SOME RESULTS ABOUT WEAKLY S-PRIMARY IDEALS OF A COMMUTATIVE RING

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Abstract. Let $R$ be a commutative ring with identity and $S \subseteq R$ a multiplicative subset. We define a proper ideal $P$ of $R$ disjoint from $S$ to be weakly $S$-primary if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in P$ then $sa \in P$ or $sb \in \sqrt{P}$. We show that weakly $S$-primary ideals enjoy analogs of many properties of weakly primary ideals and we study the form of weakly $S$-primary ideals of the amalgamation of $A$ with $B$ along an ideal $J$ with respect to $f$ (denoted by $A \triangleright J \leftarrow B$). Weakly $S$-primary ideals of the trivial ring extension are also characterized.

1. Introduction

Throughout this paper, all considered rings are assumed to be commutative with identity $1 \neq 0$ and all ring homomorphisms are assumed to be unital. If $A$ is a subring of $B$, we suppose that they have the same identity element. As usual, if $R$ is a commutative ring, then $Z(R)$ denotes the set of zero divisors of $R$ and $\text{Reg}(R) = R \setminus Z(R)$ is the set of its regular elements. Recall that a subset $S$ of a ring $R$ is called multiplicative if $1 \in S$, $0 \notin S$ and $S$ is closed under multiplication. Note that any multiplicative subset of $R$ satisfies the inclusion relations $\{1\} \subseteq S \subseteq R$. Recall also that an ideal $P$ of $R$ is said to be prime if $P \neq R$ and whenever $a$ and $b$ are elements of $R$ such that $ab \in P$, then $a \in P$ or $b \in P$. Note that $P$ is a prime ideal of $R$ if and only if $R \setminus P$ is a multiplicative subset of $R$. In [2], D. D. Anderson and E. Smith have defined a proper ideal of $R$ to be weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Some properties of weakly prime ideals have been settled. On the other hand, A. Hamed and A. Malek have introduced and investigated the concept of $S$-prime ideals which constitute a generalization of prime ideals (see [10]). More precisely, let $R$ be a commutative ring, $S$ a multiplicative subset of $R$ and $I$ an ideal of $R$ disjoint from $S$. Then, $I$ is called an $S$-prime ideal of $R$ if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in I$. Note that if $S$ consists of units of $R$, then the notions of $S$-prime and prime ideals coincide. Recall that an ideal $P$ of $R$ is said to be primary if for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in \sqrt{P}$.

\begin{itemize}
\item \textbf{Date:} Received: Sep 30, 2021; Accepted: Jan 26, 2022.
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\item \textbf{2010 Mathematics Subject Classification.} 13A15, 13B02, 13B10, 13B25, 13B30, 13F20.
\item \textbf{Key words and phrases.} $S$-primary ideal, weakly $S$-primary ideal, weakly primary ideal, weakly $S$-prime ideal, Amalgamated algebra.
\end{itemize}
In [3], S. E. Atani and F. Farzalipour have defined a proper ideal of \( R \) to be \textit{weakly primary} if \( 0 \neq ab \in P \) implies \( a \in P \) or \( b \in \sqrt{P} \). The first author in [12] introduced and investigated the concept of \( S \)-primary ideals which constitute a generalization of primary ideals. More precisely, let \( R \) be a commutative ring, \( S \) a multiplicative subset of \( R \) and \( I \) an ideal of \( R \) disjoint from \( S \). Then, \( I \) is called an \( S \)-primary ideal of \( R \) if there exists an \( s \in S \) such that for all \( a, b \in R \) if \( ab \in I \), then \( sa \in I \) or \( sb \in \sqrt{I} \). Note that if \( S \) consists of units of \( R \), then the notions of \( S \)-primary and primary ideals coincide. In [1] F. A. A. Almahdi, E. M. Bouba and M. Tamekkante have defined a proper ideal \( P \) of \( R \) disjoint from a multiplicative subset \( S \) to be \textit{weakly S-prime} if \( 0 \neq ab \in P \) implies \( sa \in P \) or \( sb \in P \). The main goal of the present paper is to complete this circle of ideas by introducing and studying the concept of weakly \( S \)-primary ideals of a commutative ring in a way that generalizes essentially all the results concerning weakly primary ideals. Let \( R \) be a commutative ring, \( S \) a multiplicative subset and \( P \) a proper ideal of \( R \) disjoint from \( S \). Then we say that \( P \) is \textit{weakly \( S \)-primary ideal} of \( R \) if there exists an \( s \in S \) such that for all \( a, b \in R \) if \( 0 \neq ab \in P \), then \( sa \in P \) or \( sb \in \sqrt{P} \). In Section 2, we study the basic properties of weakly \( S \)-primary ideals. Example 2.3 provides a weakly \( S \)-primary ideal which is not weakly \( S \)-prime. Example 2.4 gives a weakly \( S \)-primary ideal but is not \( S \)-primary. Proposition 2.6 states that \( P \) is a weakly \( S \)-primary ideal of \( R \) if and only if \((P : s)\) is a weakly primary ideal of \( R \) for some \( s \in S \) if and only if \( S^{-1}P \) is a weakly primary ideal of \( S^{-1}R \) and there is \( s \in S \) such that \((P : t) \subseteq (P : s)\) for all \( t \in S \). In Theorem 2.8, we show that a weakly \( S \)-primary ideal \( P \) that is not \( S \)-primary satisfies \( P^2 = 0 \) and \( \sqrt{P} = \sqrt{(0)} \). Theorem 2.9 provides others characterizations of weakly \( S \)-primary ideals in the case where \( S \subseteq \text{Reg}(R) \). Recall that, in general, the intersection of a family of \( S \)-primary ideals is not \( S \)-primary, but we have the following result: Let \( R \) be a commutative ring and \( S \subseteq R \) a strongly-multiplicative set. \((P_\alpha)_{\alpha \in \Lambda}\) be a chain of weakly \( S \)-primary ideals of \( R \) that are not \( S \)-primary. Then \( P = \bigcap_{\alpha \in \Lambda} P_\alpha \) is a weakly \( S \)-primary ideal of \( R \). Theorem 2.18 characterizes weakly \( S \)-primary ideals of the ring \( R = R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are commutative rings. Section 3 is devoted to study the form of weakly \( S \)-primary ideals in trivial extension and in the amalgamation of \( A \) with \( B \) along an ideal \( J \) with respect to \( f \) (such amalgamation is denoted by \( A \bowtie_f J \)). This concept has been introduced and studied by D’Anna and Fontana in [6]. Any unexplained terminology is standard as in [4], [9], [11] and [13].

2. Weakly \( S \)-primary ideals

We start this section by introducing the concept of weakly \( S \)-primary ideals of a commutative ring \( R \), where \( S \) is a multiplicative subset of \( R \). The following definition constitutes the weakly-version of \( S \)-primary ideals.

