta}^+$ and $t_2(u)u \in \mathcal{N}_{\lambda,\beta}^-$ [see Fig. 2(5)].

Case (6): $u \in \mathcal{A}_\beta^+ \cap \mathcal{E}_{0,\beta}^- \cap \mathcal{B}_{0,\beta}^+$.

In this case (4.2) has no solution. Thus no multiple of u lies in $\mathcal{N}_{\lambda,\beta}$ [see Fig. 2(6)].

Case (7): $u \in \mathcal{A}_\beta^+ \cap \mathcal{E}_\beta^- \cap \mathcal{B}_\beta^-$ or $u \in \mathcal{A}_\beta^+ \cap \mathcal{E}_{0,\beta}^+ \cap \mathcal{B}_{0,\beta}^-$.

In this case (4.2) has exactly one solution $t_0(u) > 0$. Moreover $h_u'(t_0(u)) > 0$. Then $t_0(u)u \in \mathcal{N}_{\lambda,\beta}^+$. Consequently, Φ_u has exactly one critical point $t_0(u)$ which is a global minimum point [see Fig. 2(7)].

Case (8): $u \in \mathcal{A}_\beta^- \cap \mathcal{E}_{0,\beta}^- \cap \mathcal{B}_{0,\beta}^+$ or $u \in \mathcal{A}_\beta^- \cap \mathcal{E}_\beta^+ \cap \mathcal{B}_\beta^+$ i.e. $u \in \mathcal{A}_\beta^- \cap \mathcal{B}_{0,\beta}^+$.

In this case (4.2) has exactly one solution $t_0(u) > 0$. Moreover $h_u'(t_0(u)) < 0$. Then $t_0(u)u \in \mathcal{N}_{\lambda,\beta}^-$. Consequently, Φ_u has exactly one critical point $t_0(u)$ which is a global maximum point [see Fig. 2(8)].

Case (9): $u \in \mathcal{A}_\beta^- \cap \mathcal{E}_{0,\beta}^+ \cap \mathcal{B}_{0,\beta}^-$.

In this case (4.2) has no solution. Thus no multiple of u lies in $\mathcal{N}_{\lambda,\beta}$ [see Fig. 2(9)].

Case (10): $u \in \mathcal{A}_\beta^- \cap \mathcal{E}_\beta^- \cap \mathcal{B}_\beta^-$.

In this case (4.2) has two solutions if and only if

$$\lambda \;<\; -\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-r}} \frac{[-E_{V,\beta}(u)]^{\frac{q-r}{q-p}}}{[-B_\beta(u)]^{\frac{p-r}{q-p}}}.$$

Take:

$$\begin{aligned}
\sigma_{21} &= -\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-r}}, \\
\sigma_{22} &= \inf_{A_\beta^-\cap E_\beta^-\cap B_\beta^-} \frac{[-E_{V,\beta}(u)]^{\frac{q-r}{q-p}}}{A_\beta(u)\,[-B_\beta(u)]^{\frac{p-r}{q-p}}}, \\
\lambda_1^- &= \sigma_{21}\sigma_{22}.
\end{aligned}$$

Thus, if $\lambda_1^- > 0$ then, $\forall \lambda \in (0;\lambda_1^-)$ there exists $0 < t_1(u) < t_2(u)$ such that $\Phi_u'(t_1(u)) = \Phi_u'(t_2(u)) = 0$. Consequently, $t_1(u)$ is a local maximum point and $t_2(u)$ is a local minimum point. Therefore $t_1(u)u \in \mathcal{N}_{\lambda,\beta}^-$ and $t_2(u)u \in \mathcal{N}_{\lambda,\beta}^+$ [see Fig. 2(10)].

It is also clear that, in each of the above cases, $\mathcal{N}_{\lambda,\beta}^0 = \emptyset$ if $\lambda \in (0, \min\{\lambda_1^+, \lambda_1^-\})$.

Proposition 4.2. *If u is a minimizer for $I_{\lambda,\beta}$ in any subsets of $\mathcal{N}_{\lambda,\beta}$:*

$$\begin{aligned}
&\mathcal{N}_{\lambda,\beta}^\pm \cap \mathcal{A}_\beta^+ \cap \mathcal{B}_\beta^+ \cap \mathcal{E}_\beta^+ \quad ; \quad \mathcal{N}_{\lambda,\beta}^+ \cap \mathcal{A}_\beta^+ \cap \mathcal{B}_\beta^- \cap \mathcal{E}_\beta^\pm; \\
&\mathcal{N}_{\lambda,\beta}^- \cap \mathcal{A}_\beta^- \cap \mathcal{B}_\beta^+ \cap \mathcal{E}_\beta^\pm \quad ; \quad \mathcal{N}_{\lambda,\beta}^\pm \cap \mathcal{A}_\beta^- \cap \mathcal{B}_\beta^- \cap \mathcal{E}_\beta^-,
\end{aligned}$$

then u is a critical point of $I_{\lambda,\beta}$, and (u,λ) can be assumed to be a solution of (4.1).

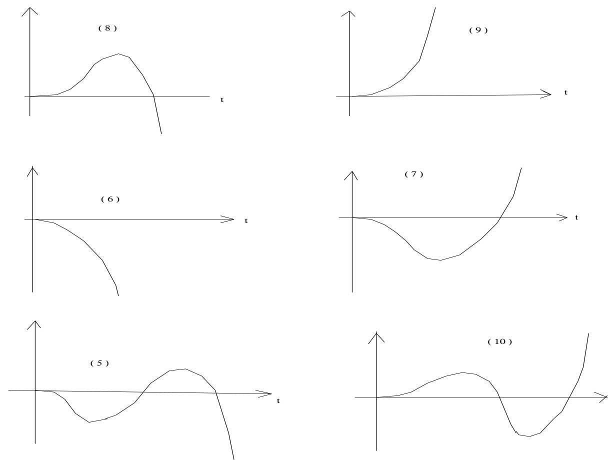

FIGURE 2. Possible forms of Φ_u

Proof. Let u_0 be a minimizer for $I_{\lambda,\beta}$ in any of above subsets of

$$\mathcal{N}_{\lambda,\beta} = \{u \in X \setminus \{0\} : K_{\lambda,\beta}(u) = E_{V,\beta}(u) - \lambda A_\beta(u) - B_\beta(u) = 0\}\,.$$

Thus, by the Lagrange multipliers rule, there exists $\delta \in \mathbb{R}$ such that $I'_{\lambda,\beta}(u_0) = \delta K'_{\lambda,\beta}(u_0)$. So, one has: $\langle I'_{\lambda,\beta}(u_0), u_0\rangle = \langle \delta K'_{\lambda,\beta}(u_0), u_0\rangle$ and $\delta = 0$ or $\delta K'_{\lambda,\beta}(u_0) = 0$.

Additionally, we have:

$$\begin{aligned}\langle K'_{\lambda,\beta}(u_0), \varphi\rangle &= p\int_\Omega e^{\beta\cdot x}|\Delta u_0|^{p-2}\Delta u_0\Delta\varphi dx + p\int_\Omega V e^{\beta\cdot x}|u_0|^{p-2}u_0\varphi dx \\ &- r\lambda\int_\Omega a e^{\beta\cdot x}|u_0|^{r-2}u_0\varphi dx - q\int_\Omega b e^{\beta\cdot x}|u_0|^{q-2}u_0\varphi dx\end{aligned}$$

for all $\varphi \in X$. Since $u_0 \notin \mathcal{N}^0_\lambda$, then we have

$$\langle K'_{\lambda,\beta}(u_0), u_0\rangle = pE_{V,\beta}(u_0) - r\lambda A_\beta(u_0) - qB_\beta(u_0) = \phi''_{u_0}(1) \neq 0.$$

Consequently, one has $\delta = 0$, $\langle I'_{\lambda,\beta}(u_0), \varphi\rangle = 0$ for all $\varphi \in X$ and we can assume that u_0 is a critical point of $I_{\lambda,\beta}$ and (u, λ) is a solution of (4.1). □

