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INVARIANCE OF CONVEX SETS: AN ALTERNATIVE PROOF AND APPLICATION TO BLACK-SCHOLES OPERATOR

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This paper is dedicated to our great Professor Abderrahim Driouich

Abstract. An alternative proof of invariance of convex sets by the solution of non autonomous Cauchy problem is given. The proof is based on the recent integral approximation of time dependent operators $\mathcal{A}(t)$ acting on Hilbert space when they are associated with smooth sesquilinear forms $\mathfrak{a}(t,.,.)$ defined on common dense domain and the known Chernoff Product Formula. An application to positivity of Black-Scholes operator is given.

1. Introduction

Let H and V be two Hilbert spaces. As integrals involved are of Banach valued functions, we assume that H is separable in order to take profit from Pettis theorem which relies integrability of an $H-$valued function to separability of its range and norm integrability. The subspace V is, as usual, densely embedded into H. The scalar products and the corresponding norms on H and V will be denoted by $(.\mid .), (.\mid \cdot)_V, \|.\|$ and $\|\cdot\|_V$, respectively. To describe such situation, we usually write $V \underset{d}{\hookrightarrow} H$ and this means in particular that there exists a constant $c > 0$ such that $\|u\| \leq c\|u\|_V \quad (\forall u \in V)$. According to [3, Remark A.4.12, page 473], this warrants the separability of V. We consider the non-autonomous evolutionary linear Cauchy-problems

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0, \tag{1.1}$$

where $u_0 \in H, f \in L^2(0,T;H)$ and the operators $\mathcal{A}(t), t \in [0,T]$ which arise from sesquilinear forms $\mathfrak{a}(t,.,.)$ all defined on the Hilbert subspace V. Throughout this paper, V' denotes the antidual of V and as usual, we will identify H with its antidual H' itself seen as an embedded space into V'. So one has $\|w\|_{V'} \leq c\|w\| \quad (\forall w \in V')$ and both of embeddings $V \underset{d}{\hookrightarrow} H \underset{d}{\hookrightarrow} V'$ are continuous. By Lions's Theorem, the problem (1.1) admits a unique solution u which belongs to $L^2(0,T;V) \cap H^1(0,T;V')$. More precise, we enunciate the result:

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Theorem 1.1 (Lions Theorem)**.** *Let $f \in L^2(0,T;V')$ and $u_0 \in H$. There is a unique solution $u \in MR(V,V') := L^2(0,T;V) \cap H^1(0,T;V')$ of*

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0.$$

An alternative proof of this interesting result was given recently in [10] or [12]. Let $\mathcal{C}$ be a closed convex subset of the Hilbert space H. To make precise our purposes, we recall the definition:

Definition 1.2. The closed convex set $\mathcal{C}$ is invariant for the Cauchy problem (1.1) if for each $u_0 \in \mathcal{C}$ the solution u of (1.1) satisfies $u(t) \in \mathcal{C}$ for all $t \in [0,T]$.

In the autonomous case (i.e $\mathcal{A}(t) = \mathcal{A}$ independent on time t), E. Ouhabbaz, in [9] proved that $\mathcal{C}$ is invariant provided that

- $PV \subset V$
- $\operatorname{Re}\mathfrak{a}(Pv, v - Pv) \geq \operatorname{Re}\langle f(t), v - Pv\rangle$

for all $v \in V$, where P denotes naturally the projection on $\mathcal{C}$. For time depending operators $\mathcal{A}(t)$, W. Arendt et al. [1] proved that the closed convex set $\mathcal{C}$ is invariant for the inhomogenous Cauchy problem provided that $PV \subset V$ and

$$\operatorname{Re}\mathfrak{a}(t, Pv, v - Pv) \geq \operatorname{Re}\langle f(t), v - Pv\rangle$$

for all $v \in V$ and for almost every $t \in [0,T]$. Recently, the same result was obtained in different way by Sani and Laasri (see [12, Theorem 4.1 and Theorem 4.2]).

