Gulf Journal of Mathematics
Vol 5, Issue 2 (2017) 20-30

THE SPECTRA OF CERTAIN NONLINEAR SUPERPOSITION OPERATORS IN THE SPACES OF SEQUENCES

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Abstract. In this paper, we consider the nonlinear superposition operator F in l_p spaces of sequences $(1 \leq p \leq \infty)$, generated by the function

$$f(s,u) = a(s) + \arctan u \quad \text{or} \quad f(s,u) = a(s) - \arctan u.$$

We find out the Rhodius spectra $\sigma_R(F)$ and the Neuberger spectra $\sigma_N(F)$ of these operators and finally the radii of these spectra. The superposition operator generated by the function $f(s,u) = a(s) \pm \operatorname{arccot} u$ appears to be a special case of above mentioned operator.

1. Introduction and preliminaries

We consider the certain nonlinear superposition operators in the Banach spaces of sequences l_p $(1 \leq p \leq \infty)$. Let $f = f(s,u)$ be a function defined on $\mathbb{N} \times \mathbb{R}$ with the values in $\mathbb{R}$. Given a function $x = x(s)$, by applying f, we get another function $y = y(s)$ on $\mathbb{N}$ by $y(s) = f(s, x(s))$. This function f generates an operator F :

$$Fx(s) = f(s, x(s)),$$

which is called superposition operator, Nemytskij operator or composition operator ([2]). The spectral theories for nonlinear operators are relatively new field of functional analysis and operator theory. The term spectrum for nonlinear operators, in the beginning, was used just for the set of eigenvalues (i.e. the point spectrum). Later it became clear that the notion spectrum need to have wider meaning and more complete description. The several ways of defining and studying spectra for nonlinear operators have been presented so far ([1], [3], [5], [9], [10]). For the class $\mathfrak{C}(X)$ of all continuous operators F on Banach space X over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), the Rhodius resolvent set ([10]) is given by:

$$\rho_R(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}(X)\}$$

and the Rhodius spectrum is the set

$$\sigma_R(F) = \mathbb{K} \setminus \rho_R(F).$$

Date: Received: Dec 8, 2016; Accepted: Jan 13, 2017.

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2010 *Mathematics Subject Classification.* Primary 47J10; Secondary 47H30.

Key words and phrases. Rhodius spectrum, Neuberger spectrum, Fréchet derivative, superposition operator, spaces of sequences.

The Rhodius spectral radius of $F \in \mathfrak{C}(X)$ is the number

$$r_R(F) = \sup\{|\lambda| : \lambda \in \sigma_R(F)\}.$$

J.W.Neuberger ([9]) introduced the notion of resolvent set and spectrum for the class of continuously Fréchet differentiable operators.

Definition 1.1. ([1], [2]) An operator $F : X \to Y$ is called Fréchet differentiable at $x_0 \in X$ if there is an operator $L : X \to Y$ such that

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|} \|F(x_0 + h) - F(x_0) - Lh\| = 0 \quad (h \in X).$$

In this case the linear operator L is called Fréchet derivative of F at x_0 and denoted by $F'(x_0)$. The value $F'(x_0)x \in Y$ for arbitrary $x \in X$, is called Fréchet derivative of operator F at x_0 along x.

If F is differentiable at each point $x \in X$ and the map $x \mapsto F'(x)$ is continuous, we write $F \in \mathfrak{C}^1(X,Y)$ and call F continuously differentiable ([12]). For an operator $F : X \to X$ which is continuously Fréchet differentiable, the Neuberger resolvent set is defined by

$$\rho_N(F) = \left\{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}^1(X)\right\}$$

and the set

$$\sigma_N(F) = \mathbb{K} \setminus \rho_N(F)$$

is called Neuberger spectrum of F. We call the number

$$r_N(F) = \sup\{|\lambda| : \lambda \in \sigma_N(F)\}$$

Neuberger spectral radius of $F \in \mathfrak{C}^1(X)$. Since $\mathfrak{C}^1(X) \subseteq \mathfrak{C}(X)$, the following inclusions hold: $\rho_N(F) \subseteq \rho_R(F)$ and $\sigma_N(F) \supseteq \sigma_R(F)$, for $F \in \mathfrak{C}^1(X)$ ([3]). Some applications of spectral theories are given in [1], [5], [8] and the Neuberger spectrum may be useful in solvability of certain operator equations and eigenvalue problems ([9]). The Rhodius and Neuberger spectrum of some nonlinear superposition operators may be found in [1], [6], [7]. The conditions of acting, continuity and differentiability of the superposition operator defined in the spaces of sequences l_p are given in the next three theorems.