**Definition 2.1.** Let \( R \) be a commutative ring, \( S \) a multiplicative subset of \( R \) and \( P \) an ideal of \( R \) disjoint from \( S \). We say that \( P \) is a weakly \( S \)-primary ideal of \( R \) if there exists an \( s \in S \) such that for all \( a, b \in R \) if \( 0 \neq ab \in P \), then \( sa \in P \) or \( sb \in \sqrt{P} \).
Remark 2.2. 1. If \( S \) consists of units of \( R \), then the weakly primary and the weakly \( S \)-primary ideals coincide.

2. Clearly, if \( P \) is a weakly \( S \)-prime ideal of a ring \( R \), then \( P \) is a weakly \( S \)-primary. The converse does not hold in general.

Example 2.3. Let \( I \) be a primary and not a prime ideal of a commutative ring \( R \) (We can take, for example \( R = \mathbb{Z} \) and \( I = 9\mathbb{Z} \)). Let \( J \) be an ideal of \( R \) and set \( S := \{(1,0), (1,1)\} \). Clearly, \( S \) is a multiplicative subset of \( R \times R \) and \((I \times J) \cap S = \emptyset \). \( I \times J \) is a weakly \( S \)-primary ideal of \( R \times R \). Indeed, let \((0,0) \neq (a,b)(c,d) \in I \times J \), then \( ac \in I \) which is a primary ideal, hence \( a \in I \) or \( c \in \sqrt{I} \), therefore \((1,0)(a,b) \in I \times J \) or \((1,0)(c,d) \in \sqrt{I} \times J \). In the other hand, there exist \( x, y \notin I \) such that \( xy \notin I \) which implies that \((x,0)(y,0) \notin I \times J \) but \((1,0)(x,0) \notin I \times J \), \((1,0)(y,0) \notin I \times J \), \((1,1)(x,0) \notin I \times J \) and \((1,1)(y,0) \notin I \times J \) that is \( I \times J \) is not \( S \)-prime.

It is clear that an \( S \)-primary ideal is a weakly \( S \)-primary ideal but the following example show that the converse is not true in general:

Example 2.4. Let \( R = \mathbb{Z}/12\mathbb{Z} \), \( S = \{1, 5, 7, 11\} \). It is clear that \((0)\) is a weakly \( S \)-primary ideal which is not \( S \)-primary. Indeed, \( \overline{4} \times \overline{3} = \overline{0} \) but, \( \overline{s4} \notin (\overline{0}) \) and \((\overline{s3^3})^n \notin (\overline{0}) \) for every \( s \in \{1,5,7,11\} \) and for each nonzero integer \( n \).

If \( P \) is a weakly primary ideal of \( R \) disjoint with \( S \), then \( P \) is weakly \( S \)-primary ideal of \( R \). The converse is not true in general:

Example 2.5. Consider the polynomial ring \( R = \mathbb{Z}[X] \) and set \( S := \{2^n \mid n \in \mathbb{N}\} \). By using [10, Example 1(3)], \( P = 4XR \) is an \( S \)-prime ideal of \( R \) and so is an \( S \)-primary ideal of \( R \). Thus, \( P \) is a weakly \( S \)-primary ideal of \( R \) but, we claim that \( P \) is not a weakly primary ideal of \( R \). Indeed, we have \( 0 \neq 4X \in P \) and \( 4 \notin P \). If \( X \in \sqrt{P} \), then there exists an integer \( n \geq 1 \) and \( a_n \in \mathbb{Z} \setminus \{0\} \) such that \( X^n = 4a_nX^n \). Hence, \( a_n = \frac{1}{4} \), the desired contradiction asserting our claim.

Our next proposition characterizes the weakly \( S \)-primary ideals of a commutative ring \( R \) but, first recall that if \( I \) is an ideal of \( R \) and \( s \in R \), then \((I : s) := \{x \in R \mid sx \in I \} \) is an ideal of \( R \) containing \( I \).

Let \( R \) be a commutative ring, \( S \) a multiplicative subset of \( R \) and \( P \) an ideal of \( R \) disjoint from \( S \). It is clear that if \((P : s)\) is a weakly primary ideal of \( R \) for some \( s \in S \), then \( P \) is a weakly \( S \)-primary ideal. However, the converse is not true in general but, if \( S \) consisting of regular elements we have the following result:

Proposition 2.6. Let \( R \) be a commutative ring and \( S \) a multiplicative subset of \( R \) consisting of regular elements and \( P \) be an ideal of \( R \) disjoint from \( S \). Then, the following assertions are equivalent:

1. \( P \) is a weakly \( S \)-primary ideal of \( R \).
2. \((P : s)\) is a weakly primary ideal of \( R \) for some \( s \in S \).
3. \( S^{-1}P \) is a weakly primary ideal of \( S^{-1}R \) and there exists \( s \in S \) such that \((P : t) \subseteq (P : s)\) for all \( t \in S \).
(4) \( S^{-1}P \) is a weakly primary ideal of \( S^{-1}R \) and \( S^{-1}P \cap R = (P : s) \) for some \( s \in S \).

Proof. (1) \( \implies \) (2). Since \( P \) is weakly \( S \)-primary, then there exists \( s \in S \) such that for all \( x, y \in R \) with \( 0 \neq xy \in P \), we have \( sx \in P \) or \( sy \in \sqrt{P} \). First we claim that for \( s \in S \) and \( m \in \mathbb{N}^* \) we have \( (P : s^m) = (P : s) \). Indeed let \( x \in (P : s) \) then, \( sx \in P \) and so \( s^{m+1}x \in P \) hence \( x \in (P : s) \). Conversely, let \( 0 \neq x \in (P : s^m) \), then \( 0 \neq s^{m+1}x \in P \), so \( s^{m+1} \in \sqrt{P} \) or \( sx \in P \), then \( sx \in P \) since \( P \cap S = \emptyset \) and hence \( x \in (P : s) \). Now let \( 0 \neq ab \in (P : s) \). Then, \( 0 \neq sab \in P \), hence \( 0 \neq s^2a \in P \) or \( sb \in \sqrt{P} \). Thus, \( sa \in P \) or \( sb \in \sqrt{P} \) since \( S \cap P = \emptyset \). If \( sb \in \sqrt{P} \), there exists an integer \( n \geq 1 \) such that \( s^nb^n \in P \) then \( b^n \in (P : s^n) = (P : s) \) and so \( b \in \sqrt{(P : s)} \). Hence, \((P : s)\) is weakly primary ideal of \( R \).

(2) \( \implies \) (1). Clear.

(1) \( \implies \) (3). As \( P \cap S = \emptyset \), we have that \( S^{-1}P \neq S^{-1}R \). Let \( 0 \neq \frac{a}{s_1} \cdot \frac{b}{s_2} \in S^{-1}P \) where \( a, b \in R \) and \( s_1, s_2 \in S \). Then, \( \frac{a}{s_1} \cdot \frac{b}{s_2} = \frac{p}{s_3} \) for some \( p \in P \) and \( s_3 \in S \). So there exists \( u \in S \) such that \( 0 \neq us_3ab = us_1s_2p \in P \). Since \( P \) is weakly \( S \)-primary, there exists \( s \in P \) such that \( sus_3 \in \sqrt{P} \) or \( 0 \neq sab \in P \). Thus \( sab \in P \) since \( sus_3 \notin \sqrt{P} \). Hence, \( 0 \neq s^2a \in P \) or \( sb \in \sqrt{P} \), and so \( sa \in P \) or \( sb \in \sqrt{P} \). This implies that \( \frac{a}{s_1} = \frac{sa}{ss_1} \in S^{-1}P \) or \( \frac{b}{s_2} = \frac{sb}{ss_2} \in S^{-1}P \), and so \( S^{-1}P \) is a weakly primary ideal of \( S^{-1}R \).