Here we present an alternative proof, for the homogeneous problem (i.e when $f = 0$ almost everywhere on $[0,T]$) based on integral product and the technique of approximation with frozen coefficients as suggested by El-Mennaoui, Keyantuo and Laasri in [6]. To this end, we begin by recalling definitions and necessary results, as preliminaries. In section 3, we enunciate and prove otherwise invariance of closed and convex sets by the solution of problem (1.1). At last, in section 4, an application to positivity of Black-Scholes operator is given. For more information on this latter operator and in particular its maximal regularity, we refer to [5].

2. Preliminaries

The main result of this paper is based on discrete approximation of operators. Precisely, the efficient tool will be a corrolary of Chernoff Product Formula ([See [7, Proposition 21, page 220]] or [8]) that we recall as a proposition:

Proposition 2.1. *Let $V : t \in \mathbb{R}^+ \mapsto \mathcal{L}(E)$ a function such that $V(0) = \mathbb{I}_E$. Assume that:*

- $\exists M > 0\ \forall (k,t) \in \mathbb{N} \times \mathbb{R}^+ \quad \left\|V(t)^k\right\| \leq M$.
- *There exists a subspace D of E such that D and $(\lambda_0 - A)D$ are dense, for some $\lambda_0 > 0$.*
- *For all $x \in D$, the limit $\lim\limits_{t \downarrow 0} \dfrac{V(t)x - x}{t} = Ax$ exists.*

Then

- *The closure $\overline{A}$ of A generates bounded strongly continuous semigroup $(\mathbb{T}(t))_{t \geq 0}$.*

- *The semigroup* $(\mathbb{T}(t))_{t\geq 0}$ *is approximated uniformly on compacts of* $\mathbb{R}^+$ *as follows:*

$$\forall x \in E \quad V(t)x = \lim_{n\to+\infty} \left(V\left(\frac{t}{n}\right)\right)^n x$$

Now, let $T > 0$ and let $\mathfrak{a}$ be a sesquilinear non-autonomous form

$$\mathfrak{a} : [0,T] \times V \times V \to \mathbb{C}$$

such that the mapping $t \in [0,T] \mapsto \mathfrak{a}(t,u,v)$ is measurable for all $(u,v) \in V^2$ and there exist real constants $M > 0, \alpha > 0$ and ω satisfying

$$|\mathfrak{a}(t,u,v)| \leq M\|u\|_V\|v\|_V \quad (t \in [0,T], u,v \in V) \tag{2.1}$$

and

$$\operatorname{Re}\mathfrak{a}(t,u,u) + \omega\|u\| \geq \alpha\|u\|_V^2 \quad (t \in [0,T], u \in V) \tag{2.2}$$

A form satisfying (2.2) are called quasicoercive and if in addition $\omega = 0$, the form is coercive. For all $t \in [0,T]$ we denote $\mathcal{A}(t) \in \mathcal{L}(V,V')$ the operator associated with the form $\mathfrak{a}(t,.,.)$ on V' and let $\mathcal{T}_t$ be the analytic semigroup generated by $-\mathcal{A}(t)$ on V'. In order to dodge the consideration of two parameter semigroup theory as raised in [4] or more general multi-parameter one related to attractors as developed in [13], we will proceed otherwise. So, consider the autonomous evolutionary problem

$$\dot{u}(t) + \mathcal{A}u(t) = f(t), \qquad \text{for a.e } t \in [0,T], \quad u(0) = u_0 \tag{2.3}$$

where the singular operator $\mathcal{A}$ is associated with autonomous bounded form $\mathfrak{a}$. It is known that the problem (5) is well posed in V' and that it has L^2-maximal regularity which simply means that its solution u belongs to the space

$$MR(V,V') = L^2(0,T,V) \bigcap W^{1,2}(0,T;V')$$

for a given f in $L^2(0,T;V')$.
When the form $\mathfrak{a}$ is piecewise defined on the horizon $[0,T]$, a simple analysis suffices to establish the well-posedness. To be more precise, consider a subdivision $\Lambda = (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T)$ of $[0,T]$. Let

