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SOME RESULTS ABOUT WEAKLY S-PRIMARY IDEALS OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity and $S \subsetneq R$ a multiplicative subset. We define a proper ideal P of R disjoint from S to be weakly S-primary if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in P$ then $sa \in P$ or $sb \in \sqrt{P}$. We show that weakly S-primary ideals enjoy analogs of many properties of weakly primary ideals and we study the form of weakly S-primary ideals of the amalgamation of A with B along an ideal J with respect to f (denoted by $A \bowtie^f J$). Weakly S-primary ideals of the trivial ring extension are also characterized.

1. Introduction

Throughout this paper, all considered rings are assumed to be commutative with identity $1 \neq 0$ and all ring homomorphisms are assumed to be unital. If A is a subring of B, we suppose that they have the same identity element. As usual, if R is a commutative ring, then $Z(R)$ denotes the set of zero divisors of R and $\text{Reg}(R) = R \setminus Z(R)$ is the set of its regular elements. Recall that a subset S of a ring R is called *multiplicative* if $1 \in S$, $0 \notin S$ and S is closed under multiplication. Note that any multiplicative subset of R satisfies the inclusion relations $\{1\} \subseteq S \subsetneq R$. Recall also that an ideal P of R is said to be *prime* if $P \neq R$ and whenever a and b are elements of R such that $ab \in P$, then $a \in P$ or $b \in P$. Note that P is a prime ideal of R if and only if $R \setminus P$ is a multiplicative subset of R. In [2], D. D. Anderson and E. Smith have defined a proper ideal of R to be *weakly prime* if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Some properties of weakly prime ideals have been settled. On the other hand, A. Hamed and A. Malek have introduced and investigated the concept of S-prime ideals which constitute a generalization of prime ideals (see [10]). More precisely, let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. Then, I is called an *S-prime ideal* of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in I$. Note that if S consists of units of R, then the notions of S-prime and prime ideals coincide. Recall that an ideal P of R is said to be *primary* if for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in \sqrt{P}$. In [3], S. E. Atani and F. Farzalipour have defined a proper ideal of R to be

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weakly primary if $0 \neq ab \in P$ implies $a \in P$ or $b \in \sqrt{P}$. The first author in [12] introduced and investigated the concept of S-primary ideals which constitute a generalization of primary ideals. More precisely, let R be a commutative ring, S a multiplicative subset of R and I an ideal of R disjoint from S. Then, I is called an S-primary ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in \sqrt{I}$. Note that if S consists of units of R, then the notions of S-primary and primary ideals coincide. In [1] F. A. A. Almahdi, E. M. Bouba and M. Tamekkante have defined a proper ideal P of R disjoint from a multiplicative subset S to be *weakly S-prime* if $0 \neq ab \in P$ implies $sa \in P$ or $sb \in P$. The main goal of the present paper is to complete this circle of ideas by introducing and studying the concept of weakly S-primary ideals of a commutative ring in a way that generalizes essentially all the results concerning weakly primary ideals. Let R be a commutative ring, S a multiplicative subset and P a proper ideal of R disjoint from S. Then we say that P is *weakly S-primary ideal* of R if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in P$, then $sa \in P$ or $sb \in \sqrt{P}$. In Section 2, we study the basic properties of weakly S-primary ideals. Example 2.3 provides a weakly S-primary ideal which is not weakly S-prime. Example 2.4 gives a weakly S-primary ideal but is not S-primary. Proposition 2.6 states that P is a weakly S-primary ideal of R if and only if $(P : s)$ is a weakly primary ideal of R for some $s \in S$ if and only if $S^{-1}P$ is a weakly primary ideal of $S^{-1}R$ and there is $s \in S$ such that $(P : t) \subseteq (P : s)$ for all $t \in S$. In Theorem 2.8, we show that a weakly S-primary ideal P that is not S-primary satisfies $P^2 = 0$ and $\sqrt{P} = \sqrt{(0)}$. Theorem 2.9 provides others characterizations of weakly S-primary ideals in the case where $S \subseteq Reg(R)$. Recall that, in general, the intersection of a family of S-primary ideals is not S-primary, but we have the following result: Let R be a commutative ring and $S \subseteq R$ a strongly-multiplicative set. $(P_\alpha)_{\alpha \in \Lambda}$ be a chain of weakly S-primary ideals of R that are not S-primary. Then $P = \bigcap_{\alpha \in \Lambda} P_\alpha$ is a weakly S-primary ideal of R. Theorem 2.18 characterizes weakly S-primary ideals of the ring $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings. Section 3 is devoted to study the form of weakly S-primary ideals in trivial extension and in the amalgamation of A with B along an ideal J with respect to f (such amalgamation is denoted by $A \bowtie^f J$). This concept has been introduced and studied by D'Anna and Fontana in [6]. Any unexplained terminology is standard as in [4], [9], [11] and [13] .

2. Weakly S-primary ideals

We start this section by introducing the concept of weakly S-primary ideals of a commutative ring R, where S is a multiplicative subset of R. The following definition constitutes the weakly-version of S-primary ideals.

Definition 2.1. Let R be a commutative ring, S a multiplicative subset of R and P an ideal of R disjoint from S. We say that P is a weakly S-primary ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in P$, then $sa \in P$ or $sb \in \sqrt{P}$.

Remark 2.2.
1. If S consists of units of R, then the weakly primary and the weakly S-primary ideals coincide.
2. Clearly, if P is a weakly S-prime ideal of a ring R, then P is a weakly S-primary. The converse does not hold in general.

Example 2.3. Let I be a primary and not a prime ideal of a commutative ring R (We can take, for example $R = \mathbb{Z}$ and $I = 9\mathbb{Z}$). Let J be an ideal of R and set $S := \{(1,0)\ ,\ (1,1)\}$. Clearly, S is a multiplicative subset of $R \times R$ and $(I \times J) \cap S = \emptyset$. $I \times J$ is a weakly S-primary ideal of $R \times R$. Indeed, let $(0,0) \neq (a,b)(c,d) \in I \times J$, then $ac \in I$ which is a primary ideal, hence $a \in I$ or $c \in \sqrt{I}$, therefore $(1,0)(a,b) \in I \times J$ or $(1,0)(c,d) \in \sqrt{I \times J}$. In the other hand, there exist $x, y \notin I$ such that $xy \in I$ which implies that $(x,0)(y,0) \in I \times J$ but $(1,0)(x,0) \notin I \times J$, $(1,0)(y,0) \notin I \times J$, $(1,1)(x,0) \notin I \times J$ and $(1,1)(y,0) \notin I \times J$ that is $I \times J$ is not S-prime.