5. THE EXISTENCE OF MINIMIZERS

Let us denote by $\varphi_{V,\beta}$ an eigenfunction of problem (1.1) (with $m \equiv 1$), associated to $\lambda_1(V,\beta) = \lambda_1(V,\beta,1)$ and introduce the following constants

$$\begin{aligned}\alpha(V,\beta,b) &= \min\{E_{V,\beta}(u); I_\beta(u) = 1, B_\beta(u) = 0\};\\ \alpha_+(V,\beta,b) &= \min\{E_{V,\beta}(u); I_\beta(u) = 1, B_\beta(u) = 0, A_\beta(u) \geq 0\};\\ \alpha_-(V,\beta,b) &= \min\{E_{V,\beta}(u); I_\beta(u) = 1, B_\beta(u) = 0, A_\beta(u) \leq 0\};\\ \alpha(V,\beta,a) &= \min\{E_{V,\beta}(u); I_\beta(u) = 1, A_\beta(u) = 0\};\\ \alpha_-(V,\beta,a) &= \min\{E_{V,\beta}(u); I_\beta(u) = 1, A_\beta(u) = 0, B_\beta(u) \leq 0\};\\ \alpha_+(V,\beta,a) &= \min\{E_{V,\beta}(u); I_\beta(u) = 1, A_\beta(u) = 0, B_\beta(u) \geq 0\}\end{aligned}$$

and

$$\begin{aligned}
c(V,\beta,a) &= \inf\{E_{V,\beta}(u); A_\beta(u)=1\};\\
d(V,\beta,b) &= \inf\{E_{V,\beta}(u); B_\beta(u)=1\};\\
c_+(V,\beta,a,b) &= \inf\{E_{V,\beta}(u); A_\beta(u)=1, B_\beta(u)>0\};\\
c_-(V,\beta,a,b) &= \inf\{E_{V,\beta}(u); A_\beta(u)=1, B_\beta(u)<0\};\\
d_+(V,\beta,a,b) &= \inf\{E_{V,\beta}(u); B_\beta(u)=1, A_\beta(u)>0\};\\
d_-(V,\beta,a,b) &= \inf\{E_{V,\beta}(u); B_\beta(u)=1, A_\beta(u)<0\}
\end{aligned}$$

thereafter

$$\begin{aligned}
\eta_1^+ &= r\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{(q-p)(q+p-r)}{qp(q-r)} \left[d_+(V,\beta,a,b)\right]^{\frac{q(p-r)}{p(q-p)}} \left[c_+(V,\beta,a,b)\right]^{\frac{r}{p}};\\
\lambda_2^* &= -\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-r}} \frac{[-\lambda_1(V,\beta)]^{\frac{q-r}{q-p}}}{A_\beta(\varphi_{V,\beta})\left[-B_\beta(\varphi_{V,\beta})\right]^{\frac{p-r}{q-p}}};\\
\eta_2^* &= -r\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{(q-p)(q+p-r)}{qp(q-r)} \frac{[-\lambda_1(V,\beta)]^{\frac{p-r}{q-p}}}{A_\beta(\varphi_{V,\beta})[-B_\beta(\varphi_{V,\beta})]^{\frac{p-r}{q-p}}}.
\end{aligned}$$

It is easy to see that:

$$\begin{aligned}
\lambda_1(V,\beta) &\leq \alpha(V,\beta,b) \leq \alpha_\pm(V,\beta,b),\\
\lambda_1(V,\beta) &\leq \alpha(V,\beta,a) \leq \alpha_\pm(V,\beta,a).
\end{aligned}$$

Moreover if $(H_{\lambda_1(V,\beta)})$ holds, then

$$\begin{aligned}
\lambda_1(V,\beta)=\alpha(V,\beta,b) &\iff B_\beta(\varphi_{V,\beta})=0,\\
\lambda_1(V,\beta)=\alpha(V,\beta,a) &\iff A_\beta(\varphi_{V,\beta})=0.
\end{aligned}$$

Lemma 5.1. .

(1) *If $\alpha(V,\beta,a)>0$ [resp. $\alpha(V,\beta,b)>0$] then $c(V,\beta,a)$ [resp. $d(V,\beta,b)$] is well defined.*
(2) *If $\lambda_1(V,\beta)>0$, then $c(V,\beta,a)>0$ and $d(V,\beta,b)>0$.*
(3) *If $\lambda_1(V,\beta)=0$, $(H_{\lambda_1(V,\beta)})$ hold and $\varphi_{V,\beta}\in\mathcal{A}_\beta^-$ [resp. $\varphi_{V,\beta}\in\mathcal{B}_\beta^-$] then $c(V,\beta,a)>0$ [resp. $d(V,\beta,b)>0$].*
(4) *If $c_+(V,\beta,a,b)>0$ and $d_+(V,\beta,a,b)>0$ then $0<\eta_1^+<\lambda_1^+$.*
(5) *If $\varphi_{V,\beta}\in\mathcal{A}_\beta^-\cap\mathcal{B}_\beta^-\cap\mathcal{E}_\beta^-$ then $0<\eta_2^*<\lambda_2^*$.*

Proof. (1) Assume that $\alpha(V,\beta,a)>0$.
It is easy to see that $E_{V,\beta}$ is bounded from below on the set:

$$A_\beta^{-1}(\{1\})=\{u\in X : A_\beta(u)=1\}.$$

Let u_n be a minimizing sequence for $E_{V,\beta}$ on $A_\beta^{-1}(\{1\})$. If u_n is unbounded we set $v_n=\frac{u_n}{\|u_n\|}$, we may assume that $v_n\rightharpoonup v_0$ and $E_{V,\beta}(v_0)\leq \liminf\limits_{n\to+\infty} E_{V,\beta}(v_n)$, $I_\beta(v_n)\longrightarrow I_\beta(v_0)$, $A_\beta(v_n)\longrightarrow A_\beta(v_0)$ and $B_\beta(v_n)\longrightarrow B_\beta(v_0)$. Since $E_{V,\beta}(u_n)$ is bounded, we get $E_{V,\beta}(v_0)\leq 0$ and $A_\beta(v_0)=0$.

Moreover, there exist two positive constants $c_1 = \min_{x\in\bar\Omega} e^{\beta\cdot x}$ and $c_2 = \|e^{\beta\cdot x}\|_\infty$ such that

$$c_1\|v_n\|_p^p \leq I_\beta(v_n) \leq c_2\|v_n\|_p^p.$$

Then

$$c_1\|v_0\|_p^p \leq I_\beta(v_0) \leq c_2\|v_0\|_p^p.$$

Consequently $I_\beta(v_0) \neq 0$ and we get a contradiction from $\alpha(V,\beta,a) \leq 0$. Thus u_n must be bounded and we assume that $u_n \rightharpoonup u_0$ and

$$\begin{aligned} E_{V,\beta}(u_0) &\leq \inf_{u\in A_\beta^{-1}(1)} E_{V,\beta}(u), \\ A_\beta(u_0) &= \lim_{n\to+\infty} A_\beta(u_n) = 1, \end{aligned}$$

i.e. $E_{V,\beta}(u_0) = \min_{u\in A_\beta^{-1}(1)} E_{V,\beta}(u) = c(V,\beta,a)$. The proof for $d(V,\beta,b)$ is totally similar.

(2) Assume that $\lambda_1(V,\beta) > 0$. Then $\alpha(V,\beta,a) > 0$ and there exists $u_0 \in A_\beta^{-1}$ such that $E_{V,\beta}(u_0) = c(V,\beta,a)$. Thus $I_\beta(u_0) > 0$ and we set $w = \dfrac{u_0}{[I_\beta(u_0)]^{\frac{1}{p}}}$. We get $I_\beta(w) = 1$ and $\lambda_1(V,\beta) \leq E_{V,\beta}(w) = \dfrac{E_{V,\beta}(u_0)}{I_\beta(u_0)}$. Consequently $c(V,\beta,a) \geq \lambda_1(V,\beta) I_\beta(u_0) > 0$.

Similar statements hold for $d(V,\beta,b) > 0$.

(3) Assume that $(H_{\lambda_1(V,\beta)})$, $\lambda_1(V,\beta) = 0$ hold and $\varphi_{V,\beta} \in \mathcal{A}_\beta^-$. Then $\lambda_1(V,\beta) = 0 < \alpha(V,\beta,a)$ and there exists $u_0 \in A_\beta^{-1}$ such that $c(V,\beta,a) \geq \lambda_1(V,\beta) I_\beta(u_0) \geq 0$. If $c(V,\beta,a) = 0$ then there exists $c \in \mathbb{R}$ such that $u_0 = c\varphi_{V,\beta}$. Thus

$$0 < A_\beta(u_0) = |c|^r A_\beta(\varphi_{V,\beta}) \leq 0,$$

so we reach a contradiction. Consequently $c(V,\beta,a) > 0$.

The proof for $d(V,\beta,b) > 0$ if $(H_{\lambda_1(V,\beta)})$, $\lambda_1(V,\beta) = 0$ hold and $\varphi_{V,\beta} \in \mathcal{B}_\beta^-$, is totally similar.