$$\mathfrak{a}_k : V \times V \to \mathbb{C} \qquad \text{pour } k = 0,1,\ldots,n$$

be finite family of continuous and H-elliptic forms [i.e that satisfy (2.1) and (2.2)]. We denote by $\mathcal{A}_k \in \mathcal{L}(V,V')$ the associate operator and by $\mathcal{T}_k$ the semigroup generated by $-\mathcal{A}_k$ on V' for all $k = 0,1\ldots n$. The mapping

$$\mathfrak{a}_\Lambda : [0,T] \times V \times V \to \mathbb{C} \tag{2.4}$$

defined by $\mathfrak{a}_\Lambda(t;u,v) := \mathfrak{a}_k(u,v)$ for $\lambda_k \leq t < \lambda_{k+1}$ and $\mathfrak{a}_\Lambda(T;u,v) := \mathfrak{a}_n(u,v)$, is manifestly strongly measurable on $[0,T]$. Consider

$$\mathcal{A}_\Lambda : [0,T] \to \mathcal{L}(V,V')$$

given by $\mathcal{A}_\Lambda(t) := \mathcal{A}_k$ if $\lambda_k \leq t < \lambda_{k+1}, k = 0,1,\ldots,n$, and $\mathcal{A}_\Lambda(T) := \mathcal{A}_n$. For all $s > 0$ and (a,b) in $\Delta = \{(x,y) \in [0,T]^2, x < y\}$ such that

$$\lambda_{m-1} \leq a < \lambda_m < \ldots < \lambda_{l-1} \leq b < \lambda_l$$

we consider the walks $\mathcal{P}_\Lambda(a,b,s) \in \mathcal{L}(V')$ defined as

$$\mathcal{P}_\Lambda(a,b,s) := \mathcal{T}_{l-1}\left((b-\lambda_{l-1})\,s\right)\mathcal{T}_{l-2}\left((\lambda_{l-1}-\lambda_{l-2})\,s\right)\ldots\mathcal{T}_m\left((\lambda_{m+1}-\lambda_m)\,s\right)\mathcal{T}_{m-1}\left((\lambda_m-a)\,s\right) \tag{2.5}$$

and

$$\mathcal{P}_\Lambda(a,b,s) := \mathcal{T}_{l-1}((b-a)s) \tag{2.6}$$

if $\lambda_{l-1} \le a \le b < \lambda_l$ For each $u_0 \in H$ and $f \in L^2(a,b,V')$ the function

$$u_\Lambda(t) = \mathcal{P}_\Lambda(a,t,1)u_0 + \int_a^t \mathcal{P}_\Lambda(r,t,1)f(r)\mathrm{d}r \tag{2.7}$$

belongs to $MR(a,b;V,V')$ and solves uniquely the approximated problem:

$$\dot{u}_\Lambda(t) + \mathcal{A}_\Lambda(t)u_\Lambda(t) = f(t), \quad \text{for a.e } t \in [a,b] \subset [0,T], \quad u_\Lambda(a) = u_0 \tag{2.8}$$

We refer to [6] and [10] for more information. Especially the convergence of the walks (finite product $P_\wedge$ above) when the modulus $|\Lambda| = \sup_k |\lambda_{k+1} - \lambda_k|$ tends to zero was the origin of π-integration theory and its recent development (One can see [10], [11] or [12]).