It is clear that an S-primary ideal is a weakly S-primary ideal but the following example show that the converse is not true in general.

Example 2.4. Let $R = \mathbb{Z}/12\mathbb{Z}$, $S = \{\overline{1},\ \overline{5},\ \overline{7},\ \overline{11}\}$. It is clear that $(\overline{0})$ is a weakly S-primary ideal which is not S-primary. Indeed, $\overline{4} \times \overline{3} = \overline{0}$ but, $\overline{s}\overline{4} \notin (\overline{0})$ and $(\overline{s}\overline{3})^n \notin (\overline{0})$ for every $s \in \{1, 5, 7, 11\}$ and for each nonzero integer n.

If P is a weakly primary ideal of R disjoint with S, then P is weakly S-primary ideal of R. The converse is not true in general:

Example 2.5. Consider the polynomial ring $R = \mathbb{Z}[X]$ and set $S := \{2^n \mid n \in \mathbb{N}\}$. By using [10, Example 1(3)], $P = 4XR$ is an S-prime ideal of R and so is an S-primary ideal of R. Thus, P is a weakly S-primary ideal of R but, we claim that P is not a weakly primary ideal of R. Indeed, we have $0 \neq 4X \in P$ and $4 \notin P$. If $X \in \sqrt{P}$, then there exists an integer $n \geq 1$ and $a_n \in \mathbb{Z} \setminus \{0\}$ such that $X^n = 4a_nX^n$. Hence, $a_n = \frac{1}{4}$, the desired contradiction asserting our claim.

Our next proposition characterizes the weakly S-primary ideals of a commutative ring R but, first recall that if I is an ideal of R and $s \in R$, then $(I : s) := \{x \in R \mid sx \in I\}$ is an ideal of R containing I.

Let R be a commutative ring, S a multiplicative subset of R and P an ideal of R disjoint from S. It is clear that if $(P : s)$ is a weakly primary ideal of R for some $s \in S$, then P is a weakly S-primary ideal. However, the converse is not true in general but, if S consisting of regular elements we have the following result:

Proposition 2.6. *Let R be a commutative ring and S a multiplicative subset of R consisting of regular elements and P be an ideal of R disjoint from S. Then, the following assertions are equivalent:*

(1) *P is a weakly S-primary ideal of R.*
(2) *$(P : s)$ is a weakly primary ideal of R for some $s \in S$.*
(3) *$S^{-1}P$ is a weakly primary ideal of $S^{-1}R$ and there exists $s \in S$ such that $(P : t) \subseteq (P : s)$ for all $t \in S$.*

(4) *$S^{-1}P$ is a weakly primary ideal of $S^{-1}R$ and $S^{-1}P \cap R = (P : s)$ for some $s \in S$.*

Proof. (1) $\Longrightarrow$ (2). Since P is weakly S-primary, then there exists $s \in S$ such that for all $x, y \in R$ with $0 \neq xy \in P$, we have $sx \in P$ or $sy \in \sqrt{P}$. First we claim that for $s \in S$ and $m \in \mathbb{N}^*$ we have $(P : s^m) = (P : s)$. Indeed let $x \in (P : s)$ then, $sx \in P$ and so $s^m x \in P$ hence $x \in (P : s^m)$. Conversely, let $0 \neq x \in (P : s^m)$, then $0 \neq s^m x \in P$, so $s^{m+1} \in \sqrt{P}$ or $sx \in P$, then $sx \in P$ since $P \cap S = \emptyset$ and hence $x \in (P : s)$. Now let $0 \neq ab \in (P : s)$. Then, $0 \neq sab \in P$, hence $0 \neq s^2 a \in P$ or $sb \in \sqrt{P}$. Thus, $sa \in P$ or $sb \in \sqrt{P}$ since $S \cap P = \emptyset$. If $sb \in \sqrt{P}$, there exists an integer $n \geqslant 1$ such that $s^n b^n \in P$ then $b^n \in (P : s^n) = (P : s)$ and so $b \in \sqrt{(P : s)}$. Hence, $(P : s)$ is weakly primary ideal of R.
(2) $\Longrightarrow$ (1). Clear.
(1) $\Longrightarrow$ (3). As $P \cap S = \emptyset$, we have that $S^{-1}P \neq S^{-1}R$. Let $0 \neq \frac{a}{s_1}\frac{b}{s_2} \in S^{-1}P$ where $a, b \in R$ and $s_1, s_2 \in S$. Then, $\frac{a}{s_1}\frac{b}{s_2} = \frac{p}{s_3}$ for some $p \in P$ and $s_3 \in S$. So there exists $u \in S$ such that $0 \neq us_3 ab = us_1 s_2 p \in P$. Since P is weakly S-primary, there exists $s \in P$ such that $sus_3 \in \sqrt{P}$ or $0 \neq sab \in P$. Thus $sab \in P$ since $sus_3 \notin \sqrt{P}$. Hence, $0 \neq s^2 a \in P$ or $sb \in \sqrt{P}$, and so $sa \in P$ or $sb \in \sqrt{P}$. This implies that $\frac{a}{s_1} = \frac{sa}{ss_1} \in S^{-1}P$ or $\frac{b}{s_2} = \frac{sb}{ss_2} \in \sqrt{S^{-1}P}$, and so $S^{-1}P$ is a weakly primary ideal of $S^{-1}R$.
Let $s \in S$ the element associated to P. Let $t \in S$ and $0 \neq a \in (P : t)$, so $0 \neq ta \in P$. Hence $st \in \sqrt{P}$ or $sa \in P$. Since $P \cap S = \emptyset$, $st \notin \sqrt{P}$ which implies that $a \in (P : s)$ consequently $(P : t) \subseteq (P : s)$.
(3) $\Longrightarrow$ (1). Let $a, b \in R$ such that $0 \neq ab \in P$. Since $0 \neq \frac{a}{1}\frac{b}{1} \in S^{-1}P$, we have $\frac{a}{1} \in S^{-1}P$ or $(\frac{b}{1})^n \in S^{-1}P$ for some integer $n \geqslant 1$. If $\frac{a}{1} \in S^{-1}P$, then $\frac{a}{1} = \frac{p}{t}$ for some $p \in P$ and $t \in S$. Hence, $ta = p$, and so $a \in (P : t) \subseteq (P : s)$, then $sa \in P$. If $(\frac{b}{1})^n \in S^{-1}P$, $\frac{b^n}{1} = \frac{q}{u}$ for some $q \in P$ and $u \in S$. Hence, $ub^n = q \in P$ and so $b^n \in (P : u) \subseteq (P : s)$ then $sb \in \sqrt{P}$ which means that P is weakly S-primary ideal of R.
(1) $\Longrightarrow$ (4). As in "(1) $\Longrightarrow$ (3)" we have , $S^{-1}P$ is weakly primary ideal of $S^{-1}R$. Let $0 \neq a \in (P : s)$. Then, $sa \in P$ and $a = \frac{sa}{s} \in S^{-1}P$. Hence $a \in S^{-1}P \cap R$, and so $(P : s) \subseteq S^{-1}P \cap R$. Now let $0 \neq a \in S^{-1}P \cap R$, then $a \in R$ and $a = \frac{p}{t}$ with $p \in P$ and $t \in S$. So, $0 \neq ta = p \in P$. Hence $st \in \sqrt{P}$ or $sa \in P$. Thus , $sa \in P$ since $S \cap P = \emptyset$. Consequently $a \in (P : s)$ and so $S^{-1}P \cap R \subseteq (P : s)$.
(4) $\Longrightarrow$ (1). Let $a, b \in R$ such that $0 \neq ab \in P$. Since $0 \neq \frac{a}{1}\frac{b}{1} \in S^{-1}P$, we have $\frac{a}{1} \in S^{-1}P$ or $\frac{b}{1} \in \sqrt{S^{-1}P}$. If $\frac{a}{1} \in S^{-1}P$, then $\frac{a}{1} = \frac{p}{t}$ for some $p \in P$ and $t \in S$. Hence, $ta = p \in P$ and so $a = \frac{ta}{t} \in S^{-1}P = (P : s)$ by hypothesis, that is $sa \in P$. Similarly, if $\frac{b}{1} \in \sqrt{S^{-1}P}$ we have $sb \in \sqrt{P}$. Thus P is a weakly S-primary ideal of R. □