(4) Assume that $c_+(V,\beta,a,b) > 0$ and $d_+(V,\beta,a,b) > 0$.

If $u \in \mathcal{A}_\beta^+ \cap \mathcal{B}_\beta^+ \cap \mathcal{E}_\beta^+$ we set $v = \dfrac{u}{[A_\beta(u)]^{\frac{1}{r}}}$ and $w = \dfrac{u}{[B_\beta(u)]^{\frac{1}{q}}}$, then

$$c_+(V,\beta,a,b) \leq E_{V,\beta}(v) = \frac{E_{V,\beta}(u)}{[A_\beta(u)]^{\frac{p}{r}}} \quad \text{i.e.} \quad E_{V,\beta}(u) \geq c_+(V,\beta,a,b)[A_\beta(u)]^{\frac{p}{r}}$$

and

$$d_+(V,\beta,a,b) \leq E_{V,\beta}(v) = \frac{E_{V,\beta}(u)}{[B_\beta(u)]^{\frac{p}{q}}} \quad \text{i.e.} \quad E_{V,\beta}(u) \geq d_+(V,\beta,a,b)[B_\beta(u)]^{\frac{p}{q}}.$$

Thus $[E_{V,\beta}(u)]^{\frac{r}{p}} \geq A_\beta(u)[c_+(V,\beta,a,b)]^{\frac{r}{p}}$ and $[E_{V,\beta}(u)]^{\frac{q}{p}} \geq B_\beta(u)[d_+(V,\beta,a,b)]^{\frac{q}{p}}$. Consequently

$$\frac{[E_{V,\beta}(u)]^{\frac{q-r}{q-p}}}{A_\beta(u)[B_\beta(u)]^{\frac{p-r}{q-p}}} = \left(\frac{[E_{V,\beta}(u)]^{\frac{q}{p}}}{B_\beta(u)}\right)^{\frac{p-r}{q-p}} \frac{[E_{V,\beta}(u)]^{\frac{r}{p}}}{A_\beta(u)} \geq [d_+(V,\beta,a,b)]^{\frac{q(p-r)}{p(q-p)}}[c_+(V,\beta,a,b)]^{\frac{r}{p}} > 0$$

and

$$\sigma_{12} = \inf_{u \in A_\beta^+ \cap B_\beta^+ \cap E_\beta^+} \frac{[E_{V,\beta}(u)]^{\frac{q-r}{q-p}}}{A_\beta(u)\,[B_\beta(u)]^{\frac{p-r}{q-p}}} \geq [d_+(V,a,b])^{\frac{q(p-r)}{p(q-p)}}\,[c_+(V,a,b)]^{\frac{r}{p}}.$$

Since

$$0 \quad < \quad r\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{(q-p)(q+p-r)}{qp(q-r)} < \left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}}$$

then we conclude that $0 < \eta_1^+ < \lambda_1^+$.

(5) If $\varphi_{V,\beta} \in \mathcal{A}_\beta^- \cap \mathcal{B}_\beta^- \cap \mathcal{E}_\beta^-$, then

$$\frac{[-\lambda_1(V,\beta)]^{\frac{p-r}{q-p}}}{A_\beta(\varphi_{V,\beta})[-B_\beta(\varphi_{V,\beta})]^{\frac{p-r}{q-p}}} < 0.$$

Moreover

$$0 > -r\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}} \frac{(q-p)(q+p-r)}{qp(q-r)} > -\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}}.$$

Consequently $0 < \eta_2^* < \lambda_2^*$.

□

Proposition 5.2. .

1. *If $\alpha_+(V,\beta,b) > 0$ [resp. $\alpha_-(V,\beta,b) > 0$] then $\mathcal{N}_{\lambda,\beta}^+ \cap \mathcal{A}_\beta^+$ [resp. $\mathcal{N}_{\lambda,\beta}^+ \cap \mathcal{A}_\beta^-$] is bounded.*
2. *$I_{\lambda,\beta}$ is positive on $\mathcal{N}_{\lambda,\beta}^- \cap \mathcal{A}_{0,\beta}^-$.*
3. *If $\alpha_+(V,\beta,b) > 0$ then $I_{\lambda,\beta}$ in bounded below on $\mathcal{N}_{\lambda,\beta}^- \cap \mathcal{A}_{0,\beta}^+$.*
4. *If $\alpha_+(V,\beta,b) > 0$, [resp. $\alpha_-(V,\beta,b) > 0$] then every minimizing sequence of $I_{\lambda,\beta}$ on any subset of $\mathcal{N}_{\lambda,\beta}^- \cap \mathcal{A}_{0,\beta}^+$ [resp. $\mathcal{N}_{\lambda,\beta}^- \cap \mathcal{A}_{0,\beta}^-$] is bounded.*
5. *If $\alpha_-(V,\beta,b) > 0$ and $\alpha_+(V,\beta,a) > 0$ then*
$$\inf_{u \in \mathcal{N}_{\lambda,\beta}^- \cap \mathcal{A}_{0,\beta}^- \cap \mathcal{B}_\beta^+} I_{\lambda,\beta}(u) > 0.$$
6. *If $\alpha_-(V,\beta,b) > 0$ and $\alpha_-(V,\beta,a) > 0$ then*
$$\inf_{u \in \mathcal{N}_{\lambda,\beta}^- \cap \mathcal{A}_{0,\beta}^- \cap \mathcal{B}_\beta^-} I_{\lambda,\beta}(u) > 0$$
for $\lambda \in (0, \lambda_1^+)$.

Proof. (1) Assume that $\alpha_+(V,\beta,b) > 0$.

If $u_n \in \mathcal{N}_{\lambda,\beta}^+ \cap \mathcal{A}_\beta^+$ and $\|u_n\| \longrightarrow \infty$ we set $v_n = \dfrac{u_n}{\|u_n\|}$ and we may assume that $v_n \rightharpoonup v_0$, $E_{V,\beta}(v_0) \leq \lim_{n\to\infty} \inf E_{V,\beta}(v_n)$, $R_\beta(v_n) \to R_\beta(v_0)$, $A_\beta(v_n) \to A_\beta(v_0)$ and $B_\beta(v_n) \to B_\beta(v_0)$.

Since $u_n \in \mathcal{N}_{\lambda,\beta}^+$, then:

$$\begin{cases} E_{V,\beta}(u_n) - \lambda A_\beta(u_n) - B_\beta(u_n) = 0 \\ (p-q)E_{V,\beta}(u_n) - \lambda(r-q)A_\beta(u_n) > 0. \end{cases} \tag{5.1}$$

Thus

$$\begin{cases} \dfrac{E_{V,\beta}(v_n)}{\|u_n\|^{q-p}} - \lambda \dfrac{A_\beta(v_n)}{\|u_n\|^{q-r}} = B_\beta(v_n) \\ E_{V,\beta}(v_n) < \lambda \left(\frac{q-r}{q-p}\right) \dfrac{A_\beta(v_n)}{\|u_n\|^{p-r}}. \end{cases} \tag{5.2}$$

Consequently $E_{V,\beta}(v_0) \leq \lim_{n\to\infty} \lambda \left(\dfrac{q-r}{q-p}\right) \dfrac{A_\beta(v_n)}{\|u_n\|^{p-r}} = 0.$

Additionally, $X \hookrightarrow L^p(\Omega)$ and there exists $s > 0$ such that $\|v_n\|_p \leq s\|v_n\|$. Then

$$|E_{V,\beta}(v_n)| \leq \|v_n\|^p + \|V\|_\infty \|v_n\|_p^p \leq 1 + s^p \|V\|_\infty$$

and the sequence $E_{V,\beta}(v_n)$ is bounded. We get by (5.2)

$$B_\beta(v_0) = \lim_{n\to\infty} B_\beta(v_n) = 0.$$

From $A_\beta(v_n) = \dfrac{A_\beta(u_n)}{\|u_n\|^r}$ and $u_n \in A^+_\beta$ we get $v_0 \in A^+_{0,\beta}$.

If $R_\beta(v_0) = 0$ then $E_{V,\beta}(v_0) = \lim_{n\to\infty} \inf E_{V,\beta}(v_n) = 0$, so that $v_n \to v_0$. But this is impossible since $\|v_n\| = 1$. Thus $R_\beta(v_0) \neq 0$ and $\alpha_+(V,\beta,b) \leq 0$, so we reach a contradiction. Consequently $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta$ is bounded if $\alpha_+(V,\beta,b) > 0$. The proof for $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^-_\beta$ bounded if $\alpha_-(V,\beta,b) > 0$ is totally similar.