Let \( s \in S \) the element associated to \( P \). Let \( t \in S \) and \( 0 \neq a \in (P : t) \), so \( 0 \neq ta \in P \). Hence \( st \in \sqrt{P} \) or \( sa \in P \). Since \( P \cap S = \emptyset \), \( st \notin \sqrt{P} \) which implies that \( a \in (P : s) \) consequently \( (P : t) \subseteq (P : s) \).

(3) \( \implies \) (1). Let \( a, b \in R \) such that \( 0 \neq ab \in P \). Since \( 0 \neq \frac{a}{1} \cdot \frac{b}{1} \in S^{-1}P \), we have \( \frac{a}{t} \in S^{-1}P \) or \( \left( \frac{b}{t} \right)^n \in S^{-1}P \) for some \( n \geq 1 \). If \( \frac{a}{t} \in S^{-1}P \), then \( \frac{a}{t} = \frac{p}{t} \) for some \( p \in P \) and \( t \in S \). Hence, \( ta = p \), and so \( a \in (P : t) \subseteq (P : s) \), then \( sa \in P \).

If \( \left( \frac{b}{t} \right)^n \in S^{-1}P \), \( \frac{b^n}{t} = \frac{a}{u} \) for some \( q \in P \) and \( u \in S \). Hence, \( ub^n = q \in P \) and so \( b^n \in (P : u) \subseteq (P : s) \) then \( sb \in \sqrt{(P : s)} \) which means that \( P \) is weakly \( S \)-primary ideal of \( R \).

(1) \( \implies \) (4). As in \((1) \implies (3)\) we have , \( S^{-1}P \) is weakly primary ideal of \( S^{-1}R \). Let \( 0 \neq a \in (P : s) \). Then, \( sa \in P \) and \( a = \frac{sa}{s} \in S^{-1}P \). Hence \( a \in S^{-1}P \cap R \), and so \( (P : s) \subseteq S^{-1}P \cap R \). Now let \( 0 \neq a \in S^{-1}P \cap R \), then \( a \in R \) and \( a = \frac{p}{t} \) with \( p \in P \) and \( t \in S \). So, \( 0 \neq ta = p \in P \). Hence \( st \in \sqrt{P} \) or \( sa \in P \). Thus , \( sa \in P \) since \( S \cap P = \emptyset \). Consequently \( a \in (P : s) \) and so \( S^{-1}P \cap R \subseteq (P : s) \).

(4) \( \implies \) (1). Let \( a, b \in R \) such that \( 0 \neq ab \in P \). Since \( 0 \neq \frac{a}{1} \cdot \frac{b}{1} \in S^{-1}P \), we have \( \frac{a}{t} \in S^{-1}P \) or \( \frac{b}{t} \in \sqrt{S^{-1}P} \). If \( \frac{a}{t} \in S^{-1}P \), then \( \frac{a}{t} = \frac{p}{t} \) for some \( p \in P \) and \( t \in S \). Hence, \( ta = p \in P \) and so \( a = \frac{ta}{t} \in S^{-1}P = (P : s) \) by hypothesis, that is \( sa \in P \).

Similarly, if \( \frac{b}{t} \in \sqrt{S^{-1}P} \) we have \( sb \in \sqrt{P} \). Thus \( P \) is a weakly \( S \)-primary ideal of \( R \).  

Example 2.7. Let \( R \) be a commutative ring, \( S \) a multiplicative set of \( R \) consisting of nonzero divisors and \( P \) an ideal of \( R \) disjoint with \( S \). If \( P \) is a weakly primary ideal of \( R \), then for any \( s \in S \), \( sP \) is a weakly \( S \)-primary ideal of \( R \). Indeed, let \( s \in S \). It is convenient to denote \( sP \) by \( I \). As \( I \subseteq P \) and \( P \cap S = \emptyset \), it follows
that $I \cap S = \emptyset$. Since $P$ is a weakly primary ideal of $R$ with $\sqrt{P} \cap S = \emptyset$, we get that $(sP : s) = P$. Hence, $(I : s) = P$ is a weakly primary ideal of $R$. Therefore, we claim that $I = sP$ is a weakly $S$-primary ideal of $R$ by Proposition 2.6.

**Theorem 2.8.** Let $R$ be a commutative ring, $S$ a multiplicative set of $R$ consisting of nonzero divisors and $P$ an ideal of $R$ disjoint with $S$. Suppose that $P$ is weakly $S$-primary and not $S$-primary. Then, $\sqrt{P} = (0)$.

**Proof.** First, we prove that $P^2 = 0$. Suppose that $P^2 \neq 0$, we show that $P$ is $S$-primary. Let $pq \in P$ where $p, q \in R$. If $pq \neq 0$, then either $sp \in P$ or $sq \in \sqrt{P}$ since $P$ is weakly $S$-primary. So, suppose that $pq = 0$. If $P \neq 0$, then there is an element $p'$ of $P$ such that $pp' \neq 0$, so $0 \neq pp' = p(p' + q) \in P$, and hence $P$ weakly $S$-primary gives either $sp \in P$ or $s(p' + q) \in \sqrt{P}$. As $p' \in P \subseteq \sqrt{P}$ we have either $sp \in P$ or $sq \in \sqrt{P}$. So, we can assume that $pP = 0$. Similarly, we can assume that $qP = 0$. Since $P^2 \neq 0$, there exist $c, d \in P$ such that $cd \neq 0$. Then, $(p + c)(q + d) = cd \in P$, so either $s(p + c) \in P$ or $s(q + d) \in \sqrt{P}$ and hence either $sp \in P$ or $sq \in \sqrt{P}$. Thus, $P$ is $S$-primary. Clearly, $\sqrt{(0)} \subseteq \sqrt{P}$.

As $P^2 = 0$, we get $P \subseteq (0)$, hence $\sqrt{P} \subseteq (0)$ as required. \hfill $\Box$

We next gives other characterisations of weakly $S$-primary ideals.

**Theorem 2.9.** Let $R$ be a commutative ring, $S$ a multiplicative set of $R$ consisting of nonzero divisors and $P$ a proper ideal of $R$ disjoint with $S$. Then, the following assertions are equivalent:

1. $P$ is weakly $S$-primary ideal of $R$.
2. There exists $s \in S$ such that for each $x \in R \setminus \sqrt{(P : s)}$, $(P : sx) = (P : s) \cup (0 : x)$ for some $s \in S$.
3. There exists $s \in S$ such that for each $x \in R \setminus \sqrt{(P : s)}$, $(P : sx) = (P : s)$ or $(P : sx) = (0 : x)$ for some $s \in S$.