Consider now the more general case of continuous form $\mathfrak{a} : [0,T]\times V\times V \to \mathbb{C}$ and as above the operator $\mathcal{A}(t) \in \mathcal{L}(V,V')$ associated with $\mathfrak{a}(t,.,.)$. In order to approximate $\mathfrak{a}$ and $\mathcal{A}$ by step functions as described before, we define for all subdivision $\Lambda := (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T)$ of $[0,T]$, the piecewise form $a_\Lambda : [0,T]\times V\times V \to \mathbb{C}$ and associate operator $\mathcal{A}_\Lambda : [0,T] \to \mathcal{L}(V,V')$ given by

$$\begin{array}{c} \mathfrak{a}_k(u,v) := \frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{\lambda_{k+1}} \mathfrak{a}(r;u,v)\mathrm{d}r \\ \text{for } u,v \in V, k = 0,1,\ldots,n \end{array} \tag{2.9}$$

and

$$\mathcal{A}_k u := \frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r)u\ \mathrm{d}r \tag{2.10}$$

Recall that it is easy to establish that the forms $\mathfrak{a}_k(t,.,.);\ k = 0,1,\ldots n$ satisfy (2.1) and (2.2) (See Section 3 below).

3. A suitable reordering of factors and main result

We begin by proving the last fact enunciated above. The main tool is the following

Proposition 3.1. *The operator $\mathcal{A} = \frac{1}{b-a}\int_a^b A(r)dr$ is associated with the form $\mathrm{a} = \frac{1}{b-a}\int_a^b a(r,.,.)dr$,*

Proof. For all $(t_1,t_2) \in [a,b]^2$, we have: $\forall (u,v) \in V^2 : \mathfrak{a}(t_1,u,v) = (A(t_1)u,v)_V$ and $a(t_2,u,v) = (A(t_2)u,v)_V$. This implies that for any $\theta \in [0,1]$:

$$\theta a(t_1,u,v) + (1-\theta)a(t_2,u,v) = (\theta A(t_1) + (1-\theta)A(t_2))u,v)_V$$

It yields that $\theta A(t_1) + (1-\theta)A(t_2)$ is associated with $\theta a(t_1,.,.) + (1-\theta)a(t_2,.,.)$. A similar argument ensures that each finite combination of $(A(t_i))_{i=1}^m$, i.e

$$\sum_{i=1}^m \theta_i A(t_i) \text{ such that } \sum_{i=1}^m \theta_i = 1 \text{ and for all } i : \theta_i \ge 0$$

is associated with discrete forms combination: $\sum_{i=1}^{m} \theta_i \mathfrak{a}(t_i, \ldots)$.

Let $(\lambda_i)_{i=0}^{i=n}$ be a subdivision of $[a,b] \subset [0,T]$ such that $a = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n = b$. It is easy to see that the operator $\frac{1}{b-a}\sum_i (\lambda_{i+1} - \lambda_i) A(\lambda_i)$ is manifestly associated with the form $\frac{1}{b-a}\sum_i (\lambda_{i+1} - \lambda_i) \mathfrak{a}(\lambda_i, ., .)$, Since $t \to \mathfrak{a}(t, ., .)$, is measurable, it is allowed to tend n to infinity. Hence the operator $\frac{1}{b-a}\int_a^b A(r)dr$ with domain $D = \cap_{t\geq 0} D(t)$ is associated with the form $\frac{1}{b-a}\int_a^b a(r, ., .)dr$. □

Remark 3.2. It is worth to mention that all properties of $\frac{1}{b-a}\int_a^b \mathfrak{a}(r, ., .)dr$ concerning sesquilinearity, boundedness and coercivity are preserved and are easy to establish.

Up to now, we will assume that $\omega = 0$. If it is not the case, one may consider the form $\mathfrak{a}^\omega = \mathfrak{a} + \omega : (u,v) \to \mathfrak{a}(u,v) + \omega(u,v)$ and it is easy to verify that $\mathfrak{a}^\omega$ allows the assumption which will be ours in the sequel. To make this remark more precise, we consider for all $t \in [0,T]$ the mappings $u : t \mapsto u(t)$ and $v : t \mapsto e^{-wt}u(t)$. Then u satisfies (1) if and only if v satisfies

$$\dot{v}(t) + (\omega + A(t))v(t) = e^{-wt}f(t) \quad t - \text{ a.e. on } [0,T], \quad v(0) = u_0 \tag{3.1}$$

Let $u_0 \in H$ and $f \in L^2(0,T;V')$. Let $u_\Lambda \in MR(V,V')$ be the solution of the approximate problem (10). According to [10, Lemma 3.1], there exists a constant $k > 0$ independent on f, u_0 and Λ such that

$$\int_0^t \|P_\Lambda(0,t,s)u_0\|_V^2 \, ds \leq k\left[\int_0^t \|f(s)\|_{V'}^2 ds + \|u_0\|^2\right] \tag{3.2}$$

for ae $t \in [0,T]$.