(2) Assume that $u \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_{0,\beta}$. Then

$$\begin{cases} A_\beta(u) \leq 0 \\ (p-q)E_{V,\beta}(u) - \lambda(r-q)A_\beta(u) < 0 \\ I_{\lambda,\beta}(u) = \left(\frac{1}{p} - \frac{1}{q}\right) E_{V,\beta}(u) - \lambda \left(\frac{1}{r} - \frac{1}{q}\right) A_\beta(u). \end{cases}$$

Thus

$$I_{\lambda,\beta}(u) > \lambda(q-r)\left(\frac{r-p}{rpq}\right) A_\beta(u) = \frac{\lambda(q-r)(r-p)}{rpq} A_\beta(u) \geq 0.$$

(3) Assume that $\alpha_+(V,\beta,b) > 0$ and $I_{\lambda,\beta}(u_n) \to -\infty$ with $u_n \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^+_{0,\beta}$.

If u_n is bounded, then one has $u_n \rightharpoonup u_0$ and $I_{\lambda,\beta}(u_0) \leq \lim_{n\to\infty} I_{\lambda,\beta}(u_n)$. This is impossible. We deduce that u_n is unbounded. We set $v_n = \dfrac{u_n}{\|u_n\|}$ and we may assume that $v_n \rightharpoonup v_0$, $E_{V,\beta}(v_0) \leq \lim_{n\to\infty} \inf E_{V,\beta}(v_n)$, $R_\beta(v_n) \to R_\beta(v_0)$, $A_\beta(v_n) \to A_\beta(v_0)$ and $B_\beta(v_n) \to B_\beta(v_0)$. One has $v_0 \in \mathcal{A}^+_{0,\beta}$. From $\lim_{n\to\infty} I_{\lambda,\beta}(u_n) = -\infty$, for n large, we get $I_{\lambda,\beta}(u_n) < 0$. Therefore we obtain

$$\left(\frac{1}{p} - \frac{1}{q}\right) E_{V,\beta}(u_n) < \lambda \left(\frac{1}{r} - \frac{1}{q}\right) A_\beta(u_n)$$

and

$$E_{V,\beta}(v_0) \leq \lim_{n\to\infty} \inf E_{V,\beta}(v_n) \leq \lim_{n\to\infty} \left[\frac{\lambda p(q-r)}{r(q-p)}\right] \frac{A_\beta(v_n)}{\|u_n\|^{p-r}} \leq 0.$$

Additionally

$$B_\beta(v_0) = \lim_{n\to\infty} B_\beta(v_n) = \lim_{n\to\infty}\left[\frac{E_{V,\beta}(v_n)}{\|u_n\|^{q-p}} - \lambda\frac{A_\beta(v_n)}{\|u_n\|^{q-r}}\right] \;=\; 0.$$

Thereby, we get: $E_{V,\beta}(v_0)\le 0$, $B_\beta(v_0)=0$, $v_0\in\mathcal{A}^+_{0,\beta}$ and $R_\beta(v_0)\neq 0$. Thus we conclude that $\alpha_+(V,\beta,b)\le 0$ and we reach a contradiction. Consequently $I_{\lambda,\beta}$ is bounded below on $\mathcal{N}^-_{\lambda\beta}\cap\mathcal{A}^+_{0,\beta}$ if $\alpha_+(V,\beta,b)>0$.

(4) Assume that $\alpha_+(V,\beta,b)>0$ [resp. $\alpha_-(V,\beta,b)>0$] and $\mathcal{W}_\beta$ a subset of $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^+_{0,\beta}$ [resp. $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}$].

Let $u_n\in\mathcal{W}_\beta$ be such that $\lim\limits_{n\to\infty} I_{\lambda,\beta}(u_n)=\inf\limits_{u\in\mathcal{W}} I_{\lambda,\beta}(u)$.

If u_n is unbounded, since above we set $v_n=\dfrac{u_n}{\|u_n\|}$ and we can assume that $v_n\rightharpoonup v_0$, $E_{V,\beta}(v_0)\le 0$, $B_\beta(v_0)=0$, $v_0\in\mathcal{A}^+_{0,\beta}$ [resp. $v_0\in\mathcal{A}^-_{0,\beta}$] and $R_\beta(v_0)\neq 0$. Consequently, $\alpha_+(V,\beta,b)\le 0$ [resp. $\alpha_-(V,\beta,b)\le 0$], so this is impossible. We deduce the result.

(5) Assume that $\alpha_-(V,\beta,b)>0$ and $\alpha_+(V,\beta,a)>0$.

Let $u_n\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta$ be a minimizing sequence of $I_{\lambda,\beta}$. Then

$$\inf_{u\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta} I_{\lambda,\beta}(u)\ge 0$$

because $I_{\lambda,\beta}(u_n)>0$, for all $u_n\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta$. Assume by contradiction that

$$\inf_{u\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta} I_{\lambda,\beta}(u)= 0.$$

Since $\alpha_-(V,\beta,b)>0$, then u_n is bounded and there exists $u_0\in X$ such that $u_n\rightharpoonup u_0$. One has $B_\beta(u_0)\ge 0$.

Case (1): $u_0\neq 0$

Since $u_n\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}$, then we have

$$\begin{cases} I_{\lambda,\beta}(u_n)>\lambda(q-r)\left(\frac{r-p}{rpq}\right)A_\beta(u_n)\\ A_\beta(u_n)\le 0\end{cases}$$

and $A_\beta(u_0)=0$ because $\lim\limits_{n\to+\infty} I_{\lambda,\beta}(u_n)=0$. Thereby we have $A_\beta(u_0)=0$, $B_\beta(u_0)\ge 0$ and $\alpha_+(V,\beta,a)>0$. We deduce that $E_{V,\beta}(u_0)>0$.

Moreover, we have:

$$\begin{aligned} 0 \;&=\; \lim_{n\to+\infty} I_{\lambda,\beta}(u_n)=\lim_{n\to+\infty}\left[\left(\frac1p-\frac1q\right)E_{V,\beta}(u_n)-\lambda\left(\frac1r-\frac1q\right)A_\beta(u_n)\right],\\ &=\; \lim_{n\to+\infty}\left(\frac1p-\frac1q\right)E_{V,\beta}(u_n).\end{aligned}$$

We deduce that $\lim\limits_{n\to+\infty}E_{V,\beta}(u_n)=0$ and $E_{V,\beta}(u_0)\le 0$. This is impossible.

Case (2): $u_0=0$

Then $u_n \longrightarrow u_0$. We set $v_n = \dfrac{u_n}{\|u_n\|}$ and assume that there exists $v_0 \in X$ such that $v_n \rightharpoonup v_0$ and thereafter $B_\beta(v_0) = 0$, $A_\beta(v_0) = 0$, $E_{V,\beta}(v_0) \leq 0$ and $R_\beta(v_0) \neq 0$.

Consequently $\alpha_+(V,\beta,a) \leq 0$. This is impossible. We conclude that

$$\inf_{u \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_{0,\beta} \cap \mathcal{B}^+_\beta} I_{\lambda,\beta}(u) > 0.$$

(6) If $\alpha_-(V,\beta,b) > 0$ and $\alpha_-(V,\beta,a) > 0$, the proof for

$$\inf_{u \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_{0,\beta} \cap \mathcal{B}^-_\beta} I_{\lambda,\beta}(u) > 0$$

is totally similar to the previous one, so we omit it.

□

Proposition 5.3. *Assume $\alpha_+(V,\beta,b) > 0$.*

(1) *If $c_-(V,\beta,a,b) > 0$ then, for any $\lambda > 0$, $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta$ and this one is negative.*

(2) *If $c_+(V,\beta,a,b) > 0$ and $d_+(V,\beta,a,b) > 0$ then, for $\lambda \in (0,\lambda_1^+)$, $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta$ and this one is negative.*

Proof. Assume that $\alpha_+(V,\beta,b) > 0$.

(1) Assume that $c_-(V,\beta,a,b) > 0$ and $\lambda \in]0,\infty[$.