**Proof.** (i) $\implies$ (ii). Since $P$ is weakly $S$-prime there exists $s \in S$, such that for each $0 \neq ab \in P$, we have $sa \in P$ or $sb \in P$.

Let $y \in (P : sx)$ where $x \in R \setminus \sqrt{(P : s)}$. If $xy \neq 0$, so $0 \neq sxy \in P$, hence $s^2x \in \sqrt{P}$ or $sy \in P$. If $s^2x \in \sqrt{P}$, then $0 \neq s^{2n+1}x^n \in P$ for some integer $n$ and hence $s^{2n+1}x^n \in P$. This is absurd, then $sy \in P$ so $y \in (P : s)$. If $xy = 0$ then $y \in (0 : x)$. In each cases $y \in (0 : x) \cup (P : s)$.

For the other inclusion, if $y \in (P : s)$, then $sy \in P$ so $sxy \in P$ and hence $y \in (P : xs)$. If $y \in (0 : x)$, then $sxy = 0$ so $y \in (P : sx)$. As the reverse containment holds for any ideal $P$, we have equality.

Hence $(P : s) \cup (0 : x) \subseteq (P : sx)$.

(ii) $\implies$ (iii). Clear.

(iii) $\implies$ (i). Let $0 \neq xy \in P$ and suppose that $sx \notin \sqrt{P}$, then for each $n \in \mathbb{N}^*$ $s^n x^n \notin P$ so $sx^n \notin P$ hence $x^n \notin (P : s)$ and so $x \in R \setminus \sqrt{(P : s)}$. Now since $0 \neq sxy \in P$ we have $y \in (P : sx) = (P : s) \cup (0 : x)$ hence $y \in (P : s)$ since $xy \neq 0$, so $sy \in P$ and hence $P$ is weakly $S$-primary. \hfill $\Box$
Remark 2.10. Let $S_1 \subseteq S_2$ be multiplicative subsets of $R$ and $P$ an ideal of $R$ disjoint from $S_2$. Clearly if $P$ is a weakly $S_1$-primary of $R$ then $P$ is a weakly $S_2$-primary. However the converse is not true in general. To see this, we consider the ideal $P = (4X)$ of the ring $\mathbb{Z}[X]$ and set $S_1 = 1$ and $S_2 = \{2^n \mid n \in \mathbb{N}\}$. By Example 2.5 $P$ is a weakly $S_2$-primary of $\mathbb{Z}[X]$ but not weakly $S_1$-primary.

**Proposition 2.11.** Let $R$ be a commutative ring, $S_1 \subseteq S_2$ be multiplicative subsets of $R$ such that for any $s \in S_2$ there exists an element $t \in S_2$ satisfying $st \in S_1$. If $P$ is a weakly $S_2$-primary ideal of $R$, then $P$ is a weakly $S_1$-primary ideal of $R$.

**Proof.** Let $a, b \in R$ such that $0 \neq ab \in P$. So there exists an $s \in S_2$ such that $sa \in P$ or $sb \in \sqrt{P}$. By assumption, $st = st \in S_1$ for some $t \in S_2$, and then $s'a \in P$ or $s'b \in \sqrt{P}$. This completes the proof. \(\square\)

Let $S$ be a multiplicative subset of $R$, $S^* = \{r \in R \mid \frac{r}{1} \text{ is unit in } S^{-1}R\}$ denotes the saturation of $S$. Note that $S^*$ is a multiplicative subset of $R$ containing $S$. A multiplicative subset of $R$ is called saturated if $S^* = S$. It is clear that $S^*$ is always a saturated multiplicative subset of $R$ [9].

**Proposition 2.12.** Let $R$ be a commutative ring, $S$ a multiplicative subset of $R$ and $P$ an ideal of $R$ disjoint from $S$. Then $P$ is a weakly $S$-primary ideal of $R$ if and only if $P$ is a weakly $S^*$-primary ideal.

**Proof.** It is clear that $S^* \cap P = \emptyset$. We will show that for any $r \in S^*$, there is $r' \in S^*$ such that $rr' \in S$. Let $r \in S^*$, then $\frac{r}{1} = 1$ for some $s \in S$ and $a \in R$. This implies that $tar = ts$ for some $t \in S$. Now take $r' = ta$, we have $r' \in S^*$ with $rr' \in S$ and so the desired condition is satisfied. Therefore, by putting $S_1 = S$ and $S_2 = S^*$ we conclude immediatly the result from the Proposition 2.11. \(\square\)

**Proposition 2.13.** Let $f : R \rightarrow T$ be a ring homomorphism and $S$ a multiplicative subset of $R$ such that $0 \notin f(S)$. Then, the following hold:

1. If $f$ is an epimorphism and $P$ is a weakly $S$-primary ideal of $R$ containing $\ker(f)$, then $f(P)$ is a weakly $f(S)$-primary ideal of $T$.

2. If $f$ is a monomorphism and $Q$ is a weakly $f(S)$-primary ideal of $T$, then $f^{-1}(Q)$ is a weakly $S$-primary ideal of $R$.

**Proof.** (1) Let $r \in f(S) \cap f(P)$. Then, $r = f(p) = f(s)$ for some $p \in P$ and $s \in S$. So $s - p \in \ker(f)$, which implies that $s \in P$, a contradiction. Hence $f(S) \cap f(P) = \emptyset$. Now let $0 \neq xy \in f(P)$. Then there is $a, b \in R$ such that $x = f(a)$, $y = f(b)$ and $0 \neq f(ab) = xy \in f(P)$. Since $\ker(f) \subseteq P$, we get $0 \neq ab \in P$, and so $sa \in P$ or $sm^ny^m \in P$ for some integer $m \geq 1$, for some $s \in S$. It means that $f(s)x \in f(P)$ or $(f(s))^m y^m \in f(P)$. Thus $f(s)x \in f(P)$ or $f(s)y \in \sqrt{f(P)}$ and hence $f(P)$ is a weakly $f(S)$-primary ideal of $T$.

(2) Since $Q$ is a weakly $f(S)$-primary ideal of $T$, there exists $s \in S$ such that, for all $x, y \in T$, $0 \neq xy \in Q$ we have either $f(s)x \in Q$ or $f(s)y \in \sqrt{Q}$. We can easily show that $f^{-1}(Q) \cap S = \emptyset$. Let $a, b \in R$ such that $0 \neq ab \in f^{-1}(Q)$. Since $\ker(f) = \{0\}$, we get $0 \neq f(ab) = f(a)f(b) \in Q$. Then $f(s)f(a) = f(sa) \in Q$.
or \((f(s)f(b))^n = f((sb)^n) \in Q\), for some integer \(n \geq 1\). Hence \(sa \in f^{-1}(Q)\) or \(sb \in \sqrt{f^{-1}(Q)}\), and so we conclude that \(f^{-1}(Q)\) is a weakly \(S\)-primary ideal of \(R\). \(\square\)

Let \(R\) be a commutative ring, \(S\) a multiplicative subset of \(R\) and \(P\) an ideal of \(R\) disjoint from \(S\). Set \(\overline{S} := \{s + P \mid s \in S\}\). It is easy to check that \(\overline{S}\) is a multiplicative subset of \(R/P\).