Example 2.7. Let R be a commutative ring, S a multiplicative set of R consisting of nonzero divisors and P an ideal of R disjoint with S. If P is a weakly primary ideal of R, then for any $s \in S$, sP is a weakly S-primay ideal of R. Indeed, let $s \in S$. It is convenient to denote sP by I. As $I \subseteq P$ and $P \cap S = \emptyset$, it follows that $I \cap S = \emptyset$. Since P is a weakly primary ideal of R with $\sqrt{P} \cap S = \emptyset$, we get that $(sP : s) = P$. Hence, $(I : s) = P$ is a weakly primary ideal of R. Therefore, we claim that $I = sP$ is a weakly S-primary ideal of R by Proposition 2.6.

Theorem 2.8. *Let R be a commutative ring, S a multiplicative set of R consisting of nonzero divisors and P an ideal of R disjoint with S. Suppose that P is weakly S-primary and not S-primary. Then, $\sqrt{P} = \sqrt{(0)}$.*

Proof. First, we prove that $P^2 = 0$. Suppose that $P^2 \neq 0$, we show that P is S-primary. Let $pq \in P$ where $p, q \in R$. If $pq \neq 0$, then either $sp \in P$ or $sq \in \sqrt{P}$ since P is weakly S-primary. So, suppose that $pq = 0$. If $pP \neq 0$, then there is an element p' of P such that $pp' \neq 0$, so $0 \neq pp' = p(p' + q) \in P$, and hence P weakly S-primary gives either $sp \in P$ or $s(p' + q) \in \sqrt{P}$. As $p' \in P \subseteq \sqrt{P}$ we have either $sp \in P$ or $sq \in \sqrt{P}$. So, we can assume that $pP = 0$. Similarly, we can assume that $qP = 0$. Since $P^2 \neq 0$, there exist $c, d \in P$ such that $cd \neq 0$. Then, $(p + c)(q + d) = cd \in P$, so either $s(p + c) \in P$ or $s(q + d) \in \sqrt{P}$ and hence either $sp \in P$ or $sq \in \sqrt{P}$. Thus, P is S-primary. Clearly, $\sqrt{(0)} \subseteq \sqrt{P}$. As $P^2 = 0$, we get $P \subseteq \sqrt{(0)}$, hence $\sqrt{P} \subseteq \sqrt{(0)}$ as required. □

We next gives other characterisations of weakly S-primary ideals.

Theorem 2.9. *Let R be a commutative ring, S a multiplicative set of R consisting of nonzero divisors and P a proper ideal of R disjoint with S. Then, the following assertions are equivalent:*

(i) *P is weakly S-primary ideal of R.*
(ii) *There exists $s \in S$ such that for each $x \in R \setminus \sqrt{(P : s)}$, $(P : sx) = (P : s) \cup (0 : x)$ for some $s \in S$.*
(iii) *There exists $s \in S$ such that for each $x \in R \setminus \sqrt{(P : s)}$, $(P : sx) = (P : s)$ or $(P : sx) = (0 : x)$ for some $s \in S$.*

Proof. $(i) \Longrightarrow (ii)$. Since P is weakly S-prime there exists $s \in S$, such that for each $0 \neq ab \in P$, we have $sa \in P$ or $sb \in P$.
Let $y \in (P : sx)$ where $x \in R \setminus \sqrt{(P : s)}$. If $xy \neq 0$, so $0 \neq sxy \in P$, hence $s^2x \in \sqrt{P}$ or $sy \in P$. If $s^2x \in \sqrt{P}$, then $0 \neq s^{2n}x^n \in P$ for some integer n and so $s^{2n+1} \in \sqrt{P}$ or $sx^n \in P$ which is absurd, then $sy \in P$ so $y \in (P : s)$. If $xy = 0$ then $y \in (0 : x)$. In each cases $y \in (0 : x) \cup (P : s)$.
For the other inclusion, if $y \in (P : s)$, then $sy \in P$ so $sxy \in P$ and hence $y \in (P : xs)$. If $y \in (0 : x)$, then $sxy = 0$ so $y \in (P : sx)$. As the reverse containment holds for any ideal P, we have equality.
Hence $(P : s) \cup (0 : x) \subseteq (P : sx)$.
$(ii) \Longrightarrow (iii)$. Clear.
$(iii) \Longrightarrow (i)$. Let $0 \neq xy \in P$ and suppose that $sx \notin \sqrt{P}$, then for each $n \in \mathbb{N}^*$ $s^nx^n \notin P$ so $sx^n \notin P$ hence $x^n \notin (P : s)$ and so $x \in R \setminus \sqrt{(P : s)}$. Now since $0 \neq sxy \in P$ we have $y \in (P : sx) = (P : s) \cup (0 : x)$ hence $y \in (P : s)$ since $xy \neq 0$, so $sy \in P$ and hence P is weakly S-primary.