Theorem 1.2. *([4]) Let $1 \le p, q < \infty$. Then the following properties are equivalent:*

(1) *the operator F acts from l_p to l_q;*
(2) *there are functions $a(s) \in l_q$ and constants $\delta > 0, n \in \mathbb{N}, b \ge 0$, for which $|f(s,u)| \le a(s) + b|u|^{\frac{p}{q}} \quad (s \ge n, |u| < \delta)$;*
(3) *for any $\varepsilon > 0$ there exists a function $a_\varepsilon \in l_q$ and constants $\delta_\varepsilon > 0$, $n_\varepsilon \in \mathbb{N}, b_\varepsilon \ge 0$, for which $\|a_\varepsilon(s)\|_q < \varepsilon$ and*

$$|f(s,u)| \le a_\varepsilon(s) + b_\varepsilon |u|^{\frac{p}{q}} \;\; (s \ge n_\varepsilon, \;\; |u| \le \delta_\varepsilon).$$

Theorem 1.3. *([4]) Let $1 \le p, q < \infty$ and let the superposition operator* (1)*, generated by the function $f(s,u)$, acts from l_p to l_q. Then this operator is continuous if and only if each of the functions is continuous for every $s \in \mathbb{N}$.*

Theorem 1.4. *([4], [12]) Let $1 \leq p, q < \infty$ and the operator F generated by the function $f(s,u)$ acts from l_p into l_q. The operator F is differentiable at $x_0 \in l_p$ if and only if $f'_u(s,\cdot)$ is continuous at x_0 for almost all $s \in \mathbb{N}$.*

2. Main Results

We consider the superposition operator $F : l_p \to l_p$, generated by the function $f(s,u) = a(s) \pm \arctan u$, where $a = (a(s))_{s\in\mathbb{N}}$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. Firstly we will show that this operator really acts from the space l_p to the space l_p.

a) Case $1 \leq p < \infty$

$$|f(s,u)| = |a(s) \pm \arctan u| \leq |a(s)| + |\arctan u|.$$

For $|u| < 1$ we certainly have $|\arctan u| \leq |u|$, so

$$|f(s,u)| \leq |a(s)| + |\arctan u| \leq d(s) + 1 \cdot |u|, \tag{2.1}$$

where $d(s) = |a(s)|$. Since $a \in l_q$ and $l_q \subseteq l_p$ we conclude that $d \in l_p$. Now we can see there exists constants $\delta = 1, n = 1, b = 1$ such that $\forall s \geq n, \quad |u| < \delta$, inequality (2.1) holds, so from the Theorem 1.2 it follows that $F : l_p \to l_p$.

b) Case $p = l_\infty$

$a \in l_q \subseteq l_\infty \Rightarrow \sup\limits_{s\in\mathbb{N}} |a(s)| = A < \infty.$

For arbitrary $x = (x_1, x_2, \ldots) \in l_\infty$ we have

$$\sup_{s\in\mathbb{N}}|Fx(s)| = \sup_{s\in\mathbb{N}}|a(s) \pm \arctan x_s| \leq \sup_{s\in\mathbb{N}}|a(s)| + \sup_{s\in\mathbb{N}}|\arctan x_s| \leq$$

$$\leq A + \frac{\pi}{2} < \infty.$$

We see that for every $x \in l_\infty$ it holds $Fx \in l_\infty$, so indeed F acts from l_∞ to l_∞. It is clear that $f(s,u) = a(s) + \arctan u$ and $f(s,u) = a(s) - \arctan u$ are continuous functions for every $s \in \mathbb{N}$, so the operator F is continuous (from the Theorem 1.3) and we write $F \in \mathfrak{C}(l_p)$.

