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PERFECT POWERS BETWEEN TWO CONSECUTIVE SQUARES

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Abstract. In this note we obtain an upper bound for the number of perfect powers between two consecutive squares. We also prove some theorems on perfect powers between two consecutive squares. In this way we improve former results of the author.

1. Introduction

A natural number of the form m^n where m and $n \geq 2$ are positive integers is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$$

Let $A(n)$ be the number of perfect powers in the open interval $((n-1)^2, n^2)$, where $n \geq 2$ is a positive integer. Clearly a trivial upper bound for $A(n)$ is $2n-2$, since the number of integers in the open interval $((n-1)^2, n^2)$ is $n^2-(n-1)^2-1 = 2n-2$. In this article we obtain a better upper bound. It is well-known that $A(n) = 0$ for almost all intervals $((n-1)^2, n^2)$, since we have the theorem (see [2, Theorem 3.10])

Theorem 1.1. *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \ldots, ((n-1)^2, n^2)$. Let $S(n)$ be the number of these n open intervals that contain some perfect power. The following limit holds*

$$\lim_{n\to\infty} \frac{S(n)}{n} = 0$$

2. Main Results

We have the following lemma.

Lemma 2.1. *Let $h \geq 3$ an arbitrary but fixed positive integer. Let us consider the sequence of disjoint intervals*

$$[k^2, (k+1)^2) \qquad (k = 0, 1, 2, \ldots) \tag{2.1}$$

If $n \geq 2$ is a positive integer then n^h and $(n+1)^h$ pertain to different and not consecutive intervals of the sequence (2.1).

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Proof. We have

$$\lfloor n^{h/2} \rfloor \leq n^{h/2} < \lfloor n^{h/2} \rfloor + 1 \tag{2.2}$$

Equation (2.2) gives

$$\left(\lfloor n^{h/2} \rfloor\right)^2 \leq n^h < \left(\lfloor n^{h/2} \rfloor + 1\right)^2 \tag{2.3}$$

In the same way we obtain

$$\left(\lfloor (n+1)^{h/2} \rfloor\right)^2 \leq (n+1)^h < \left(\lfloor (n+1)^{h/2} \rfloor + 1\right)^2 \tag{2.4}$$

We have the equation

$$\left(n^{h/2} + 2\right)^2 = n^h + 4n^{h/2} + 4 \tag{2.5}$$

From the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

we obtain the equation

$$\left((n+1)^{h/2}\right)^2 = (n+1)^h = \sum_{k=0}^{h} \binom{h}{k} n^{h-k} = n^h + hn^{h-1} + \frac{h(h-1)}{2} n^{h-2} + A \tag{2.6}$$

where $(A \geq 1)$.

Clearly we have the inequality

$$4 < 6 \leq \frac{h(h-1)}{2} n^{h-2} \qquad (h \geq 3) \qquad (n \geq 2) \tag{2.7}$$

and the inequality

$$\frac{4}{h} < n^{(h/2)-1} \qquad (h \geq 3) \qquad (n \geq 2) \tag{2.8}$$

since if $h = 3$ we have $\frac{4}{3} < \sqrt{2}$ and if $h \geq 4$ the left side is less than or equal to 1 and the right side if greater than 1.

Inequality (2.8) is equivalent to the inequality

$$4n^{h/2} < hn^{h-1} \qquad (h \geq 3) \qquad (n \geq 2) \tag{2.9}$$

Equations (2.5), (2.6), (2.7) and (2.9) give us the inequality

$$n^{(h/2)} + 2 < (n+1)^{h/2} \tag{2.10}$$

Inequality (2.10) gives

$$\lfloor n^{h/2} \rfloor + 1 \leq n^{(h/2)} + 1 < (n+1)^{h/2} - 1 < \lfloor (n+1)^{h/2} \rfloor \tag{2.11}$$

Now, the lemma is an immediate consequence of (2.3), (2.4) and (2.11). □

Corollary 2.2. *Let $h \geq 3$ an arbitrary but fixed positive integer. Let us consider the closed interval $[k^2, (k+1)^2]$, the number of h-th perfect powers in this interval is either 0 or 1. Let us consider the open interval $(k^2, (k+1)^2)$, the number of h-th perfect powers in this interval is either 0 or 1.*

Proof. It is an immediate consequence of Lemma 2.1. However, we shall give a simple and direct proof of this corollary. We have (mean value Theorem)

$$(k+1)^{2/h} - k^{2/h} = \frac{2}{h}(k+\epsilon)^{(2/h)-1} \leq \frac{2}{3}(2)^{(2/3)-1} < 1 \quad (k \geq 2) \quad (h \geq 3) \tag{2.12}$$

where $0 < \epsilon < 1$.