□

Remark 2.10. Let $S_1 \subseteq S_2$ be multiplicative subsets of R and P an ideal of R disjoint from S_2. Clearly if P is a weakly S_1-primary of R then P is a weakly S_2-primary. However the converse is not true in general. To see this, we consider the ideal $P = (4X)$ of the ring $\mathbb{Z}[X]$ and set $S_1 = 1$ and $S_2 = \{2^n \mid n \in \mathbb{N}\}$. By Example 2.5 P is a weakly S_2-primary of $\mathbb{Z}[X]$ but not weakly S_1-primary.

Proposition 2.11. *Let R be a commutative ring, $S_1 \subseteq S_2$ be multiplicative subsets of R such that for any $s \in S_2$ there exists an element $t \in S_2$ satisfying $st \in S_1$. If P is a weakly S_2-primary ideal of R, then P is a weakly S_1-primary ideal of R.*

Proof. Let $a, b \in R$ such that $0 \neq ab \in P$. So there exists an $s \in S_2$ such that $sa \in P$ or $sb \in \sqrt{P}$. By assumption, $s' = st \in S_1$ for some $t \in S_2$, and then $s'a \in P$ or $s'b \in \sqrt{P}$. This completes the proof. □

Let S be a multiplicative subset of R, $S^* = \{r \in R \mid \frac{r}{1} \textit{ is unit in } S^{-1}R\}$ denotes the saturation of S. Note that S^* is a multiplicative subset of R containing S. A multiplicative subset of R is called saturated if $S^* = S$. It is clear that S^* is always a saturated multiplicative subset of R [9]

Proposition 2.12. *Let R be a commutative ring, S a multiplicative subset of R and P an ideal of R disjoint from S. Then P is a weakly S-primary ideal of R if and only if P is a weakly S^*-primary ideal.*

Proof. It is clear that $S^* \cap P = \emptyset$. We will show that for any $r \in S^*$, there is $r' \in S^*$ such that $rr' \in S$. Let $r \in S^*$, then $\frac{r}{1}\frac{a}{s} = 1$ for some $s \in S$ and $a \in R$. This implies that $tar = ts$ for some $t \in S$. Now take $r' = ta$, we have $r' \in S^*$ with $rr' \in S$ and so the desired condition is satisfied. Therefore, by putting $S_1 = S$ and $S_2 = S^*$ we conclude immediatly the result from the Proposition 2.11. □

Proposition 2.13. *Let $f : R \longrightarrow T$ be a ring homomorphism and S a multiplicative subset of R such that $0 \notin f(S)$. Then, the following hold:*

1. *If f is an epimorphism and P is a weakly S-primary ideal of R containing $Ker(f)$, then $f(P)$ is a weakly $f(S)$-primary ideal of T.*
2. *If f is a monomorphism and Q is a weakly $f(S)$-primary ideal of T, then $f^{-1}(Q)$ is a weakly S-primary ideal of R.*

Proof. (1) Let $r \in f(S) \cap f(P)$. Then, $r = f(p) = f(s)$ for some $p \in P$ and $s \in S$. So $s - p \in Ker(f) \subseteq P$, which implies that $s \in P$, a contradiction. Hence $f(S) \cap f(P) = \emptyset$. Now let $0 \neq xy \in f(P)$. Then there is $a, b \in R$ such that $x = f(a)$, $y = f(b)$ and $0 \neq f(ab) = xy \in f(P)$. Since $Ker(f) \subseteq P$, we get $0 \neq ab \in P$, and so $sa \in P$ or $s^m b^m \in P$ for some integer $m \geqslant 1$, for some $s \in S$. It means that $f(s)x \in f(P)$ or $(f(s))^m y^m \in f(P)$. Thus $f(s)x \in f(P)$ or $f(s)y \in \sqrt{f(P)}$ and hence $f(P)$ is a weakly $f(S)$-primary ideal of T.
(2) Since Q is a weakly $f(S)$-primary ideal of T, there exists $s \in S$ such that, for all $x, y \in T$, $0 \neq xy \in Q$ we have either $f(s)x \in Q$ or $f(s)y \in \sqrt{Q}$. We can easily show that $f^{-1}(Q) \cap S = \emptyset$. Let $a, b \in R$ such that $0 \neq ab \in f^{-1}(Q)$. Since $Ker(f) = \{0\}$, we get $0 \neq f(ab) = f(a)f(b) \in Q$. Then $f(s)f(a) = f(sa) \in Q$

or $(f(s)f(b))^n = f((sb)^n) \in Q$, for some integer $n \geqslant 1$. Hence $sa \in f^{-1}(Q)$ or $sb \in \sqrt{f^{-1}(Q)}$, and so we conclude that $f^{-1}(Q)$ is a weakly S-primary ideal of R. □

Let R be a commutative ring, S a multiplicative subset of R and P an ideal of R disjoint from S. Set $\overline{S} := \{s + P \mid s \in S\}$. It is easy to check that $\overline{S}$ is a multiplicative subset of R/P.