Theorem 2.1. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = a(s) + \arctan u$, where $(a(s))_s$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = [0, 1)$.*

Proof. Denote $a = (a_1, a_2, \ldots) \in l_q$, and $x = (x_1, x_2, \ldots) \in l_p$. We have
$Fx = F(x_1, x_2, \ldots) = (a_1 + \arctan x_1, a_2 + \arctan x_2, \ldots)$.
Consider the operator $\lambda I - F$. For $\lambda = 0$ it becomes

$$(\lambda I - F)x = -Fx = (-a_1 - \arctan x_1, -a_2 - \arctan x_2, \ldots). \tag{2.2}$$

From $-Fx = -Fy$ $(x,\, y \in l_p)$, we have

$$(-a_1 - \arctan x_1, -a_2 - \arctan x_2, \ldots) = (-a_1 - \arctan y_1, -a_2 - \arctan y_2, \ldots)$$

$$-a_i - \arctan x_i = -a_i - \arctan y_i, \quad \forall i \in \mathbb{N}$$

$$\arctan x_i = \arctan y_i, \quad \forall i \in \mathbb{N}. \tag{2.3}$$

Since the function $f(x) = \arctan x$, $x \in \mathbb{R}$, is injective, from (2.3) we get
$x_i = y_i$, $\forall i \in \mathbb{N}$, i.e. $x = y$, so this operator (2.2) is injective. This is not surjective

operator since $-a_i - \arctan x_i \in (-\frac{\pi}{2} - a_i, \frac{\pi}{2} - a_i)$, $(i \in \mathbb{N})$. The operator (2.2) is not bijection and it follows that

$$0 \in \sigma_R(F). \tag{2.4}$$

Let $\lambda \neq 0$ and consider the operator

$$(\lambda I - F)x = (\lambda x_1 - a_1 - \arctan x_1, \lambda x_2 - a_2 - \arctan x_2, \ldots). \tag{2.5}$$

From $(\lambda I - F)x = (\lambda I - F)y$ $(x,\, y \in l_p)$, we have

$$(\lambda x_1 - a_1 - \arctan x_1, \lambda x_2 - a_2 - \arctan x_2, \ldots) =$$
$$= (\lambda y_1 - a_1 - \arctan y_1, \lambda y_2 - a_2 - \arctan y_2, \ldots)$$
$$\lambda x_i - a_i - \arctan x_i = \lambda y_i - a_i - \arctan y_i, \quad \forall i \in \mathbb{N} \Leftrightarrow$$
$$\lambda x_i - \arctan x_i = \lambda y_i - \arctan y_i, \quad \forall i \in \mathbb{N}.$$

Now we wonder whether the function

$$f(x) = \lambda x - \arctan x$$

is injective or not. Since $f(0) = 0$ it is enough to investigate if the equation $f(x) = \lambda x - \arctan x = 0$ has any nontrivial solution. Denote $f_1(x) = \lambda x$ and $f_2(x) = \arctan x$. It is easy to see that the function f_2 is an odd function and has the tangent $y = x$, in point 0. The function f_2 is under the tangent for $x > 0$ and above the tangent for $x < 0$. This leads us to conclusion that the equation $f_1(x) = f_2(x)$ for $\lambda = 1$ has only trivial solution $x = 0$ (f is injective) and for $\lambda \in (0, 1)$ there are also two nontrivial solutions. It means that $f(x) = 0$ has nontrivial solutions for $\lambda \in (0, 1)$, so from

$$\lambda x_i - \arctan x_i = \lambda y_i - \arctan y_i \quad (i \in \mathbb{N}),$$

it does not have to follow that $x_i = y_i$.

We show that for $\lambda \in (0, 1)$, from $(\lambda I - F)x = (\lambda I - F)y$ it does not follow $x = y$ and so the operator (2.5) is not injective in the case $\lambda \in (0, 1)$. We just find out that

$$(0, 1) \subset \sigma_R(F). \tag{2.6}$$

The superposition operator $(\lambda I - F)$ is generated by the function

$$f(s, u) = \lambda u - a(s) - \arctan u. \tag{2.7}$$

We can consider the function (2.7) as the function of one variable u, where $a(s)$ is a real constant for fixed $s \in \mathbb{N}$. The first derivative of the function (2.7) is

$$f'_u(s, u) = \lambda - \frac{1}{1 + u^2} = \frac{\lambda u^2 + \lambda - 1}{1 + u^2}.$$

For $\lambda > 1$ we see that $f'_u(s, u) > 0$, so f is strictly increasing function; also $u \to \pm\infty \Rightarrow f(s, u) \to \pm\infty$ holds. Hence f is bijective function for every $s \in \mathbb{N}$, so the operator $\lambda I - F$ is bijective for $\lambda > 1$.

For $\lambda < 0$ we see that $f'_u(s, u) < 0$, so f is strictly decreasing function; also $u \to \pm\infty \Rightarrow f(s, u) \to \mp\infty$ holds, so the function f is bijective (for every $s \in \mathbb{N}$). It follows that the operator $\lambda I - F$ is bijective for $\lambda < 0$.