In the sequel, we discuss the reordering impact of operators $(A(t))_{t\geq 0}$ on the estimate (3.2) which will play an important role to establish the main result. More precise, let B be a family of operators defined on $[0,1]$ as follows:

$$B(t) = \sum_{k=0}^{n-1} A\left(t - \frac{k}{n}\right)\chi_{[\frac{k}{n};\frac{k+1}{n}[}(t) + \delta_{tT}A(t)$$

and on $(a,b) \in \Delta$ as

$$B(t) = \sum_{k=0}^{n-1} A\left(t - k\frac{b-a}{n}\right)\chi_{[a+k\frac{b-a}{n};a+(k+1)\frac{b-a}{n}[}(t) + \delta_{tb}A(t)$$

Let U_j and $T_{j;k}$ be respectively the semigroups generated by $A(t_j)$ and $B\left(t_k^j\right)$ where $t_j \in \left[a, a + \frac{b-a}{n}\right]$ and $t_j^k \in \left[a + k\frac{b-a}{n}; a + (k+1)\frac{b-a}{n}\right]$. Then $t_j^k = kt_j \to$

$U_j = T_{j;k}$ For all $s > 0$, one may easily do the following calculation

$$\begin{aligned} P_{\wedge}^{B}(0,1,s) &= \prod_{j=0}^{n-1} T_j\left(\left(\lambda_{j+1}-\lambda_j\right)s\right) \\ &= \prod_{j=0}^{n-1} T_j\left(n\left(\lambda_{j+1}-\lambda_j\right)\frac{s}{n}\right) \\ &= \prod_{j=0}^{n-1} U_j\left(\left(\lambda_{j+1}-\lambda_j\right)\frac{s}{n}\right) \\ &= \left[P_{\wedge}^{A}(0,1,s/n)\right]^n \end{aligned}$$

This yields estimates for $\left[P_{\wedge}^{A}(a,t,s/n)\right]^n$ and $P_{\wedge}^{B}(a,t,s)$ similar to (3.2). That is

$$\left\|\left[P_{\wedge}^{A}(a,.,s/n)\right]^n u_0\right\|_{L^2(a,t;V)} \leq K\left\|u_0\right\| \tag{3.3}$$

where the constant K depends merely on M, α and c. We can now prove otherwise the following theorem known in literature by Beurling Deny condition which gives a practical criteria of invariance.

Theorem 3.3. *Let $\mathcal{C} \subset H$ a closed convex set and $P: \quad H \mapsto \mathcal{C}$ be the projection on C. The following two propositions are equivalent*

i) $P(V) \subset V$ and $\operatorname{Re}\mathfrak{a}(t, Pv, v-Pv) \geq 0$ for all $v \in V$ and $t \geq 0$
ii) For all $a \in [0,T]: u(a) \in \mathcal{C} \Longrightarrow u(t) \in \mathcal{C} \quad$ for all $t \geq a$.