**Corollary 2.14.** Let \(R\) be a commutative ring and \(S\) a multiplicative subset of \(R\).

1. If \(I \subseteq P\) be two ideals of \(R\) such that \(P \cap S = \emptyset\). If \(P\) is a weakly \(S\)-primary ideal of \(R\), then \(P/I\) is a weakly \(\overline{S}\)-primary ideal of \(R/I\).
2. If \(R\) is a subring of \(T\) and \(Q\) is a weakly \(S\)-primary ideal of \(T\), then \(Q \cap R\) is a weakly \(S\)-primary ideal of \(R\).

**Proof.** (1) Follows by applying Proposition 2.13(1) to the canonical surjection \(\pi : R \rightarrow R/I\).

(2) It suffices to apply Proposition 2.13(2) to the naturel injection \(\iota : R \hookrightarrow T\), since \(\iota^{-1}(Q) = Q \cap R\). \(\square\)

**Proposition 2.15.** Let \(R\) be a commutative ring, \(S\) a multiplicative subset of \(R\) and \(P\) an ideal of \(R\) disjoint from \(S\). If \(J\) is an ideal of \(R\) such that \(J \cap S \neq \emptyset\) and \(P\) is a weakly \(S\)-primary ideal of \(R\), then so are \(J \cap P\) and \(JP\).

**Proof.** It is obvious that \(JP \cap S = \emptyset\) and \((J \cap P) \cap S = \emptyset\) since \(JP \subseteq P\) and \(P \cap S = \emptyset\). As \(P\) is a weakly \(S\)-primary ideal of \(R\), then there exists \(s \in S\) such that \(sa \in P\) or \(sb \in \sqrt{P}\).

First we will prove that \(J \cap P\) is a weakly \(S\)-primary ideal of \(R\). Pick \(t \in J \cap S\) (such \(t\) exists since \(J \cap S \neq \emptyset\)) and let \(a, b \in R\) such that \(0 \neq ab \in J \cap P(\subseteq P)\). Thus \(sta \in P \cap J\) and \(stb \in \sqrt{P} \cap \sqrt{J} = \sqrt{P \cap J}\). Consequently \(P \cap J\) is a weakly \(S\)-primary ideal of \(R\).

We prove now that \(JP\) is a weakly \(S\)-primary ideal, let \(x, y \in R\) such that \(0 \neq xy \in JP(\subseteq P)\) If \(sx \in P\), then \((ts)x = t(sx) \in JP\). Assume now that \(sy \in \sqrt{P}\). Then \(s^ny^n = (sy)^n \in P\) for some integer \(n \geq 1\). Thus \(((ts)y)^n = t^ns^ny^n \in JP\). It follows that \((ts)y \in \sqrt{JP}\). Therefore, \(JP\) is a weakly \(S\)-primary ideal of \(R\). \(\square\)

Recall that, in general, the intersection of a family of \(S\)-primary ideals is not \(S\)-primary, but we have the following results:

**Proposition 2.16.** Let \(R\) be a commutative ring, \(S\) a multiplicative set of \(R\). Let \(n \geq 1\). let \(i \in \{1, \ldots, n\}\). Let \(P_i\) be an ideal of \(R\) with \(P_i \cap S = \emptyset\). If \(P_i\) is a weakly \(S\)-primary ideal of \(R\) for each \(i \in \{1, \ldots, n\}\) with \(\sqrt{P_i} = \sqrt{P_j}\) for all \(i, j \in \{1, \ldots, n\}\), then \(\cap_{i=1}^nP_i\) is a weakly \(S\)-primary ideal of \(R\).

**Proof.** Let \(i \in \{1, \ldots, n\}\). Since \(P_i\) is a weakly \(S\)-primary ideal of \(R\), there exists \(s_i \in S\) such that for all \(a, b \in R\) with \(0 \neq ab \in P_i\), we have either \(sa \in P_i\) or \(sb \in \sqrt{P_i}\). Let \(s = \prod_{i=1}^n s_i\). Then \(s \in S\). Let \(a, b \in R\) be such that \(0 \neq ab \in \cap_{i=1}^nP_i\). Suppose that \(sa \not\in \cap_{i=1}^nP_i\). Then \(sa \not\in P_k\) for some
k ∈ \{1, \ldots, n\}. Hence, \( s_k a \notin P_k \). From \( 0 \neq ab \in P_k \), it follows that \( s_k b \in \sqrt{P_k} \). Therefore, \( sb \in \sqrt{P_k} \). By hypothesis, \( \sqrt{P_i} = \sqrt{P_j} \) for all \( i \in \{1, \ldots, n\} \). Thus \( sb \in \sqrt{P_i} = \cap_{i=1}^n \sqrt{P_i} = \sqrt{\cap_{i=1}^n P_i} \). This shows that the element \( s \in S \) is such that for all \( a, b \in R \) with \( 0 \neq ab \in \cap_{i=1}^n P_i \), we have either \( sa \in \cap_{i=1}^n P_i \) or \( sb \in \sqrt{\cap_{i=1}^n P_i} \). This proves that \( \cap_{i=1}^n P_i \) is a weakly \( S \)-primary ideal of \( R \).

\[ \square \]

Recall from [10] that if \( R \) is a commutative ring with identity and \( S \) a multiplicative set of \( R \). We say that \( S \) is a strongly-multiplicative set if for each family \( (s_\alpha)_{\alpha \in \Lambda} \) of element of \( S \) we have

\[ (\bigcap_{\alpha \in \Lambda} s_\alpha R) \cap S \neq \emptyset \]

**Theorem 2.17.** Let \( R \) be a commutative ring and \( S \subseteq R \) a strongly-multiplicative set. If \( (P_\alpha)_{\alpha \in \Lambda} \) is a chain of weakly \( S \)-primary ideals of \( R \) that are not \( S \)-primary, then \( P = \bigcap_{\alpha \in \Lambda} P_\alpha \) is a weakly \( S \)-primary ideal of \( R \).

**Proof.** First, we show that \( \sqrt{P} = \bigcap_{\alpha \in \Lambda} \sqrt{P_\alpha} \). Clearly, \( \sqrt{P} \subseteq \bigcap_{\alpha \in \Lambda} \sqrt{P_\alpha} \). For the other inclusion suppose that \( r \in \bigcap_{\alpha \in \Lambda} \sqrt{P_\alpha} \), so \( r^m = 0 \) for some \( m \) since \( \bigcap_{\alpha \in \Lambda} \sqrt{P_\alpha} = \sqrt{(0)} \) by Theorem 2.4. It follows that \( r^m \in P_\alpha \) for each \( \alpha \in \Lambda \) and hence \( r \in \sqrt{P} \).