Corollary 2.14. *Let R be a commutative ring and S a multiplicative subset of R.*

1. *If $I \subseteq P$ be two ideals of R such that $P \cap S = \emptyset$. If P is a weakly S-primary ideal of R, then P/I is a weakly $\overline{S}$-primary ideal of R/I.*
2. *If R is a subring of T and Q is a weakly S-primary ideal of T, then $Q \cap R$ is a weakly S-primary ideal of R.*

Proof. (1) Follows by applying Proposition 2.13(1) to the canonical surjection$\pi : R \longrightarrow R/I$.
(2) It suffices to apply Proposition 2.13(2) to the naturel injection
$\imath : R \hookrightarrow T$, since $\imath^{-1}(Q) = Q \cap R$. □

Proposition 2.15. *Let R be a commutative ring, S a multiplicative subset of R and P an ideal of R disjoint from S. If J is an ideal of R such that $J \cap S \neq \emptyset$ and P is a weakly S-primary of R, then so are $J \cap P$ and JP.*

Proof. It is obvious that $JP \cap S = \emptyset$ and $(J \cap P) \cap S = \emptyset$ since $JP \subseteq P$ and $P \cap S = \emptyset$. As P is a weakly S-primary ideal of R, then there exists $s \in S$ such that $sa \in P$ or $sb \in \sqrt{P}$.
First we will prove that $J \cap P$ is a weakly S-primary ideal of R. Pick $t \in J \cap S$ (such t exists since $J \cap S \neq \emptyset$) and let $a, b \in R$ such that $0 \neq ab \in J \cap P (\subseteq P)$. Thus $sta \in P \cap J$ or $stb \in \sqrt{P} \cap \sqrt{J} = \sqrt{P \cap J}$ and $st \in S$. Consequently $P \cap J$ is a weakly S-primary ideal of R.
We prove now that JP is a weakly S-primary ideal, let $x, y \in R$ such that $0 \neq xy \in JP (\subseteq P)$ If $sx \in P$, then $(ts)x = t(sx) \in JP$. Assume now that $sy \in \sqrt{P}$. Then $s^n y^n = (sy)^n \in P$ for some integer $n \geqslant 1$. Thus $((ts)y)^n = t^n s^n y^n \in JP$. It follows that $(ts)y \in \sqrt{JP}$. Therefore, JP is a weakly S-primary ideal of R. □

Recall that, in general, the intersection of a family of S-primary ideals is not S-primary, but we have the following results:

Proposition 2.16. *Let R be a commutative ring, S a multiplicative set of R. Let $n \geqslant 1$. let $i \in \{1, \dots, n\}$. Let P_i be an ideal of R with $P_i \cap S = \emptyset$. If P_i is a weakly S-primary ideal of R for each $i \in \{1, \dots, n\}$ witn $\sqrt{P_i} = \sqrt{P_j}$ for all $i, j \in \{1, \dots, n\}$, then $\cap_{i=1}^n P_i$ is a weakly S-primary ideal of R.*

Proof. Let $i \in \{1, \dots, n\}$. Since P_i is a weakly S-primary ideal of R, there exists $s_i \in S$ such that for all $a, b \in R$ with $0 \neq ab \in P_i$, we have either $s_i a \in P_i$ or $s_i b \in \sqrt{P_i}$. Let $s = \prod_{i=1}^n s_i$. Then $s \in S$. Let $a, b \in R$ be such that $0 \neq ab \in \cap_{i=1}^n P_i$. Suppose that $sa \notin \cap_{i=1}^n P_i$. Then $sa \notin P_k$ for some

$k \in \{1 , ... , n\}$. Hence, $s_k a \notin P_k$. From $0 \neq ab \in P_k$, it follows that $s_k b \in \sqrt{P_k}$. Therefore, $sb \in \sqrt{P_k}$. By hypothesis, $\sqrt{P_1} = \sqrt{P_i}$ for all $i \in \{1 , ... , n\}$. Thus $sb \in \sqrt{P_1} = \cap_{i=1}^n \sqrt{P_i} = \sqrt{\cap_{i=1}^n P_i}$. This shows that the element $s \in S$ is such that for all $a, b \in R$ with $0 \neq ab \in \cap_{i=1}^n P_i$, we have either $sa \in \cap_{i=1}^n P_i$ or $sb \in \sqrt{\cap_{i=1}^n P_i}$. This proves that $\cap_{i=1}^n P_i$ is a weakly S-primary ideal of R.

□

Recall from [10] that if R is a commutative ring with identity and S a multiplicative set of R. We say that S is a strongly-multiplicative set if for each family $(s_\alpha)_{\alpha\in\Lambda}$ of element of S we have

$$(\bigcap_{\alpha\in\Lambda} s_\alpha R) \cap S \neq \emptyset$$

Theorem 2.17. *Let R be a commutative ring and $S \subseteq R$ a strongly-multiplicative set. If $(P_\alpha)_{\alpha\in\Lambda}$ is a chain of weakly S-primary ideals of R that are not S-primary, then $P = \bigcap_{\alpha\in\Lambda} P_\alpha$ is a weakly S-primary ideal of R.*

Proof. First, we show that $\sqrt{P} = \bigcap_{\alpha\in\Lambda} \sqrt{P_\alpha}$. Clearly, $\sqrt{P} \subseteq \bigcap_{\alpha\in\Lambda} \sqrt{P_\alpha}$. For the other inclusion suppose that $r \in \bigcap_{\alpha\in\Lambda} \sqrt{P_\alpha}$, so $r^m = 0$ for some m since $\bigcap_{\alpha\in\Lambda} \sqrt{P_\alpha} = \sqrt{(0)}$ by Theorem 2.4. It follows that $r^m \in P_\alpha$ for each $\alpha \in \Lambda$ and hence $r \in \sqrt{P}$.
For each $\alpha \in \Lambda$, there exists $s_\alpha \in S$ such that for all $a, b \in R$, $0 \neq ab \in P_\alpha$ we have $s_\alpha a \in P_\alpha$ or $s_\alpha^n b^n \in P_\alpha$ for some $n \geqslant 1$.
Since S is strongly-multiplicative set , then $(\bigcap_{\alpha\in\Lambda} s_\alpha R) \cap S \neq \emptyset$. Let $t \in (\bigcap_{\alpha\in\Lambda} s_\alpha R) \cap S$. we will show that for all $a, b \in R$ such that $0 \neq ab \in P$ we have $ta \in P$ or $tb \in \sqrt{P}$.
Let $a, b \in R$, such that $0 \neq ab \in P$ and suppose that $ta \notin P$. Then, $ta \notin P_\beta$ for some $\beta \in \Lambda$. Let $\alpha \in \Lambda$. We have $P_\alpha \subseteq P_\beta$ or $P_\beta \subseteq P_\alpha$.
First case, $P_\alpha \subseteq P_\beta$. Since $ta \notin P_\beta$ then $ta \notin P_\alpha$, so $s_\alpha a \notin P_\alpha$. This implies that $s_\alpha b \in \sqrt{P_\alpha}$ and $tb \in \sqrt{P_\alpha}$.
Second case, $P_\beta \subseteq P_\alpha$. As $ab \in P_\beta$ and $ta \notin P_\beta$, then $s_\beta a \notin P_\beta$ so $(s_\beta b)^m \in P_\beta \subseteq P_\alpha$ for some $m \geqslant 1$. this implies that $(tb)^m \in P_\alpha$ for some $m \geqslant 1$ and hence $tb \in \sqrt{P}$. □