Suppose that there are two different h-th perfect powers s^h and r^h in the closed interval $[k^2.(k+1)^2]$, then we have

$$k^2 \leq s^h < r^h \leq (k+1)^2$$

Therefore

$$k^{2/h} \leq s < r \leq (k+1)^{2/h}$$

and consequently

$$(k+1)^{2/h} - k^{2/h} \geq r - s \geq 1$$

since r and s are different positive integers. This is an evident contradiction with inequality (2.12). □

Let us consider the prime factorization of a positive integer $n > 1$

$$n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$$

where $p_1, p_2, \ldots, p_s$ are the different primes and $a_1, a_2, \ldots, a_s$ are the exponents. Then, n is a perfect power if and only if $\gcd(a_1, a_2, \ldots, a_s) > 1$. Therefore, if n is a perfect power then we can writte

$$n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} = A^m$$

where A is not a perfect power and $m = \gcd(a_1, a_2, \ldots, a_s) > 1$. The integer A will be called principal basis of the perfect power n and the integer $m > 1$ will be called the principal exponent of the perfect power n.

Corollary 2.3. *Suppose that in the open interval* $(k^2, (k+1)^2)$ *there are* $t \geq 2$ *different perfect powers* $A_1^{m_1}$, $A_2^{m_2}$, $\ldots$, $A_t^{m_t}$, *where* $A_1, A_2, \ldots, A_t$ *are the principal basis and* $m_1, m_2, \ldots, m_t$ *are the principal exponents. Then the principal exponents are odd numbers and* $\gcd(m_i, m_j) = 1$, $(i \neq j)$, $(i = 1, 2, \ldots, t)$, $(j = 1, 2, \ldots, t)$.

Proof. It is an immediate consequence of Corollary 2.2. □

Let $\pi(x)$ be the prime counting function. We have the following lemma.

Lemma 2.4. *The following inequality holds*

$$\pi(x) \leq 6\frac{x}{\log x} \qquad (x \geq 3)$$

Proof. We have the inequality (see [1, Chapter 4])

$$\pi(n) \leq 6\frac{n}{\log n} \qquad (n \geq 2)$$

Now, the function $f(x) = \frac{x}{\log x}$ is strictly increasing in the interval $[3, \infty)$, since $f'(x) = \frac{\log x - 1}{\log^2 x} > 0 \quad (x \geq 3)$. □

Now, we can prove the following theorem.

Theorem 2.5. *The following inequalities hold*

$$A(n) \leq \pi\left(\left\lfloor \frac{2\log n}{\log 2} \right\rfloor\right) - 1 \qquad (n \geq 2) \tag{2.13}$$

$$A(n) \leq \frac{12}{\log 2}\frac{\log n}{\log\log n} \qquad (n \geq 3) \tag{2.14}$$

Proof. We have

$$2^{\frac{2\log n}{\log 2}} = n^2$$

Therefore

$$2^{\left(\left\lfloor \frac{2\log n}{\log 2} \right\rfloor + 1\right)} > n^2$$

and consequently

$$s^k > n^2 \qquad (s \geq 2) \qquad \left(k \geq \left\lfloor \frac{2\log n}{\log 2} \right\rfloor + 1\right)$$

This fact implies that if there exists a perfect power a^p such that

$$(n-1)^2 < a^p < n^2$$

where p is a positive prime (note that all perfect power can be written with prime exponent) then

$$3 \leq p \leq \left\lfloor \frac{2\log n}{\log 2} \right\rfloor$$

Consequently Corollary 2.2 implies the inequality

$$A(n) \leq \pi\left(\left\lfloor \frac{2\log n}{\log 2} \right\rfloor\right) - 1$$

That is, inequality (2.13).

On the other hand, inequality (2.13) and Lemma 2.4 give

$$\begin{aligned} A(n) &\leq \pi\left(\left\lfloor \frac{2\log n}{\log 2} \right\rfloor\right) - 1 \leq \pi\left(\left\lfloor \frac{2\log n}{\log 2} \right\rfloor\right) \leq \pi\left(\frac{2\log n}{\log 2}\right) \leq \frac{\frac{12\log n}{\log 2}}{\log\left(\frac{2\log n}{\log 2}\right)} \\ &= \frac{12}{\log 2}\frac{\log n}{\log\left(\frac{2}{\log 2}\right) + \log\log n} = \frac{12}{\log 2}\frac{1}{1 + \frac{\log\left(\frac{2}{\log 2}\right)}{\log\log n}}\frac{\log n}{\log\log n} \\ &\leq \frac{12}{\log 2}\frac{\log n}{\log\log n} \end{aligned}$$

That is, inequality (2.14). □

Theorem 2.6. *Suppose that in the closed interval* $[n^2, (n+1)^2]$ $(n \geq 3)$ *there are* $t \geq 2$ *different perfect powers* $n^2 = A_1^{m_1}$, $A_2^{m_2}$, $\ldots$, $A_t^{m_t} = (n+1)^2$, *where* $A_1, A_2, \ldots, A_t$ *are not necessarily the principal basis and consequently* $m_1, m_2, \ldots, m_t$ *are not necessarily the principal exponents. Then* $A_i \neq A_j$, $(i \neq j)$, $(i = 1, 2, \ldots, t)$, $(j = 1, 2, \ldots, t)$.