If $u \in \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta$ we know that Φ_u has a negative minimum at some $t_0 > 0$ so that $t_0 u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta$ and $\Phi_u(t_0) = I_{\lambda,\beta}(t_0 u) < 0$. Thus

$$\inf_{u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta} I_{\lambda,\beta}(u) < 0.$$

Let $u_n \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta \subset \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta$ be a minimizing sequence of $I_{\lambda,\beta}$. Then u_n is bounded and there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. Since $u_n \in \mathcal{N}^+_{\lambda,\beta}$ then

$$I_{\lambda,\beta}(u_n) = \left(\frac{1}{p} - \frac{1}{q}\right) E_{V,\beta}(u_n) - \lambda \left(\frac{1}{r} - \frac{1}{q}\right) A_\beta(u_n).$$

We deduce that

$$\lambda \left(\frac{1}{r} - \frac{1}{q}\right) A_\beta(u_0) \geq - \inf_{u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+} I_{\lambda,\beta}(u) > 0$$

and $A_\beta(u_0) > 0$. We have $B_\beta(u_0) \leq 0$ and thereafter $E_{V,\beta}(u_0) > 0$ because $A_\beta(u_0) > 0$, $u_0 \neq 0$, $\alpha_+(V,\beta,b) > 0$ and $c_-(V,\beta,a,b) > 0$.

Thus: $[A_\beta(u_0) > 0,\ B_\beta(u_0) = 0$ and $E_{V,\beta}(u_0) > 0]$ or $[A_\beta(u_0) > 0,\ B_\beta(u_0) < 0$ and $E_{V,\beta}(u_0) > 0]$. In both cases, there exists a global minimum point $t_0(u_0) > 0$ for Φ_{u_0} i.e. $t_0(u_0)u_0 \in \mathcal{N}^+_{\lambda,\beta}$. Then we have $\Phi_{u_0}(t_0(u_0)) \leq \Phi_{u_0}(1)$ i.e. $I_{\lambda,\beta}(t_0(u_0)u_0) \leq I_{\lambda,\beta}(u_0)$.

Assume by contradiction that $u_n \not\longrightarrow u_0$. Then we get

$$I_{\lambda,\beta}(t_0(u_0)u_0) \leq I_{\lambda,\beta}(u_0) \quad < \quad \lim_{n\to+\infty} I_{\lambda,\beta}(u_n) = \inf_{u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta} I_{\lambda,\beta}(u)$$

and we reach a contradiction. Consequently $u_n \longrightarrow u_0$, $u_0 \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta$ and

$$I_{\lambda,\beta}(u_0) = \inf_{u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap E^+_\beta} I_{\lambda,\beta}(u).$$

(2) Assume that $c_+(V,\beta,a,b) > 0$, $d_+(V,\beta,a,b) > 0$ and $\lambda \in (0,\lambda_1^+)$.

If $u \in \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta$, we know that Φ_u has a negative minimum at some $t > 0$ so that $tu \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta$. Thus $I_{\lambda,\beta}(tu) = \Phi_u(t) < 0$ and

$$\inf_{u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta} I_{\lambda,\beta}(u) < 0.$$

Let $u_n \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta \subset \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta$ be a minimizing sequence of $I_{\lambda,\beta}$. Then u_n is bounded and there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. As above, we have $A_\beta(u_0) > 0$, $B_\beta(u_0) \geq 0$ and $E_{V,\beta}(u_0) > 0$ because $\alpha_+(V,\beta,b) > 0$ and $c_+(V,\beta,a,b) > 0$. We deduce that, there exists a global minimum point $t_1(u_0) > 0$ for Φ_{u_0} if $B_\beta(u_0) = 0$ or a local minimum point $t_1(u_0)$ and a global maximum point $t_2(u_0)$ for Φ_{u_0} if $B_\beta(u_0) > 0$ with $0 < t_1(u_0) < t_2(u_0)$.

Assume by contradiction that $u_n \not\longrightarrow u_0$. It is easy to see that $t_1(u_0) > 1$ if $B_\beta(u_0) = 0$ or $1 < t_1(u_0) < t_2(u_0)$ if $B_\beta(u_0) > 0$ such that $\Phi_{u_0}(t_1(u_0)) \leq \Phi_{u_0}(1)$ i.e.

$$I_{\lambda,\beta}(t_1(u_0)u_0) \quad \leq \quad I_{\lambda,\beta}(u_0) < \inf_{u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta} I_{\lambda,\beta}(u).$$

This is impossible because $t_0(u_0)u_0 \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta$. Consequently, we have $u_n \longrightarrow u_0$, $A_\beta(u_0) > 0$, $B_\beta(u_0) \geq 0$, $E_{V,\beta}(u_0) > 0$, $u_0 \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta$ and

$$I_{\lambda,\beta}(u_0) = \inf_{u \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta} J_\lambda(u).$$

□

Corollary 5.4. *Assume either $\alpha(V,\beta,a) > 0$, $\alpha(V,\beta,b) > 0$, $c(V,\beta,a) > 0$ and $d(V,\beta,b) > 0$ or $[(H_{\lambda_1(V,\beta)})$, $\lambda_1(V,\beta) = 0$ hold and $\varphi_{V,\beta} \in \mathcal{A}^-_\beta \cap \mathcal{B}^+_{0,\beta}]$. Then $\lambda_1^+ > 0$ and $I_{\lambda,\beta}$ achieves its infimum in $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{E}^+_\beta$, for $\lambda \in (0,\lambda_1^+)$.*

Proof. Case 1: $\alpha(V,\beta,a) > 0$, $\alpha(V,\beta,b) > 0$, $c(V,\beta,a) > 0$ and $d(V,\beta,b) > 0$.

Then, we have $c_+(V,\beta,a,b) > 0$ and $d_+(V,\beta,a,b) > 0$. Thus, by Lemma 5.1 one has $\lambda_1^+ > 0$. Moreover we have $\alpha_+(V,\beta,b) > 0$ and $c_-(V,\beta,a,b) > 0$. Then, by Proposition 5.3 $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^-_{0,\beta} \cap \mathcal{E}^+_\beta$ for all $\lambda \in]0,\infty[$ and on $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{B}^+_{0,\beta} \cap \mathcal{E}^+_\beta$ for all $\lambda \in]0,\lambda_1^+[$ and this one is negative. We deduce that $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{E}^+_\beta$ and this one is negative for all $\lambda \in (0,\lambda_1^+)$.

Case 2: $(H_{\lambda_1(V,\beta)})$, $\lambda_1(V,\beta) = 0$ hold and $\varphi_{V,\beta} \in \mathcal{A}^-_\beta \cap \mathcal{B}^+_{0,\beta}$.

Then $c(V,\beta,a) > 0$ and $c_\pm(V,\beta,a) > 0$. If $B_\beta(\varphi_{V,\beta}) = 0$, then we have

$$\lambda_1(V,\beta) = 0 = \alpha(V,\beta,b) < \alpha_+(V,\beta,b).$$

Otherwise, we have $\lambda_1(V,\beta) = 0 < \alpha(V,\beta,b)$. Thus $\alpha_+(V,\beta,b) > 0$. We set

$$e(V,\beta,b,a) = \min\{E_{V,\beta}(u) : B_\beta(u) = 1, A_\beta(u) \geq 0\}$$

and assume that $e(V,\beta,b,a) > 0$.

We deduce that $d_+(V,\beta,a,b) \geq e(V,\beta,a,b) > 0$. So $\alpha_+(V,\beta,b) > 0$, $c_\pm(V,\beta,a,b) > 0$ and $d_+(V,\beta,a,b) > 0$. Consequently by Lemma 5.1 and Proposition 5.3, $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^+_\beta \cap \mathcal{E}^+_\beta$ and this one is negative for all $\lambda \in (0,\lambda_1^+)$. $\square$

Proposition 5.5. *Assume that $\alpha_+(V,\beta,a) > 0$ and $\alpha_-(V,\beta,b) > 0$.*

1. *If $d_-(V,\beta,a,b) > 0$ then, for any $\lambda > 0$, $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$ and this one is positive.*
2. *If $d_+(V,\beta,a,b) > 0$ and $c_+(V,\beta,a,b) > 0$ then there exists $\sigma_+ > 0$ such that $I_{\lambda,\beta}(u) \geq \sigma_+$ and $u \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^+_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$ for $\lambda \in (0,\eta_1^+)$.*

   Moreover, $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^+_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$.

Proof. Suppose that $\alpha_+(V,\beta,a) > 0$ and $\alpha_-(V,\beta,b) > 0$.