Let R be a commutative ring. R is called *decomposable* if $R = R_1 \times R_2$ for some commutative rings R_1 and R_2. If I_1 is an ideal of R_1, then $I_1 \times R_2$ is an ideal of $R_1 \times R_2$ and $\sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$. Similarly, if I_2 is an ideal of R_2, then $R_1 \times I_2$ is an ideal of $R_1 \times R_2$ and $\sqrt{(R_1 \times I_2)} = R_1 \times \sqrt{I_2}$.

Now, we establish the following result.

Theorem 2.18. *Let R_1 and R_2 be commutative rings and let S_1 and S_2 be multiplicative subsets of R_1 and R_2 respectively. Set $R = R_1 \times R_2$ and $S = S_1 \times S_2$ and P_1, P_2 are nonzero ideals of R_1 and R_2 respectively. Then the following assertions are equivalent:*

(1) *$P := P_1 \times P_2$ is a weakly S-primary ideal of R.*

(2) *P_1 is an S_1-primary ideal of R_1 and $S_2 \cap P_2 \neq \emptyset$ or P_2 is an S_2-primary ideal of R_2 and $S_1 \cap P_1 \neq \emptyset$.*
(3) *$P := P_1 \times P_2$ is an S-primary ideal of R.*

Proof. (1) $\Longrightarrow$ (2). Let $0_R \neq (p,q) \in P$, with $p \in R_1$ and $q \in R_2$. Then, $0_R \neq (p,q) = (p,1)(1,q) \in P$. Since P is weakly S-primary ideal of R, then there exists $s = (s_1, s_2) \in S$ such that $s(p,1) = (s_1 p, s_2) \in P$or $s(1,q) = (s_1, s_2 q) \in \sqrt{P} = \sqrt{P_1} \times \sqrt{P_2}$. Thus $S_2 \cap P_2 \neq \emptyset$ or $S_1 \cap P_1 \neq \emptyset$. Assume $S_2 \cap P_2 \neq \emptyset$. As $P \cap S = \emptyset$, we have $S_1 \cap P_1 = \emptyset$. Now we show that P_1 is an S_1-primary ideal of R_1. Let $ab \in P_1$ for some $a, b \in R_1$. Since $S_2 \cap P_2 \neq \emptyset$, then there exists $0_{R_2} \neq t \in S_2 \cap P_2$, and so we have $0_R \neq (a,t)(b,1) \in P$. Hence $s(a,t) = (s_1 a, s_2 t) \in P$ or $s(b,1) = (s_1 b, s_2) \in \sqrt{P} = \sqrt{P_1} \times \sqrt{P_2}$. So $s_1 a \in P_1$ or $s_1 b \in \sqrt{P_1}$ as desired.
(2) $\Longrightarrow$ (3). Suppose that P_1 is an S_1-primary ideal of R_1 and $S_2 \cap P_2 \neq \emptyset$, let $t \in S_2 \cap P_2$. Let $(a,b)(c,d) = (ac, bd) \in P_1 \times P_2$, then $ac \in P_1$ and $bd \in P_2$, for some $s_1 \in S_1$ so $s_1 a \in P_1$ or $s_1 c \in \sqrt{P_1}$ and hence $(s_1, t)(a,b) = (s_1 a, tb) \in P$ or $(s_1, t)(c,d) = (s_1 c, td) \in \sqrt{P_1} \times P_2 \subseteq \sqrt{P}$, then P is an S-primary ideal of R.
(3) $\Longrightarrow$ (1). Clear. □

3. Weakly S-primary ideals in trivial ring extensions and amalgamations

Definition 3.1. Let R be a commutative ring with identity, S a multiplicative set and P an ideal of R disjoint from S. We say that P is weakly S-primary with the coefficient $s \in S$ if for each $a, b \in R$, we have $0 \neq ab \in P \Rightarrow sa \in P$ or $sb \in \sqrt{P}$.

Let R be a commutative ring with identity and let M be a unitary R-module. The idealization of M in R (or trivial extension of R by M) is the commutative ring $R(+)M = \{(r,m) \mid r \in R \text{ and } m \in M\}$ under the usual addition and the multiplication defined as

$$(r_1, m_1)(r_2, m_2) = (r_1 r_1, r_1 m_2 + r_2 m_1),$$

for all $(r_1, m_1), (r_2, m_2) \in R(+)M$.

It is easy to show that if S is a multiplicative subset of R, then $S(+)M$ and $S(+)0$ are multiplicative subset of $R(+)M$.

Theorem 3.2. *Let R be a commutative ring, M a unitary R-module, I an ideal of R and S a multiplicative subset of R with $S \cap I = \emptyset$. Then the following are equivalent:*
(1) *$I(+)M$ is a weakly $S(+)M$-primary ideal of $R(+)M$.*
(2) *$I(+)M$ is a weakly $S(+)0$-primary ideal of $R(+)M$.*
(3) *I is a weakly S-primary ideal of R with the coefficient $s \in S$, and if $a, b \in R$ with $ab = 0$, but $sa \notin I$ and $sb \notin \sqrt{I}$ then $a \in ann_R(M)$ and $b \in ann_R(M)$.*

Proof. It is not difficult to check that

$$(S(+)M) \cap (I(+)M) = \emptyset \Longleftrightarrow S \cap I = \emptyset \Longleftrightarrow (S(+)0) \cap (I(+)M) = \emptyset.$$