For each \( \alpha \in \Lambda \), there exists \( s_\alpha \in S \) such that for all \( a, b \in R \), \( 0 \neq ab \in P_\alpha \) we have \( s_\alpha a \in P_\alpha \) or \( s_\alpha b^n \in P_\alpha \) for some \( n \geq 1 \).

Since \( S \) is strongly-multiplicative set , then \( (\bigcap_{\alpha \in \Lambda} s_\alpha R) \cap S \neq \emptyset \). Let \( t \in (\bigcap_{\alpha \in \Lambda} s_\alpha R) \cap S \). we will show that for all \( a, b \in R \) such that \( 0 \neq ab \in P \) we have \( ta \in P \) or \( tb \in \sqrt{P} \).

Let \( a, b \in R \), such that \( 0 \neq ab \in P \) and suppose that \( ta \notin P \). Then, \( ta \notin P_\beta \) for some \( \beta \in \Lambda \). Let \( \alpha \in \Lambda \). We have \( P_\alpha \subseteq P_\beta \) or \( P_\beta \subseteq P_\alpha \).

**First case**, \( P_\alpha \subseteq P_\beta \). Since \( ta \notin P_\beta \) then \( ta \notin P_\alpha \), so \( s_\alpha a \notin P_\alpha \). This implies that \( s_\beta b \in \sqrt{P_\alpha} \) and \( tb \in \sqrt{P_\alpha} \).

**Second case**, \( P_\beta \subseteq P_\alpha \). As \( ab \in P_\beta \) and \( ta \notin P_\beta \), then \( s_\beta a \notin P_\beta \) so \( (s_\beta b)^m \in P_\beta \subseteq P_\alpha \) for some \( m \geq 1 \). this implies that \( (tb)^m \in P_\alpha \) for some \( m \geq 1 \) and hence \( tb \in \sqrt{P} \). \[ \square \]

Let \( R \) be a commutative ring. \( R \) is called decomposable if \( R = R_1 \times R_2 \) for some commutative rings \( R_1 \) and \( R_2 \). If \( I_1 \) is an ideal of \( R_1 \), then \( I_1 \times R_2 \) is an ideal of \( R_1 \times R_2 \) and \( \sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2 \). Similarly, if \( I_2 \) is an ideal of \( R_2 \), then \( R_1 \times I_2 \) is an ideal of \( R_1 \times R_2 \) and \( \sqrt{(R_1 \times I_2)} = R_1 \times \sqrt{I_2} \).

Now, we establish the following result.

**Theorem 2.18.** Let \( R_1 \) and \( R_2 \) be commutative rings and let \( S_1 \) and \( S_2 \) be multiplicative subsets of \( R_1 \) and \( R_2 \) respectively. Set \( R = R_1 \times R_2 \) and \( S = S_1 \times S_2 \) and \( P_1, P_2 \) are nonzero ideals of \( R_1 \) and \( R_2 \) respectively. Then the following assertions are equivalent:

1. \( P := P_1 \times P_2 \) is a weakly \( S \)-primary ideal of \( R \).
(2) $P_1$ is an $S_1$-primary ideal of $R_1$ and $S_2 \cap P_2 \neq \emptyset$ or $P_2$ is an $S_2$-primary ideal of $R_2$ and $S_1 \cap P_1 \neq \emptyset$.

(3) $P := P_1 \times P_2$ is an $S$-primary ideal of $R$.

Proof. (1) $\implies$ (2). Let $0_R \neq (p, q) \in P$, with $p \in R_1$ and $q \in R_2$. Then, $0_R \neq (p, q) = (p, 1)(1, q) \in P$. Since $P$ is weakly $S$-primary ideal of $R$, then there exists $s = (s_1, s_2) \in S$ such that $s(p, 1) = (s_1p, s_2) \in P$ or $s(1, q) = (s_1, s_2q) \in \sqrt{P} = \sqrt{P_1} \times \sqrt{P_2}$. Thus $S_2 \cap P_2 \neq \emptyset$ or $S_1 \cap P_1 \neq \emptyset$. Assume $S_2 \cap P_2 \neq \emptyset$. As $P \cap S = \emptyset$, we have $S_1 \cap P_1 = \emptyset$. Now we show that $P_1$ is an $S_1$-primary ideal of $R_1$. Let $ab \in P_1$ for some $a, b \in R_1$. Since $S_2 \cap P_2 \neq \emptyset$, then there exists $0_{R_2} \neq t \in S_2 \cap P_2$, and so we have $0_R \neq (a, t)(b, 1) \in P$. Hence $(a, t) = (s_1a, s_2t) \in P$ or $s(b, 1) = (s_1b, s_2) \in \sqrt{P} = \sqrt{P_1} \times \sqrt{P_2}$. So $s_1a \in P_1$ or $s_1b \in \sqrt{P_1}$ as desired.

(2) $\implies$ (3). Suppose that $P_1$ is an $S_1$-primary ideal of $R_1$ and $S_2 \cap P_2 \neq \emptyset$, let $t \in S_2 \cap P_2$. Let $(a, b)(c, d) = (ac, bd) \in P_1 \times P_2$, then $ac \in P_1$ and $bd \in P_2$, for some $s_1 \in S_1$ so $s_1a \in P_1$ or $s_1b \in \sqrt{P_1}$ and hence $(s_1, t)(a, b) = (s_1a, tb) \in P$ or $(s_1, t)(c, d) = (s_1c, td) \in \sqrt{P_1} \times P_2 \subseteq \sqrt{P}$, then $P$ is an $S$-primary ideal of $R$.

(3) $\implies$ (1). Clear.

3. Weakly $S$-primary ideals in trivial ring extensions and amalgamations

**Definition 3.1.** Let $R$ be a commutative ring with identity, $S$ a multiplicative set and $P$ an ideal of $R$ disjoint from $S$. We say that $P$ is weakly $S$-primary with the coefficient $s \in S$ if for each $a, b \in R$, we have $0 \neq ab \in P \Rightarrow sa \in P$ or $sb \in \sqrt{P}$.

Let $R$ be a commutative ring with identity and let $M$ be a unitary $R$-module. The idealization of $M$ in $R$ (or trivial extension of $R$ by $M$) is the commutative ring $R(+)M = \{(r, m) \mid r \in R$ and $m \in M\}$ under the usual addition and the multiplication defined as

$$(r_1, m_1)(r_2, m_2) = (r_1r_1, r_1m_2 + r_2m_1),$$

for all $(r_1, m_1), (r_2, m_2) \in R(+)M$.

It is easy to show that if $S$ is a multiplicative subset of $R$, then $S(+)M$ and $S(+)0$ are multiplicative subset of $R(+)M$.