1. Suppose that $d_-(V,\beta,a,b) > 0$ and $\lambda \in (0,\infty)$. Then $I_{\lambda,\beta}$ is positive on $\mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^+_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$. We deduce that:

$$\inf_{u\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta} I_{\lambda,\beta}(u) > 0.$$

Let $u_n \in \mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta$ be a minimizing sequence for $I_{\lambda,\beta}$. Then (u_n) is bounded so there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. Then: $E_{V,\beta}(u_0) \leq \liminf\limits_{n\to+\infty} E_{V,\beta}(u_n)$, $R_\beta(u_n) \longrightarrow R_\beta(u_0)$, $A_\beta(u_n) \longrightarrow A_\beta(u_0)$ et $B_\beta(u_n) \longrightarrow B_\beta(u_0)$. Additionally, we have:

$$I_{\lambda,\beta}(u_n) = \left(\frac{1}{p} - \frac{1}{r}\right) E_{V,\beta}(u_n) + \left(\frac{1}{r} - \frac{1}{q}\right) B_\beta(u_n).$$

We infer:

$$\begin{aligned}\left(\frac{1}{r} - \frac{1}{q}\right) B_\beta(u_n) &= \left(\frac{1}{r} - \frac{1}{p}\right) E_{V,\beta}(u_n) + I_{\lambda,\beta}(u_n)\\ &\geq I_{\lambda,\beta}(u_n)\end{aligned}$$

and

$$\left(\frac{1}{r} - \frac{1}{q}\right) B(u_0) \geq \lim_{n\longrightarrow+\infty} I_{\lambda,\beta}(u_n) > 0.$$

Thus $B_\beta(u_0) > 0$.

If $A_\beta(u_0) = 0$ or $A_\beta(u_0) < 0$ then we have $E_{V,\beta}(u_0) > 0$. So we have $B_\beta(u_0) > 0$, $A_\beta(u_0) \leq 0$ and $E_{V,\beta}(u_0) > 0$. Then Φ_{u_0} has a global maximum at some $t_0(u_0) > 0$, so that $t_0(u_0)u_0 \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$.

One has $\Phi_{u_0}(t_0(u_0)) \geq \Phi_{u_0}(t)$, $\forall t \in (0,+\infty)$. Assume by contradiction that $u_n \not\longrightarrow u_0$. Then, we have

$$I_{\lambda,\beta}(t_0(u_0)u_0) = \Phi_{u_0}(t_0(u_0)) < \lim_{n\to+\infty} \Phi_{u_n}(t_0(u_0)). \tag{5.3}$$

Since $u_n \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$, then Φ_{u_n} has a global maximum at some $t = 1$. Thus $\Phi_{u_n}(1) \geq \Phi_{u_n}(t_0(u_0))$ and

$$\lim_{n\to+\infty} \Phi_{u_n}(t_0(u_0)) \leq \lim_{n\to+\infty} \Phi_{u_n}(1) = \lim_{n\to+\infty} I_{\lambda,\beta}(u_n). \tag{5.4}$$

By (5.3) and (5.4) we have

$$I_{\lambda,\beta}(t_0(u_0)u_0) < \lim_{n\to+\infty} I_{\lambda,\beta}(u_n) = \inf_{u\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta} I_{\lambda,\beta}(u).$$

This is impossible. Consequently $u_n \longrightarrow u_0$.

We deduce that $\lim_{n\to+\infty} I_{\lambda,\beta}(u_n) = I_{\lambda,\beta}(u_0)$, $u_0 \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}_{0,\beta} \cap \mathcal{B}^+ \cap E^+_\beta$ and

$$I_{\lambda,\beta}(u_0) = \inf_{u\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta} I_{\lambda,\beta}(u) > 0.$$

(2) Suppose that $d_+(V,\beta,a,b) > 0$, $c_+(V,\beta,a,b) > 0$ and $\lambda \in (0,\eta_1^+)$.

If $u \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^+_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$, then Φ_u has a global maximum at some $t = 1$ such that $t_0 < 1$ with $t_0 = \left(\frac{(p-r)E_{V,\beta}(u)}{(q-r)B_\beta(u)}\right)^{\frac{1}{q-p}}$. Then we have

$$\begin{aligned}
I_{\lambda,\beta}(u) &= \Phi_u(1) > \Phi_u(t_0) = \frac{t_0^p}{p} E_{V,\beta}(u) - \frac{t_0^q}{q} B_\beta(u) - \lambda \frac{t_0^r}{r} A_\beta(u) \\
&> \frac{1}{p}\left(\frac{(p-r)E_{V,\beta}(u)}{(q-r)B_\beta(u)}\right)^{\frac{p}{q-p}} E_{V,\beta}(u) - \frac{1}{q}\left(\frac{(p-r)E_{V,\beta}(u)}{(q-r)B_\beta(u)}\right)^{\frac{q}{q-p}} B_\beta(u) \\
&- \frac{1}{r}\left(\frac{(p-r)E_{V,\beta}(u)}{(q-r)B_\beta(u)}\right)^{\frac{r}{q-p}} A_\beta(u) \\
&= \frac{(q-p)(q+p-r)}{qp(q-r)}\left(\frac{p-r}{q-r}\right)^{\frac{p}{q-p}}\left(\frac{[E_{V,\beta}(u)]^q}{[B_\beta(u)]^p}\right)^{\frac{1}{q-p}} \\
&- \frac{\lambda A_\beta(u)}{r}\left(\frac{p-r}{q-r}\right)^{\frac{r}{q-p}}\left(\frac{E_{V,\beta}(u)}{B_\beta(u)}\right)^{\frac{r}{q-p}}.
\end{aligned}$$

Additionally $A_\beta(u) \leq \frac{[E_{V,\beta}(u)]^{\frac{r}{p}}}{[c_+(V,\beta,a,b)]^{\frac{r}{p}}}$ and $B_\beta(u) \leq \frac{[E_{V,\beta}(u)]^{\frac{q}{p}}}{d_+(V,\beta,a,b)^{\frac{q}{p}}}$ so

$$\left(\frac{[E_{V,\beta}(u)]^q}{[B_\beta(u)]^p}\right)^{\frac{1}{q-p}} \geq [d_+(V,\beta,a,b)]^{\frac{q}{q-p}}.$$

Thus, we have

$$\begin{aligned} I_{\lambda,\beta}(u) &\geq \frac{(q-p)(q+p-r)}{qp(q-r)}\left(\frac{p-r}{q-r}\right)^{\frac{p}{q-p}}\left(\frac{[E_{V,\beta}(u)]^q}{[B_\beta(u)]^p}\right)^{\frac{1}{q-p}} \\ &- \frac{\lambda}{r}\left(\frac{p-r}{q-r}\right)^{\frac{r}{q-p}}\left(\frac{E_{V,\beta}(u)}{B_\beta(u)}\right)^{\frac{r}{q-p}}\frac{[E_{V,\beta}(u)]^{\frac{r}{p}}}{[c_+(V,\beta,a,b)]^{\frac{r}{p}}} \\ &= \frac{(q-p)(q+p-r)}{qp(q-r)}\left(\frac{p-r}{q-r}\right)^{\frac{p}{q-p}}\left(\frac{[E_{V,\beta}(u)]^q}{[B_\beta(u)]^p}\right)^{\frac{p-r}{p(q-p)}}\left(\frac{[E_{V,\beta}(u)]^q}{[B_\beta(u)]^p}\right)^{\frac{r}{p(q-p)}} \\ &- \frac{\lambda}{r}\left(\frac{p-r}{q-r}\right)^{\frac{r}{q-p}}\left(\frac{[E_{V,\beta}(u)]^q}{[B_\beta(u)]^p}\right)^{\frac{r}{p(q-p)}}[c_+(V,\beta,a,b)]^{\frac{-r}{p}} \\ &= \left(\frac{[E_{V,\beta}(u)]^q}{[B_\beta(u)]^p}\right)^{\frac{r}{p(q-p)}}\left(\frac{p-r}{q-r}\right)^{\frac{r}{q-p}} Y_{\lambda,\beta}(u) \end{aligned}$$

with

$$\begin{aligned} Y_{\lambda,\beta}(u) &= \frac{(q-p)(q+p-r)}{qp(q-r)}\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}}\left(\frac{[E_V(u)]^q}{[B(u)]^p}\right)^{\frac{p-r}{p(q-p)}} - \frac{\lambda}{rc_+(V,\beta,a,b)^{\frac{r}{p}}} \\ &\geq \frac{(q-p)(q+p-r)}{qp(q-r)}\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-p}}\left[d_+(V,\beta,a,b)^{\frac{q}{q-p}}\right]^{\frac{p-r}{p}} - \frac{\lambda}{r[c_+(V,\beta,a,b)]^{\frac{r}{p}}} \\ &= \frac{1}{r[c_+(V,\beta,a,b)]^{\frac{r}{p}}}\left(\eta_1^+ - \lambda\right) > 0. \end{aligned}$$