Proof. *i) $\Rightarrow$ ii)* : This direct implication was proved in [12]. For the converse, assume that the convex $\mathcal{C}$ is NCP-invariant. As mentioned at the preamble, we will use Chernoff product formula (Proposition (2.1) above). To make precise the purpose, for all $r \in [0,T]$, we define the potential Φ as follows

$$\begin{aligned} \Phi: \mathbb{R}^+ &\longrightarrow \mathcal{L}\left(L^2(a,T;V)\right) \\ s &\longmapsto \Phi(s): u \mapsto P_{\wedge}(a,.,s)u_0 \end{aligned}$$

By contractivity estimate (15), there exists $M > 0$ such that for all $x \in V$ one has

$$\left\|\left(\Phi\left(\frac{s}{n}\right)\right)^n x\right\|_{L^2(a,T;V)} \leq M\|x\|_V$$

which implies

$$\left\|\left(\Phi\left(\frac{s}{n}\right)\right)^n\right\|_{\mathcal{L}(L^2(a,T;V))} = \sup_{\|x\|=1}\left\|\left(\Phi\left(\frac{s}{n}\right)\right)^n x\right\|_{L^2(a,T;V)} \leq M$$

This yields to following properties

- $\Phi(0) = Id_V$
- $\left\|\left(\Phi\left(\frac{s}{n}\right)\right)^n\right\|_{\mathcal{L}(L^2(a,T;V))} \leq M$
- For all $x \in V$ and $r > 0$: $\frac{d\Phi(s)}{ds}x\Big|_{s=0} = \frac{d}{ds}P(a,r,s)x\Big|_{s=0} = \int_a^r A(\theta)x d\theta.$

Let us clarify the last point. For all $r > 0$ we define $\Phi_\wedge(s)x = P_\wedge(a, r, s)x$. It is easy to see that

$$\frac{d\Phi_\wedge(s)}{ds}x = \frac{d}{ds}P_\wedge(a, r, s)x = \int_a^r P_\wedge(\theta, r, s)A \wedge P_\wedge(\theta, r, s)xd\theta$$

So

$$\dot{\Phi}(s)x\Big|_{s=0} = \int_a^r A_\wedge(\theta)d\theta$$

Recall that $\int_a^r A_\wedge(\theta)d\theta$ is the generator associated with the form $\mathfrak{a}_\wedge$, and its domain D is, by assumption, dense in H. Hence, the Chernoff product formula ([8, page 150]) warrants that

$$\lim_{n\to\infty}\left[P_\wedge\left(a, r, \frac{s}{n}\right]^n = e^{s\int_a^r A_\wedge(\theta)d\theta}$$

When the modulus $|\wedge|$ is small enough, one has

$$\lim_{n\to\infty}\left[P\left(a, r, \frac{s}{n}\right]^n = e^{s\int_a^r A_\wedge(\theta)d\theta}$$

As $\mathcal{C}$ is invariant, $P(a, r, s)x \in \mathcal{C}$ for all $s > 0$ and $r \geq a$ provided that $x \in \mathcal{C}$. By induction, one establishes

$$u(a) = x \in \mathcal{C} \Longrightarrow e^{s\int_a^r A_\wedge(\theta)d\theta}x \in \mathcal{C}$$

In other words, the closed convex $\mathcal{C}$ is invariant by the solution of autonomous Cauchy problem associated with the operator $\int_a^r A(\theta)d\theta$. Results obtained in the autonomous case (in particular [9]) prove that $P(V) \subset V$ and for all $r \geq a$, the forms $(\mathfrak{a}(r, ., .))_r$ satisfy: $\mathfrak{a}(r, Pv, v - Pv) \geq 0$ for all $v \in V$. □

Finally, we recall the big interest of studying invariance of convex set in many fields of mathematic analysis mainly in positivity and then continuity of solutions of evolutionary problems. This explains henceforth the interest of the current result and it will be illustrated in the the following section.