**Theorem 3.2.** Let $R$ be a commutative ring, $M$ a unitary $R$-module, $I$ an ideal of $R$ and $S$ a multiplicative subset of $R$ with $S \cap I = \emptyset$. Then the following are equivalent:

1. $I(+)M$ is a weakly $S(+)M$-primary ideal of $R(+)M$.
2. $I(+)M$ is a weakly $S(+)0$-primary ideal of $R(+)M$.
3. $I$ is a weakly $S$-primary ideal of $R$ with the coefficient $s \in S$, and if $a, b \in R$ with $ab = 0$, but $sa \not\in I$ and $sb \not\in \sqrt{I}$ then $a \in \ann_R(M)$ and $b \in \ann_R(M)$.

Proof. It is not difficult to check that

$$(S(+)M) \cap (I(+)M) = \emptyset \iff S \cap I = \emptyset \iff (S(+)0) \cap (I(+)M) = \emptyset.$$
Assume that \( a, b \in R \) such that \( 0 \neq ab \in I \). Then \( (0, 0) \neq (a, 0)(b, 0) \in I(+)M \). As \( I(+)M \) is a weakly \( S(+)M \)-primary ideal of \( R(+)M \), there exists \( (s, n) \in S(+)M \) such that \( (s, n)(a, 0) = (sa, an) \in I(+)M \) or \( (s, n)(b, 0)^k = (sb, bn)^k = ((sb)^k, ksbn^k) \in I(+)M \) for some \( k \in \mathbb{N}^* \). Thus \( sa \in I \) or \( sb \in I \) and so \( sa \in I \) or \( sb \in \sqrt{I} \), hence \( I \) is weakly \( S \)-primary ideal of \( R \). Now suppose that \( ab = 0 \), \( sa \notin I \) and \( sb \notin \sqrt{I} \). Assume that \( a \notin \text{ann}_R(M) \). Then there exists \( m \in M \) such that \( am \neq 0 \) and so we have \( (0, 0) \neq (a, 0)(b, m) = (0, am) \in I(+)M \). Hence \( (s, n)(a, 0) = (sa, na) \in I(+)M \) or \( (s, n)^l(b, m)^l = (sb^l, l(n + m)s^l) \in I(+)M \) for some \( l \in \mathbb{N}^* \), which is contradiction.

\( (3) \implies (2) \). Let \( (0, 0) \neq (a, m)(b, n) \in I(+)M \). If \( ab \neq 0 \), then \( sa \in I \) or \( sb \in \sqrt{I} \), and hence \( (s, 0)(a, m) = (sa, sm) \in I(+)M \) or \( (s, 0)(b, n) = (sb, sn) \in \sqrt{I}(+)M \). Assume that \( ab = 0 \), but \( sa \notin I \) and \( sb \notin \sqrt{I} \), then \( a, b \in \text{ann}_R(M) \) consequently, we get \( (a, m)(b, n) = (ab, an + bm) = (0, 0) \), a contradiction.

\( \Box \)

Let \( A \) and \( B \) be commutative rings with identity, let \( J \) be an ideal of \( B \), and let \( f : A \to B \) be a ring homomorphism. We set

\[ A \mathrel{\perp} f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}. \]

\( A \mathrel{\perp} f J \) is a subring of \( A \times B \) called the amalgamation of \( A \) with \( B \) along \( J \) with respect to \( f \). Such construction was introduced and studied by D’Anna, Finacchiaro and Fontana in [3, 5]. This construction is in fact a generalization of the amalgamated duplication of a ring along an ideal (cf. [5, 6, 7]).

For a multiplicative subset \( S \) of \( A \), put \( S \perp f J = \{(s, f(s) + j) \mid s \in S \text{ and } j \in J\}, S \perp f 0 = \{(s, f(s)) \mid s \in S\} \). Clearly, \( S \perp f J \) and \( S \perp f 0 \) are a multiplicative subsets of \( A \perp f J \) and if \( f(S) \) does not contain zero, then \( f(S) \) is a multiplicative subset of \( B \). Let \( I \) be an ideal of \( A \) and \( K \) be an ideal of \( f(A) + J \). Notice that

\[ I \perp f J = \{(i, f(i) + j) \mid i \in I, j \in J\} \]

and

\[ K \perp f = \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in K\} \]

are ideals of \( A \perp f J \).

Our next result characterizes weakly \( S \)-primary ideals of the form \( I \perp f J \) and \( K \perp f \) of the amalgamation \( A \perp f J \).

**Theorem 3.3.** Let \( A \) and \( B \) be commutative rings with identity, let \( J \) be an ideal of \( B \), and let \( f : A \to B \) be a ring homomorphism, and \( I \) an ideal of \( A \) disjoint from \( S \). Then the following are equivalent:

1. \( I \perp f J \) is a weakly \( S \perp f J \)-primary ideal of \( A \perp f J \) with the coefficient \( (s, f(s) + j_0) \).
2. \( I \perp f J \) is a weakly \( S \perp f (0) \)-primary ideal of \( A \perp f J \) with the coefficient \( (s, f(s)) \).
(3) $I$ is a weakly $S$-primary ideal of $A$ with the coefficient $s$ and for $a, b \in A$ with $ab = 0$, but $sa \notin I$, $sb \notin \sqrt{I}$, then $jf(a) + if(b) + ij = 0$ for every $i, j \in J$.

Proof. $(2) \implies (1)$ Follows from Remark 2.10 since $S \gg J(0) \subseteq S \gg J$.

$(1) \implies (3)$ Assume that $I \gg J$ is a weakly $S \gg J$-primary ideal of $A \gg J$ with the coefficient $(s, f(s) + j_0)$. Let $ab \in I \setminus \{0\}$ where $a, b \in A$. Then, $(a, f(a))(b, f(b)) \in I \gg J \setminus \{(0, 0)\}$ and so, $(s, f(s) + j_0)(a, f(a)) \in I \gg J$ or $(s, f(s) + j_0)(b, f(b)) \in \sqrt{I} \gg J$. Hence $sa \in I$ or $sb \in \sqrt{I}$. Now, we claim that if $sa \notin I, sb \notin \sqrt{I}$ with $ab = 0$, then $f(a)j + f(b)i + ij = 0$ for every $i, j \in J$. Deny. There exist $i, j \in J$ such that $f(a)j + f(b)i + ij \neq 0$ and so $(0, 0) \neq (a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + f(a)j + f(b)i + ij) = (0, f(a)j + f(b)i + ij) \in I \gg J$, which is a contradiction since $(s, f(s) + j_0)(a, f(a) + i) \notin I \gg J, (s, f(s) + j_0)(b, f(b) + j) \notin \sqrt{I} \gg J$ and $I \gg J$ is a weakly $S \gg J$-primary ideal of $A \gg J$ with the coefficient $(s, f(s) + j_0)$.

$(3) \implies (2)$ let $(a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + f(a)j + f(b)i + ij) \in I \gg J \setminus \{(0, 0)\}$. Hence, $ab \in I$. Two cases are then possible:

Case 1: $ab \neq 0$. Hence, $sa \in I$ or $sb \in \sqrt{I}$ since $I$ is a weakly $S$-primary ideal of $A$. So, $(s, f(s))(a, f(a) + i) = (sa, f(sa) + f(s)i) \in I \gg J$ or $(s, f(s))(b, f(b) + j) = (sb, f(sb) + f(s)j) \in \sqrt{I} \gg J$.