$(2) \Longrightarrow (1)$. Follows from Remark 2.10 since $S(+)0 \subseteq S(+)M$.
$(1) \Longrightarrow (3)$. Let $a, b \in R$ such that $0 \neq ab \in I$. Then $(0,0) \neq (a,0)(b,0) \in I(+)M$. As $I(+)M$ is a weakly $S(+)M$-primary ideal of $R(+)M$, there exists $(s,n) \in S(+)M$ such that $(s,n)(a,0) = (sa, an) \in I(+)M$ or $(s,n)^k(b,0)^k = (sb, bn)^k = ((sb)^k, ksb^kn) \in I(+)M$ for some $k \in \mathbb{N}^*$. Thus $sa \in I$ or $s^kb^k \in I$ and so $sa \in I$ or $sb \in \sqrt{I}$, hence I is weakly S-primary ideal of R. Now suppose that $ab = 0$, $sa \notin I$ and $sb \notin \sqrt{I}$. Assume that $a \notin ann_R(M)$. Then there exists $m \in M$ such that $am \neq 0$ and so we have $(0,0) \neq (a,0)(b,m) = (0, am) \in I(+)M$. Hence $(s,n)(a,0) = (sa, na) \in I(+)M$ or $(s,n)^l(b,m)^l = (s^lb^l, l(n+m)s^lb^l) \in I(+)M$ for some $l \in \mathbb{N}^*$, which is contraduction.
$(3) \Longrightarrow (2)$. Let $(0,0) \neq (a,m)(b,n) \in I(+)M$. If $ab \neq 0$, then $sa \in I$ or $sb \in \sqrt{I}$, and hence $(s,0)(a,m) = (sa, sm) \in I(+)M$ or $(s,0)(b,n) = (sb, sn) \in \sqrt{I(+)M}$. Assume that $ab = 0$, but $sa \notin I$ and $sb \notin \sqrt{I}$, then $a, b \in ann_R(M)$ consequently, we get $(a,m)(b,n) = (ab, an+bm) = (0,0)$, a contradiction.

□

Let A and B be commutative rings with identity, let J be an ideal of B, and let $f : A \longrightarrow B$ be a ring homomorphism. We set

$$A \bowtie^f J = \{(a, f(a)+j) \mid a \in A \text{ and } j \in J\}.$$

$A \bowtie^f J$ is a subring of $A \times B$ called *the amalgamation of A with B along J with respect to f*. Such construction was introduced and studied by D'Anna, Finacchiaro and Fontana in [3, 5]. This construction is in fact a generalization of the amalgamated duplication of a ring along an ideal (cf. [5, 6, 7]).

For a multiplicative subset S of A, put $S \bowtie^f J = \{(s, f(s)+j) \mid s \in S \text{ and } j \in J\}$, $S \bowtie^f 0 = \{(s, f(s)) \mid s \in S\}$. Clearly, $S \bowtie^f J$ and $S \bowtie^f 0$ are a multiplicative subsets of $A \bowtie^f J$ and if $f(S)$ does not contain zero, then $f(S)$ is a multiplicative subset of B. Let I be an ideal of A and K be an ideal of $f(A)+J$. Notice that

$$I \bowtie^f J = \{(i, f(i)+j) \mid i \in I, j \in J\}$$

and

$$K^f = \{(a, f(a)+j) \mid a \in A, j \in J, f(a)+j \in K\}$$

are ideals of $A \bowtie^f J$.

Our next result characterizes weakly S-primary ideals of the form $I \bowtie^f J$ and K^f of the amalgamation $A \bowtie^f J$.

Theorem 3.3. *Let A and B be commutative rings with identity, let J be an ideal of B, and let $f : A \longrightarrow B$ be a ring homomorphism, and I an ideal of A disjoint from S. Then the following are equivalent:*

1. *$I \bowtie^f J$ is a weakly $S \bowtie^f J$-primary ideal of $A \bowtie^f J$ with the coefficient $(s, f(s)+j_0)$.*
2. *$I \bowtie^f J$ is a weakly $S \bowtie^f (0)$-primary ideal of $A \bowtie^f J$ with the coefficient $(s, f(s))$.*

(3) *I is a weakly S-primary ideal of A with the coefficient s and for $a, b \in A$ with $ab = 0$, but $sa \notin I, sb \notin \sqrt{I}$, then $jf(a) + if(b) + ij = 0$ for every i, j in J.*

Proof. (2) $\Longrightarrow$ (1) Follows from Remark 2.10 since $S \bowtie^f (0) \subseteq S \bowtie^f J$.
(1) $\Longrightarrow$ (3) Assume that $I \bowtie^f J$ is a weakly $S \bowtie^f J$-primary ideal of $A \bowtie^f J$ with the coefficient $(s, f(s) + j_0)$. Let $ab \in I \setminus \{0\}$ where $a, b \in A$. Then, $(a, f(a))(b, f(b)) \in I \bowtie^f J \setminus \{(0, 0)\}$ and so, $(s, f(s) + j_0)(a, f(a)) \in I \bowtie^f J$ or $(s, f(s) + j_0)(b, f(b)) \in \sqrt{I \bowtie^f J}$. Hence $sa \in I$ or $sb \in \sqrt{I}$. Now, we claim that if $sa \notin I, sb \notin \sqrt{I}$ with $ab = 0$, then $f(a)j + f(b)i + ij = 0$ for every $i, j \in J$. Deny. There exist $i, j \in J$ such that $f(a)j + f(b)i + ij \neq 0$ and so $(0, 0) \neq (a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + f(a)j + f(b)i + ij) = (0, f(a)j + f(b)i + ij) \in I \bowtie^f J$, which is a contradiction since $(s, f(s) + j_0)(a, f(a) + i) \notin I \bowtie^f J, (s, f(s) + j_0)(b, f(b) + j) \notin \sqrt{I \bowtie^f J}$ and $I \bowtie^f J$ is a weakly $S \bowtie^f J$-primary ideal of $A \bowtie^f J$ with the coefficient $(s, f(s) + j_0)$.
(3) $\Longrightarrow$ (2) let $(a, f(a) + i)(b, f(b) + j) = (ab, f(ab) + f(a)j + f(b)i + ij) \in I \bowtie^f J \setminus \{(0, 0)\}$. Hence, $ab \in I$. Two cases are then possible:
Case 1: $ab \neq 0$. Hence, $sa \in I$ or $sb \in \sqrt{I}$ since I is a weakly S-primary ideal of A. So, $(s, f(s))(a, f(a) + i) = (sa, f(sa) + f(s)i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b) + j) = (sb, f(sb) + f(s)j) \in \sqrt{I \bowtie^f J}$.
Case 2: $ab = 0$. We claim that $sa \in I$ or $sb \in \sqrt{I}$. Deny. We have $f(a)j + f(b)i + ij = 0$, contradiction with $(a, f(a) + i)(b, f(b) + j) \neq (0, 0)$. Therefore, in all cases, $sa \in I$ or $sb \in \sqrt{I}$ and so $(s, f(s))(a, f(a) + i) \in I \bowtie^f J$ or $(s, f(s))(b, f(b) + j) \in \sqrt{I \bowtie^f J}$, as desired. □