Proof. Suppose that there are two perfect powers A^{h_1} and A^{h_2} such that the following inequality holds

$$n^2 \leq A^{h_1} < A^{h_2} \leq (n+1)^2 \tag{2.15}$$

Note that then $h_2 > h_1$.

Therefore we have

$$\frac{A^{h_2}}{A^{h_1}} \leq \frac{(n+1)^2}{n^2} = \left(1 + \frac{1}{n}\right)^2$$

That is

$$A^{h_2 - h_1} \leq \left(1 + \frac{1}{n}\right)^2$$

That is

$$h_2 - h_1 \leq \frac{1}{\log A} \log\left(\left(1 + \frac{1}{n}\right)^2\right) \tag{2.16}$$

Let us consider the inequality

$$\log\left(\left(1 + \frac{1}{n}\right)^2\right) < \log 2 \tag{2.17}$$

that is

$$\left(1 + \frac{1}{n}\right)^2 < 2$$

that is

$$1 + \frac{1}{n} < \sqrt{2}$$

This inequality, and consequently inequality (2.17), has the solutions $n \geq 3$.

Inequality (2.16) and inequality (2.17) give the inequality $h_2 - h_1 < 1$ $(n \geq 3)$, since $A \geq 2$. Now, the inequality $h_2 - h_1 < 1$ is impossible since $h_2 \geq 2$ and $h_1 \geq 2$ are different positive integers. Therefore the inequality (2.15) is impossible. $\square$

Theorem 2.7. *Let us consider two perfect powers $A_1^{h_1} < A_2^{h_2}$ in the open interval $(n^2, (n+1)^2)$, that is*

$$n^2 < A_1^{h_1} < A_2^{h_2} < (n+1)^2 \tag{2.18}$$

where A_1 and A_2 are not necessarily the principal basis and consequently h_1 and h_2 are not necessarily the principal exponents. We have

$$h_1 > h_2 \Leftrightarrow A_1 < A_2 \tag{2.19}$$

$$h_1 < h_2 \Leftrightarrow A_1 > A_2 \tag{2.20}$$

Proof. Suppose that $h_1 > h_2$ and $A_1 > A_2$. Then

$$h_1 > h_2 \Rightarrow A_1^{h_1} > A_1^{h_2} > A_2^{h_2} \Rightarrow A_1^{h_1} > A_2^{h_2}$$

that is, an evident contradiction with (2.18). Therefore if $h_1 > h_2$ then $A_1 < A_2$. That is

$$h_1 > h_2 \Rightarrow A_1 < A_2 \tag{2.21}$$

Now, suppose that $h_1 < h_2$ and $A_1 < A_2$. Then

$$h_1 < h_2 \Rightarrow A_1^{h_1} < A_1^{h_2} < A_2^{h_2}$$

Therefore $n^2 \leq A_1^{h_1} < A_1^{h_2} \leq (n+1)^2$, an evident contradiction with Theorem 2.6. Hence if $h_1 < h_2$ then $A_1 > A_2$. That is

$$h_1 < h_2 \Rightarrow A_1 > A_2 \tag{2.22}$$

Now, (2.19) and (2.20) are an immediate consequence of (2.21) and (2.22). ☐

The following theorem is well-known (see [2, Theorem 3.13])

Theorem 2.8. *Let us consider the n open intervals $(0,1^2),(1^2,2^2),\ldots,((n-1)^2,n^2)$. Let $S(n)$ be the number of these n open intervals that contain some perfect power. We have the following asymptotic formula*

$$S(n) \sim n^{2/3}.$$

Let $S_1(n)$ be the number of these n open intervals that contain only one perfect power. We have the following asymptotic formula

$$S_1(n) \sim n^{2/3}.$$

Let $S_2(n)$ be the number of these n open intervals that contain at least two perfect powers. We have the following formula

$$S_2(n) = o\left(n^{2/3}\right).$$

Note that this theorem say us that almost all the n intervals are empty and that almost all the intervals that contain some perfect powers have only one perfect power. Therefore the intervals that contain at least two perfect powers are very scarce.

Now, we prove the following more precise theorem.