Case 2: $ab = 0$. We claim that $sa \in I$ or $sb \in \sqrt{I}$. Deny. We have $f(a)j + f(b)i + ij = 0$, contradiction with $(a, f(a) + i)(b, f(b) + j) \neq (0, 0)$. Therefore, in all cases, $sa \in I$ or $sb \in \sqrt{I}$ and so $(s, f(s))(a, f(a) + i) \in I \gg J$ or $(s, f(s))(b, f(b) + j) \in \sqrt{I} \gg J$, as desired. \hfill \Box

Theorem 3.4. Let $A$ and $B$ be commutative rings with identity, let $J$ be an ideal of $B$, and let $f : A \rightarrow B$ be a ring homomorphism, and $K$ be an ideal of $f(A) + J$ disjoint from $f(S)$. Then $K^f$ is a weakly $S \gg J(0)$-primary ideal of $A \gg J$ with the coefficient $(s, f(s))$ if and only if $K$ is a weakly $f(S)$-primary ideal of $f(A) + J$ with the coefficient $f(s)$ and when $f(s)(f(a) + j) \notin K, f(s)(f(b) + k) \notin \sqrt{K}$ with $a, b \in A, j, k \in J$ and $(f(a) + j)(f(b) + k) = 0$, then $ab = 0$.

Proof. Assume that $K^f$ is a weakly $S \gg J(0)$-primary ideal of $A \gg J$. We claim that $K$ is a weakly $f(S)$-primary ideal of $f(A) + J$. Indeed, let $xy \in K \setminus \{0\}$ with $x, y \in f(A) + J$. Then, $x = f(a) + j$ and $y = f(b) + k$ for some $a, b \in A$ and $j, k \in J$. Therefore, $(a, f(a) + j)(b, f(b) + k) = (ab, f(a) + j)(f(b) + k) \in K^f \setminus \{(0, 0)\}$ which is a weakly $S \gg J(0)$-primary ideal of $A \gg J$. Consequently, $(s, f(s))(a, f(a) + j) \in K^f$ or $(s, f(s))(b, f(b) + j) \in \sqrt{K}$, making $f(s)(f(a) + j) \in K$ or $f(s)(f(b) + k) \in \sqrt{K}$. Hence, $K$ is a weakly $f(S)$-primary ideal of $f(A) + J$. Now, let $f(s)(f(a) + j) \notin K, f(s)(f(b) + k) \notin \sqrt{K}$ with $(f(a) + j)(f(b) + k) = 0$. We claim that $ab = 0$. Deny, $(a, f(a) + j)(b, f(b) + k) = (ab, 0) \in K^f \setminus \{(0, 0)\}$, which is weakly $S \gg J(0)$-primary ideal of $A \gg J$, a contradiction since $(s, f(s))(a, f(a) + j) \notin K^f$ and $(s, f(s))(b, f(b) + k) \notin \sqrt{K}$. Hence, $ab = 0$. \hfill \Box
Case 1: \((f(a)+j)(f(b)+k) \neq 0\), then \(f(s)(f(a)+j) \in K\) or \(f(s)(f(b)+k) \in \sqrt{K}\). Hence, \((s,f(s))a(f(a)+j) \in K^f\) or \((s,f(s))b(f(b)+k) \in \sqrt{K^f}\), as desired.

Case 2: \((f(a)+j)(f(b)+k) = 0\). We claim that \(f(s)(f(a)+j) \in K\) or \(f(s)(f(b)+k) \in \sqrt{K}\). Deny, it follows that \(ab = 0\), which is absurd since \((ab,(f(a)+j)(f(b)+k)) \in K^f \setminus \{(0,0)\}\). Hence \((s,f(s))a(f(a)+j) \in K^f\) or \((s,f(s))b(f(b)+k) \in \sqrt{K^f}\), making \(K^f\) a weakly \(S \bowtie f\) (0)-primary ideal, as desired.

Let \(I\) be a proper ideal of \(A\). The (amalgamated) duplication of \(A\) along \(I\) is a special amalgamation given by

\[
A \bowtie I = \{(a,a+i) \mid a \in A, i \in I\}.
\]

The next corollary is an immediate consequence of assertion (1) of Theorem 3.2 on the transfer of weakly \(S\)-primary property to duplications.

**Corollary 3.5.** Let \(R\) be a ring, \(I\) an ideal of \(R\), \(S\) be a multiplicative subset of \(R\), and \(P\) an ideal of \(R\) disjoint from \(S\). Then, the following statements are equivalent:

1. \(P \bowtie I\) is a weakly \((S \bowtie I)\)-primary ideal of \(R \bowtie I\).
2. \(P \bowtie I\) is a weakly \((S \bowtie 0)\)-primary ideal of \(R \bowtie I\).
3. \(P\) is a weakly \(S\)-primary ideal of \(R\) associated to \(s\), and if there exist \(a, b \in R\) with \(ab = 0\), but \(\text{sa} \notin P\) and \(\text{sb} \notin \sqrt{P}\), then \(aj + bi + ij = 0\) for all \(i, j \in I\).

**Remark 3.6.** Let \(f : AB \rightarrow B\) be a ring homomorphism and \(J\) be an ideal of \(B\). Consider \(I\) (resp., \(H\)) be an ideal of \(A\) (resp., \(f(A) + J\)) such that \(f(I)J \subseteq H \subseteq J\). Observe that

\[
I \bowtie f H = \{(i,f(i)+h) \mid i \in I, h \in H\}
\]

is an ideal of \(A \bowtie f J\).

Using the previous remark, we get immediately the next result.

**Proposition 3.7.** Under the above notation, \(I \bowtie f H\) is a weakly \(S \bowtie f\) (0)-primary ideal of \(A \bowtie f J\) with the coefficient \((s, f(s)\) if and only if \(I\) is a weakly \(S\)-primary ideal of \(A\) and for each \(i, j \in A\) such that \(ij = 0, si \notin I\) and \(sj \notin \sqrt{I}\) we have \(kf(i) + hf(j) + hk = 0\) for each \(h, k \in H\).

**Proof.** Argue as in the Theorem 3.3. \(\square\)

The next corollary is an immediate consequence of Proposition 3.6 concerning duplications.

**Corollary 3.8.** Let \(A\) be a ring and \(I, H, J\) be ideals of \(A\) such that \(IJ \subseteq H \subseteq J\). \(I \bowtie H\) is a weakly \(S \bowtie J\) (primary ideal of \(A \bowtie J\) with the coefficient \((s, f(s)\) if and only if \(I\) is a weakly \(S\)-primary ideal of \(A\) with the coefficient \(s\) and for each \(i, j \in A\) such that \(ij = 0, si \notin I\) and \(sj \notin \sqrt{I}\) we have \(ki + hj + hk = 0\) for each \(h, k \in H\).
Acknowledgement. The authors would like to thank the editor and the anonymous referee who kindly reviewed the earlier version of this manuscript and provided valuable suggestions and comments, which improve the presentation of the paper.

References


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