□

Theorem 3.4. *Let A and B be commutative rings with identity, let J be an ideal of B, and let $f : A \longrightarrow B$ be a ring homomorphism, and K be an ideal of $f(A)+J$ disjoint from $f(S)$. Then K^f is a weakly $S \bowtie^f (0)$-primary ideal of $A \bowtie^f J$ with the coefficient $(s, f(s))$ if and only if K is a weakly $f(S)$-primary ideal of $f(A)+J$ with the coefficient $f(s)$ and when $f(s)(f(a)+j) \notin K, f(s)(f(b)+k) \notin \sqrt{K}$ with $a, b \in A, j, k \in J$ and $(f(a) + j)(f(b) + k) = 0$, then $ab = 0$.*

Proof. Assume that K^f is a weakly $S \bowtie^f (0)$-primary ideal of $A \bowtie^f J$. We claim that K is a weakly $f(S)$-primary ideal of $f(A) + J$. Indeed, let $xy \in K \setminus \{0\}$ with $x, y \in f(A) + J$. Then, $x = f(a) + j$ and $y = f(b) + k$ for some $a, b \in A$ and $j, k \in J$. . Therefore, $(a, f(a) + j)(b, f(b) + k) = (ab, (f(a) + j)(f(b)+k)) \in K^f \setminus \{(0, 0)\}$ which is a weakly $S \bowtie^f (0)$-primary ideal of $A \bowtie^f J$. Consequently, $(s, f(s))(a, f(a)+j) \in K^f$ or $(s, f(s))(b, f(b)+j) \in \sqrt{K^f}$, making $f(s)(f(a)+j) \in K$ or $f(s)(f(b)+k) \in \sqrt{K}$. Hence, K is a weakly $f(S)$-primary ideal of $f(A) + J$. Now, let $f(s)(f(a) + j) \notin K, f(s)(f(b) + k) \notin \sqrt{K}$ with $(f(a)+j)(f(b)+k) = 0$. We claim that $ab = 0$. Deny, $(a, f(a)+j)(b, f(b)+k) = (ab, 0) \in K^f \setminus (0, 0)$, which is weakly $S \bowtie^f (0)$-primary ideal of $A \bowtie^f J$, a contradiction since $(s, f(s))(a, f(a) + j) \notin K^f$ and $(s, f(s))(b, f(b) + k) \notin \sqrt{K^f}$. Hence, $ab = 0$.

Conversely, let $(a, f(a)+j)(b, f(b)+k) = (ab, (f(a)+j)(f(b)+k)) \in K^f \setminus \{(0,0)\}$. So, $(f(a)+j)(f(b)+k)$ is an element of K which is weakly $f(S)$-primary ideal of $f(A)+J$. Two cases are then possible:
Case 1: $(f(a)+j)(f(b)+k) \neq 0$, then $f(s)(f(a)+j) \in K$ or $f(s)(f(b)+k) \in \sqrt{K}$. Hence, $(s, f(s))(a, f(a)+j) \in K^f$ or $(s, f(s))(b, f(b)+k) \in \sqrt{K^f}$, as desired.
Case 2: $(f(a)+j)(f(b)+k) = 0$. We claim that $f(s)(f(a)+j) \in K$ or $f(s)(f(b)+k) \in \sqrt{K}$. Deny, it follows that $ab = 0$, which is absurd since $(ab, (f(a)+j)(f(b)+k)) \in K^f \setminus \{(0,0)\}$. Hence $(s, f(s))(a, f(a)+j) \in K^f$ or $(s, f(s))(b, f(b)+k) \in \sqrt{K^f}$, making K^f a weakly $S \bowtie^f (0)$-primary ideal, as desired. □ □

Let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I = \{(a, a+i) \mid a \in A, i \in I\}.$$

The next corollary is an immediate consequence of assertion (1) of Theorem 3.2 on the transfer of weakly S-primary property to duplications.

Corollary 3.5. *Let R be aring, I an ideal of R, S be a multiplicative subset of R, and P an ideal of R disjoint from S. Then, the following statements are equivalent :*

1. *$P \bowtie I$ is a weakly $(S \bowtie I)$-primary ideal of $R \bowtie I$.*
2. *$P \bowtie I$ is a weakly $(S \bowtie 0)$-primary ideal of $R \bowtie I$.*
3. *P is a weakly S-primary ideal of R associated to s, and if there exist $a, b \in R$ with $ab = 0$, but $sa \notin P$ and $sb \notin \sqrt{P}$, then $aj + bi + ij = 0$ for all i, j in I.*

Remark 3.6. Let $f : A ß \longrightarrow B$ be a ring homomorphism and J be an ideal of B. Consider I (resp., H) be an ideal of A (resp., $f(A)+J$) such that $f(I)J \subseteq H \subseteq J$. Observe that

$$I \bowtie^f H = \{(i, f(i)+h) \mid i \in I, h \in H\}$$

is an ideal of $A \bowtie^f J$.

Using the previous remark, we get immediately the next result.

Proposition 3.7. *Under the above notation, $I \bowtie^f H$ is a weakly $S \bowtie^f (0)$-primary ideal of $A \bowtie^f J$ with the coefficient $(s, f(s)$ if and only if I is a weakly S-primary ideal of A and for each $i, j \in A$ such that $ij = 0, si \notin I$ and $sj \notin \sqrt{I}$ we have $kf(i) + hf(j) + hk = 0$ for each $h, k \in H$.*

Proof. Argue as in the Theorem 3.3. □

The next corollary is an immediate consequence of Proposition 3.6 concerning duplications.

Corollary 3.8. *Let A be a ring and I, H, J be ideals of A such that $IJ \subseteq H \subset J$. $I \bowtie H$ is a weakly $S \bowtie J$-primary ideal of $A \bowtie J$ with the coefficient $(s, f(s))$ if and only if I is a weakly S-primary ideal of A with the coefficient s and for each $i, j \in A$ such that $ij = 0, si \notin I$ and $sj \notin \sqrt{I}$ we have $ki + hj + hk = 0$ for each $h, k \in H$.*

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