4. Application to time-dependent Black-Scholes

To take profit from results obtained from [5] and all technical tools developed in [2], we situate ourselves in the suitable space $H = L^2(0, +\infty)$. Let us first recall the basic notions related to Black-scholes operator. For $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$ we consider the operator A :

$$Au = ax^2D^2u + bxDu - bu$$

with natural domain $D(A) = H^2(0, +\infty)$. We summarize principal results on the operator A when it acts on L^2. The most important one is the explicit expression of the form associated to the operator A given by $\mathfrak{a}$:

$$\mathfrak{a}(u, v) = (xDu, xDv) + (2a - b)(xDu, v) + b(u, v)$$

with domain

$$V = \{u \in W^{1,1}_{loc}((0, +\infty),\ xDu \in H\}.$$

The form $\mathfrak{a}$ is regular enough in the sense that it is elliptic as stated by the following proposition

Proposition 4.1. [5, Proposition 3.6] *The form $\mathfrak{a}$ is continuous, densely defined and elliptic (on H). Moreover, if $b \geq \frac{2}{3}a$ then $\mathfrak{a}$ is coercive.*

In the current paper, we consider the particular non autonomous Black-Scholes operator $A(t)_{t\geq 0}$ satisfying the following equation:

$$A(t)u = (a+t)x^2D^2u + bxDu - bu. \tag{4.1}$$

It is obvious that the operators $A(t)$ have the same domain $D = D(t) = H^2(0,+\infty)$ and the corresponding forms are all defined on $V = \{u \in W^{1,1}_{loc}((0,+\infty),\ xDu \in H\}$.
These facts are so obvious as one may choose $a' = a + t$ and consider all results obtained in the autonomous case. This remark allows in particular to prove the punctual maximal regularity of the family $A(t)$. Nevertheless, in order to ensure the uniform boundedness of the family $(\mathfrak{a}(t,.,.))_{t\geq 0}$, it is necessary to restrict the treatment to finite horizons and we consider henceforth $t \in [0,T]$ for some fixed horizon $T > 0$. Under this hypothesis, the forms $(\mathfrak{a}(t,.,.))_{t\geq 0}$ satisfy (see proof of [5, Proposition 3.6,])

$$\mathfrak{a}(t,u,v) \leq (a+t+|2a+2t-b|+|b|)\|u\|\|v\| \quad \text{for all} \quad (u,v) \in V^2.$$

Putting $M = a+T+|2a+2T|+2|b|$, one obtains the desired uniform boundedness. Clearly said, for all $t \in [0,T]$ and all $(u,v) \in V^2$ one has $\mathfrak{a}(t,u,v) \leq M\|u\|\|v\|$.
Let us now examine the uniform ellipticity of the forms. Always according to [5, Proposition 3.6,]), the single form $\mathfrak{a}(0,u,v)$ satisfies

$$\mathfrak{a}(0,u,u) = a\|xDu\|^2_H + (\frac{3}{2}b - a)\|u\|^2_H.$$

For the autonomous $\mathfrak{a}(t,.,.)$, the following estimation is an immediate consequence of the latter identity satisfied by $\mathfrak{a}(0,.,.)$,

$$\mathfrak{a}(t,u,u) \geq a\|xDu\|^2_H + (\frac{3}{2}b - a - T)\|u\|^2_H.$$

All conditions required for well-posedness of the non autonomous evolution problem (4.1) are satisfied. In fact, the results established suffice to affirm that correspondent non autonomous BS-operator $A(t)_{t\in[0,T]}$ has the *maximal regularity property*, according to results of frozen coefficient approximation as stated in [12]. Indeed, the additional hypothesis $\frac{3}{2}b - a - T > 0$ is necessary to ensure positivity. This constrains the form to be positive for some but restricted horizons T (those for which $\frac{3}{2}b - a - T > 0$). But for practical problems in finance, predictions are credible for short term forecasts, so the choice of T such that $0 < T < \frac{3}{2}b - a$ is large enough and is largely satisfactory.
Comment: The financial interpretation consists in considering the convex closed set $\mathcal{C} = \{u \in L^2(0,T;D) \cap H^2(0,T;H); u \geq 0\}$ where u is a solution of the problem (4.1). Naturally, a trader in a stock exchange always desires to keep his portfolio positive. As a direct application of theorem (3.3) established above, he (the trader) is almost sure that his portfolio will not be negative. Anyhow, it is not certain that his wealth will grow automatically.

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