Consequently

$$\begin{aligned} I_{\lambda,\beta}(u) &\geq \left(\frac{p-r}{q-r}\right)^{\frac{r}{q-p}}\left([d_+(V,\beta,a,b)]^{\frac{q}{q-p}}\right)^{\frac{r}{p}}\frac{(\eta_1^+ - \lambda)}{r[c_+(V,\beta,a,b)]^{\frac{r}{p}}} \\ &= \sigma_+ > 0. \end{aligned}$$

Let $u_n \in \mathcal{N}^-_{\lambda,\beta} \cap A^+_{0,\beta} \cap B^+_\beta \cap E^+_\beta$ be a minimizing sequence for $I_{\lambda,\beta}$. Then (u_n) is bounded in X so there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. Thus $E_{V,\beta}(u_0) \leq \lim_{n\to+\infty}\inf E_{V,\beta}(u_n)$, $A_\beta(u_n) \longrightarrow A_\beta(u_0)$ and $B_\beta(u_n) \longrightarrow B_\beta(u_0)$. As before, we can prove that $B_\beta(u_0) > 0$ because $\lim_{n\to+\infty} I_{\lambda,\beta}(u_n) > 0$ and $A_\beta(u_0) \geq 0$. Since $\alpha_+(V,\beta,a) > 0$ and $d_+(V,\beta,a,b) > 0$ then $E_{V,\beta}(u_0) > 0$ and $u_0 \in \mathcal{A}^+_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$. Thus Φ_{u_0} has a global maximum at some $t_2(u_0) > 0$ with $t_2(u_0)u_0 \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^+_{0,\beta} \cap \mathcal{B}^+_\beta \cap \mathcal{E}^+_\beta$. One has $\Phi_{u_0}(t_2(u_0)) \geq \Phi_{u_0}(t)$ for all $t \in (0,+\infty)$.

Assume by contradiction that $u_n \not\longrightarrow u_0$. Then, we have $\Phi_{u_0}(t) < \lim_{n\to+\infty} \Phi_{u_n}(t)$ for all $t \in (0,+\infty)$. Thus $I_{\lambda,\beta}(t_2(u_0)u_0) = \Phi_{u_0}(t_2(u_0)) < \lim_{n\to+\infty} \Phi_{u_n}(t_2(u_0))$.

Since

$$\lim_{n\to+\infty} \Phi_{u_n}(t_2(u_0)) \leq \lim_{n\to+\infty} \Phi_{u_n}(1) = \lim_{n\to+\infty} I_{\lambda,\beta}(u_n)$$

then

$$I_{\lambda,\beta}(t_0(u_0)u_0) < \inf_{u\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^+_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta} I_{\lambda,\beta}(u)$$

and this is impossible. We deduce that $u_n \longrightarrow u_0$. Consequently $\lim_{n\to+\infty} I_{\lambda,\beta}(u_n) = I_{\lambda,\beta}(u_0)$ and

$$I_{\lambda,\beta}(u_0) = \inf_{u\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^+_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta} I_{\lambda,\beta}(u) > 0.$$

□

Corollary 5.6. *Assume that $\alpha(V,\beta,a) > 0$, $\alpha(V,\beta,b) > 0$, $c(V,\beta,a) > 0$ and $d(V,\beta,b) > 0$. Then $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta$ for $\lambda\in(0,\eta_1^+)$.*

Proof. Suppose that $\alpha(V,\beta,a) > 0$, $\alpha(V,\beta,b) > 0$, $c(V,\beta,a) > 0$ and $d(V,\beta,b) > 0$. Then $\alpha_+(V,\beta,a) > 0$, $\alpha_-(V,\beta,b) > 0$, $c_+(V,\beta,a) > 0$ and $d_\pm(V,\beta,b) > 0$. Consequently, by Proposition 5.5 $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta$ and $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^+_{0,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta$, and this one is positive for all $\lambda\in]0,\eta_1^+[$. We deduce that $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta$ for all $\lambda\in(0,\eta_1^+)$. □

We can combine both Corollary 5.4 and Corollary 5.6 to get:

Corollary 5.7. *Assume that $\alpha(V,\beta,a) > 0$, $\alpha(V,\beta,b) > 0$, $c(V,\beta,a) > 0$ and $d(V,\beta,b) > 0$. Then $I_{\lambda,\beta}$ achieves its infimum on $\mathcal{N}^+_{\lambda,\beta}\cap\mathcal{A}^+_\beta\cap\mathcal{E}^+_\beta$ and $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{B}^+_\beta\cap\mathcal{E}^+_\beta$ for $\lambda\in(0,\eta_1^+)$.*

Proposition 5.8. *Assume either [$\alpha(V,\beta,a) > 0$ and $\alpha(V,\beta,b) > 0$] or [$\alpha(V,\beta,a) = 0 < \alpha_-(V,\beta,a)$ or $\alpha(V,\beta,b) = 0 < \alpha_-(V,\beta,b)$]. If $\varphi_{V,\beta}\in\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta$ then:*

1. *$\forall\lambda\in(0,\lambda_2^*)$, $I_{\lambda,\beta}$ achieves its infimum in $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta$ and this one is positive.*
2. *$\forall\lambda\in(0,\eta_2^*)$, $I_{\lambda,\beta}$ achieves its infimum in $\mathcal{N}^+_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta$ and this one is negative.*

Proof. Assume either [$\alpha(V,\beta,a) > 0$ and $\alpha(V,\beta,b) > 0$] or [$\alpha(V,\beta,a) = 0 < \alpha_-(V,\beta,a)$ and $\alpha(V,\beta,b) = 0 < \alpha_-(V,\beta,b)$]. Suppose that $\varphi_{V,\beta}\in\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta$.

(1) For $\lambda\in(0,\lambda_2^*)$ we can prove that $h_{\varphi_{V,\beta}}$ has a global maximum at some

$$t_0 = t_0(\varphi_{V,\beta}) = \left[\frac{(p-r)(-\lambda_1(V,\beta))}{(q-r)(-B_\beta(\varphi_{V,\beta}))}\right]^{\frac{1}{q-p}}$$

Then, we have

$$h_{\varphi_{V,\beta}}(t_0) \;=\; -\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q-r}\right)^{\frac{p-r}{q-r}}\frac{[-\lambda_1(V,\beta)]^{\frac{q-r}{q-p}}}{[-B_\beta(\varphi_{V,\beta})]^{\frac{p-r}{q-p}}} = \lambda_2^* A_\beta(\varphi_{V,\beta}) < 0.$$

Consequently $h_{\varphi_{V,\beta}}(t_0) - \lambda A_\beta(\varphi_{V,\beta}) < 0$ for all $\lambda\in(0,\lambda_2^*)$. We deduce that $\Phi_{\varphi_{V,\beta}}$ has local maximum at $t_1\in(0,t_0)$ and global minimum at $t_2\in(t_0,\infty)$ and $\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta\neq\emptyset$.

Let $u_n\in\mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta$ be a minimizing sequence for $I_{\lambda,\beta}$. Then (u_n) is bounded so there exists $u_0\in X$ such that $u_n\rightharpoonup u_0$. Thus

$E_{V,\beta}(u_0) \leq \lim_{n\to+\infty} \inf E_{V,\beta}(u_n)$, $A_\beta(u_n) \longrightarrow A_\beta(u_0)$ and $B_\beta(u_n) \longrightarrow B_\beta(u_0)$. We hnow that

$$I_{\lambda,\beta}(u_n) = \left(\frac{1}{p} - \frac{1}{q}\right) E_{V,\beta}(u_n) - \lambda \left(\frac{1}{r} - \frac{1}{q}\right) A_\beta(u_n).$$

Then, we have

$$\lambda \left(\frac{1}{r} - \frac{1}{q}\right) A_\beta(u_n) \leq -I_{\lambda,\beta}(u_n)$$

and

$$\lambda \left(\frac{1}{r} - \frac{1}{q}\right) A_\beta(u_0) \leq - \inf_{u\in \mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta} I_{\lambda,\beta}(u) < 0.$$

We deduce that $A_\beta(u_0) < 0$ and $B_\beta(u_0) \leq 0$.

If $B_\beta(u_0) = 0$, then $\alpha_-(V, \beta, b) \leq 0$. This is impossible. Thus $B_\beta(u_0) < 0$.