Theorem 2.9. *Let us consider the n open intervals $(0,1^2),(1^2,2^2),\ldots,((n-1)^2,n^2)$. Let $S(n)$ be the number of these n open intervals that contain some perfect power. We have the following asymptotic formula*

$$S(n) = n^{2/3} + O\left(n^{2/5}\right)$$

and the number of perfect powers contained in these intervals is $n^{2/3} + n^{2/5} - n^{2/6} + o\left(n^{2/6}\right) = n^{2/3} + O\left(n^{2/5}\right)$.

Let $S_1(n)$ be the number of these n open intervals that contain only one perfect power. We have the following asymptotic formula

$$S_1(n) = n^{2/3} + O\left(n^{2/5}\right)$$

and the number of perfect powers contained in these intervals is $n^{2/3} + O\left(n^{2/5}\right)$.

Let $S_2(n)$ be the number of these n open intervals that contain at least two perfect powers. We have the following formula

$$S_2(n) = O\left(n^{2/5}\right)$$

and the number of perfect powers contained in these intervals is $O\left(n^{2/5}\right)$.

Proof. Let $N(x)$ be the number of perfect powers not exceeding x. We have the formula (see [3])

$$N(x) = x^{1/2} + x^{1/3} + x^{1/5} - x^{1/6} + o\left(x^{1/6}\right)$$

Therefore

$$N(n^2) = n + n^{2/3} + n^{2/5} - n^{2/6} + o\left(n^{2/6}\right) \tag{2.23}$$

where n are the n squares $1^2, 2^2, \ldots, n^2$ that do not pertain to the n open intervals. Hence the number of perfect powers in the n open intervals is $n^{2/3} + n^{2/5} - n^{2/6} + o\left(n^{2/6}\right) = n^{2/3} + O\left(n^{2/5}\right)$.

Let us consider the inequality $i^3 \leq n^2$, this inequality has the solutions $i = 1, 2, \ldots, \left\lfloor n^{2/3} \right\rfloor$. Now, Let us consider the inequality $i^6 \leq n^2$, this inequality has the solutions $i = 1, 2, \ldots, \left\lfloor n^{2/6} \right\rfloor$. Hence the number of cubes in these n open intervals is

$$\left\lfloor n^{2/3} \right\rfloor - \left\lfloor n^{2/6} \right\rfloor = n^{2/3} - n^{2/6} + O(1) \tag{2.24}$$

Consequently equation (2.23) can be written in the form

$$\begin{aligned} N(n^2) &= n + \left(n^{2/3} - n^{2/6} + O(1)\right) + \left(n^{2/5} + o\left(n^{2/6}\right) - O(1)\right) \\ &= n + \left(n^{2/3} - n^{2/6} + O(1)\right) + \left(n^{2/5} + o\left(n^{2/6}\right)\right) \end{aligned} \tag{2.25}$$

where $n^{2/5} + o\left(n^{2/6}\right)$ is the number of perfect powers in the n open intervals that are not squares and are not cubes.

Let $B(n)$ be the number of open intervals that contain only a perfect power different of a cube, then (see (2.25))

$$0 \leq B(n) \leq n^{2/5} + o\left(n^{2/6}\right)$$

that is

$$0 \leq \frac{B(n)}{n^{2/5}} \leq 1 + \epsilon \qquad (\epsilon > 0)$$

that is

$$B(n) = O\left(n^{2/5}\right) \tag{2.26}$$

Let $C(n)$ be the number of open intervals that contain a cube and at least other perfect power. Then (see (2.25))

$$0 \leq C(n) \leq n^{2/5} + o\left(n^{2/6}\right)$$

that is

$$C(n) = O\left(n^{2/5}\right) \tag{2.27}$$

Therefore (see (2.24), (2.26), (2.27) and Corollary 2.2)

$$\begin{aligned} S_1(n) &= n^{2/3} - n^{2/6} + O(1) + B(n) - C(n) \\ &= n^{2/3} - n^{2/6} + O(1) + O\left(n^{2/5}\right) - O\left(n^{2/5}\right) = n^{2/3} + O\left(n^{2/5}\right) \end{aligned}$$

Let $D(n)$ be the number of open intervals that contain at least two perfect powers and do not contain cubes. Then (see (2.25))

$$0 \leq D(n) \leq n^{2/5} + o\left(n^{2/6}\right)$$

that is

$$D(n) = O\left(n^{2/5}\right) \tag{2.28}$$

Therefore (see (2.27) and (2.28))

$$S_2(n) = C(n) + D(n) = O\left(n^{2/5}\right)$$

and consequently

$$S(n) = S_1(n) + S_2(n) = n^{2/3} + O\left(n^{2/5}\right)$$

□

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