Additionally, one has

$$\begin{aligned} E_{V,\beta}(u_0) &\leq \lim_{n\to+\infty} \inf E_{V,\beta}(u_n) \leq \lim_{n\to+\infty} \lambda A_\beta(u_n) + B_\beta(u_n) \\ &= \lambda A_\beta(u_0) + B_\beta(u_0) < 0. \end{aligned}$$

So we have $u_0 \in \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ and $t_2(u_0)u_0 \in \mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta$ for some $t_2(u_0) > 0$. Assume by contradiction that $u_n \not\longrightarrow u_0$. Then, we have:

$$0 = \Phi'_{u_0}(t_2(u_0)) < \lim\inf \Phi'_{u_n}(t_2(u_0))$$

so $\Phi'_{u_n}(t_2(u_0)) > 0$ for n sufficiently large. We deduce that $t_2(u_0) < 1$ or $t_2(u_0) > t_1(u_n)$.

If $t_2(u_0) < 1$ then

$$\begin{aligned} I_{\lambda,\beta}(t_2(u_0)u_0) &= \Phi_{u_0}(t_2(u_0)) < \lim\inf \Phi_{u_n}(t_2(u_0)) \\ &\leq \lim_{n\to+\infty} \Phi_{u_n}(1) = \lim_{n\to+\infty} I_{\lambda,\beta}(u_n). \end{aligned}$$

This is impossible.

If $t_2(u_0) > t_1(u_n)$ then

$$\Phi'_{u_0}(t_1(u_n)) < \lim\inf \Phi'_{u_n}(t_1(u_n)) = 0.$$

Consequently Φ'_{u_0} and $u_n \longrightarrow u_0$. Finally $\lim_{n\to+\infty} I_{\lambda,\beta}(u_n) = I_{\lambda,\beta}(u_0)$ and

$$I_{\lambda,\beta}(u_0) = \inf_{u\in \mathcal{N}^-_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta} I_{\lambda,\beta}(u) > 0.$$

(2) For $\lambda \in (0, \eta_2^*)$, it is easy to see that $\mathcal{N}^+_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta \neq \emptyset$. Let $u_n \in \mathcal{N}^+_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta$ be a minimizing sequence for $I_{\lambda,\beta}$. Then (u_n) is bounded so there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. Thus $E_{V,\beta}(u_0) \leq \lim_{n\to+\infty} \inf E_{V,\beta}(u_n)$, $A_\beta(u_n) \longrightarrow A_\beta(u_0)$ and $B_\beta(u_n) \longrightarrow B_\beta(u_0)$. One has $E_{V,\beta}(u_0) \leq 0$, $A_\beta(u_0) \leq 0$ and $B_\beta(u_0) \leq 0$. Since

$$I_{\lambda,\beta}(u_n) = \left(\frac{1}{p} - \frac{1}{r}\right) E_{V,\beta}(u_n) - \left(\frac{1}{q} - \frac{1}{r}\right) B_\beta(u_n)$$

then we have

$$\left(\frac{1}{r}-\frac{1}{q}\right) B_\beta(u_n) \leq I_{\lambda,\beta}(u_n)$$

and

$$\left(\frac{1}{r}-\frac{1}{q}\right) B_\beta(u_0) \leq \inf_{u\in\mathcal{N}^+_{\lambda,\beta}\cap A^-_\beta\cap B^-_\beta\cap E^-_\beta} I_{\lambda,\beta}(u) < 0.$$

Consequently $B_\beta(u_0) < 0$, $R_\beta(u_0) \neq 0$ and $A_\beta(u_0) \leq 0$. If $A_\beta(u_0) = 0$, then $\alpha_-(V,\beta,a) \leq 0$. This is impossible. Therefore $u_0 \in \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ and $t_1(u_0)u_0 \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ for some $t_1(u_0) > 0$.

Assume by contradiction that $u_n \not\longrightarrow u_0$. Then, we have

$$\begin{aligned} I_{\lambda,\beta}(t_1(u_0)u_0) &= \Phi_{u_0}(t_1(u_0)) < \liminf \Phi_{u_n}(t_1(u_0)) \\ &\leq \lim_{n\to+\infty} \Phi_{u_n}(1) = \lim_{n\to+\infty} I_{\lambda,\beta}(u_n). \end{aligned}$$

This is impossible. Consequently $u_n \longrightarrow u_0$. We deduce $\lim_{n\to+\infty} I_{\lambda,\beta}(u_n) = I_{\lambda,\beta}(u_0)$ and

$$I_{\lambda,\beta}(u_0) = \inf_{u\in\mathcal{N}^+_{\lambda,\beta}\cap\mathcal{A}^-_\beta\cap\mathcal{B}^-_\beta\cap\mathcal{E}^-_\beta} I_{\lambda,\beta}(u) > 0.$$

□

Corollary 5.9. *Assume either [$\alpha(V,\beta,a) > 0$ and $\alpha(V,\beta,b) > 0$] or [$\alpha(V,\beta,a) = 0 < \alpha_-(V,\beta,a)$ or $\alpha(V,\beta,b) = 0 < \alpha_-(V,\beta,b)$]. If $\varphi_{V,\beta} \in \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ then $I_{\lambda,\beta}$ achieves its infimum in $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ and $\mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ for $\lambda \in (0,\eta^*_2)$.*

Proof. If either [$\alpha(V,\beta,a) > 0$ and $\alpha(V,\beta,b) > 0$] or [$\alpha(V,\beta,a) = 0 < \alpha_-(V,\beta,a)$ and $\alpha(V,\beta,b) = 0 < \alpha_-(V,\beta,b)$] and $\varphi_{V,\beta} \in \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$, then we have $\eta^*_2 < \lambda^*_2$ and by Proposition 5.8, $I_{\lambda,\beta}$ achieves its infimum in $\mathcal{N}^+_{\lambda,\beta} \cap \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ and $\mathcal{N}^-_{\lambda,\beta} \cap \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta \cap \mathcal{E}^-_\beta$ for all $\lambda \in (0,\eta^*_2)$. □

We set $\eta = \min\{\eta^+_1, \eta^*_2\}$.

Theorem 5.10. .

(1) *If either $\lambda_1(V,\beta) > 0$ or [$\lambda_1(V,\beta) = 0$, $(H_{\lambda_1(V,\beta)})$ hold and $\varphi_{V,\beta} \in \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta$] then (1.2) has at least two solutions (u_1,β,λ) and (u_2,β,λ) such that $u_1 \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{E}^+_\beta$, $u_2 \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{E}^+_\beta$ for $\lambda \in (0,\eta^+_1)$.*

(2) *If $\lambda_1(V,\beta) = 0$, $(H_{\lambda_1(V,\beta)})$ hold and $\varphi_{V,\beta} \in \mathcal{A}^-_\beta \cap \mathcal{B}^+_{0,\beta}$ then (1.2) has at least one solution (λ,β,u_0) such that $u_0 \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{E}^+_\beta$ for $\lambda \in (0,\lambda^+_1)$.*

(3) *Assume $\lambda_1(V,\beta) < 0$ and $\varphi_{V,\beta} \in \mathcal{A}^-_\beta \cap \mathcal{B}^-_\beta$.*

   (a) *If [$\alpha(V,\beta,a) > 0$ and $\alpha(V,\beta,b) > 0$] or [$\alpha(V,\beta,a) = 0 < \alpha_-(V,\beta,a)$ or $\alpha(V,\beta,b) = 0 < \alpha_-(V,\beta,b)$] then (1.2) has at least two solutions (u_3,β,λ) and (u_4,β,λ) such that $u_3 \in \mathcal{N}^+_{\lambda,\beta} \cap \mathcal{E}^-_\beta$ and $u_4 \in \mathcal{N}^-_{\lambda,\beta} \cap \mathcal{E}^-_\beta$ for all $\lambda \in (0,\eta^*_2)$.*

(b) *If* $\alpha(V,\beta,a) > 0$, $\alpha(V,\beta,b) > 0$, $c(V,\beta,a) > 0$ *and* $d(V,\beta,b) > 0$ *then* (1.2) *has at least four solutions* (u_1,β,λ), (u_2,β,λ), (u_3,β,λ) *and* (u_4,β,λ) *for* $\lambda \in (0,\eta)$.

Proof. (1) is a consequence of Lemma 5.1, Corollary 5.7 and Lemma 4.1.

(2) is a consequence of Corollary 5.4 and Lemma 4.1.

(3) Assume $\lambda_1(V,\beta) < 0$ and $\varphi_{V,\beta} \in \mathcal{A}_\beta^- \cap \mathcal{B}_\beta^-$.

(a) is consequence of Corollary 5.9 and Lemma 4.1.

(b) holds from Corollaries 5.9 and 5.7 and Lemma 4.1.

□

Acknowledgement. The authors wish to express their gratitude to the referees for valuable comments and suggestions.

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