For $\lambda = 1$ we have already shown that f is an injective function. The first

derivative $f'_u(s,u) = \frac{u^2}{1+u^2}$ is positive except at $u = 0$ where $f'_u(s,0) = 0$. The function f is strictly increasing and $u = 0$ is a point of inflection, also $u \to \pm\infty \Rightarrow f(s,u) \to \pm\infty$ holds. That is why the function f as well as the operator $I - F$ is bijective.
We have shown that $\lambda I - F$ is bijective operator for $\lambda < 0$ and $\lambda \geq 1$ and now we have to find out if $(\lambda I - F)^{-1}$ is a continuous operator for these values of λ. For $\lambda \geq 1$ the function (2.7) is bijective, increasing and continuous function, so there exists its inverse $f^{-1}(s,u)$ which is also bijective, increasing and continuous function (for every $s \in \mathbb{N}$) ([11]). Now from the Theorem 1.3 follows that operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(s,u)$ is continuous operator and thus

$$[1,\infty) \subset \rho_R(F). \tag{2.8}$$

For $\lambda < 0$ the function (2.7) is bijective, decreasing and continuous function, so there exists its inverse $f^{-1}(s,u)$ which is also bijective, decreasing and continuous function (for every $s \in \mathbb{N}$). Again from the Theorem 1.3 follows that operator $(\lambda I - F)^{-1}$ is continuous operator and hence

$$(-\infty,0) \subset \rho_R(F). \tag{2.9}$$

Finally, from (2.4), (2.6), (2.8) and (2.9) we get the Rhodius resolvent set of F is

$$\rho_R(F) = (-\infty,0) \cup [1,\infty)$$

and the Rhodius spectrum of F is

$$\sigma_R(F) = [0,1).$$

□

Theorem 2.2. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = a(s) - \arctan u$, where $(a(s))_s$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. Then the Rhodius spectrum of F is $\sigma_R(F) = (-1,0]$.*

Proof. Operator F is given by

$$Fx = F(x_1,x_2,\ldots) = (a_1 - \arctan x_1, a_2 - \arctan x_2, \ldots),$$

where $x = (x_1,x_2,\ldots) \in l_p$ and $a = (a_1,a_2,\ldots) \in l_q$.
For $\lambda = 0$ operator $\lambda I - F$ becomes

$$-Fx = (-a_1 + \arctan x_1, -a_2 + \arctan x_2, \ldots). \tag{2.10}$$

From $-Fx = -Fy$ $(x,\, y \in l_p)$, we have

$$(-a_1 + \arctan x_1, -a_2 + \arctan x_2, \ldots) = (-a_1 + \arctan y_1, -a_2 + \arctan y_2, \ldots)$$
$$-a_i + \arctan x_i = -a_i + \arctan y_i, \quad \forall i \in \mathbb{N}$$
$$\arctan x_i = \arctan y_i, \quad \forall i \in \mathbb{N}.$$

As in the proof of the previous Theorem 2.1 we can conclude that this is an injective operator (since the function $f(x) = \arctan x$ is injective). This is not a surjective operator because $-a_i + \arctan x_i \in (-\frac{\pi}{2} - a_i, \frac{\pi}{2} - a_i)$, $(i \in \mathbb{N})$. So the operator (2.10) is not bijection and hence

$$0 \in \sigma_R(F). \tag{2.11}$$

Consider the operator

$$(\lambda I - F)x = (\lambda x_1 - a_1 + \arctan x_1, \lambda x_2 - a_2 + \arctan x_2, \ldots), \quad \lambda \neq 0. \tag{2.12}$$

From $(\lambda I - F)x = (\lambda I - F)y$ (where $x,\ y \in l_p$), we have

$$(\lambda x_1 - a_1 + \arctan x_1, \lambda x_2 - a_2 + \arctan x_2, \ldots) =$$
$$= (\lambda y_1 - a_1 + \arctan y_1, \lambda y_2 - a_2 + \arctan y_2, \ldots)$$
$$\lambda x_i - a_i + \arctan x_i = \lambda y_i - a_i + \arctan y_i, \quad \forall i \in \mathbb{N} \Leftrightarrow$$
$$\lambda x_i + \arctan x_i = \lambda y_i + \arctan y_i, \quad \forall i \in \mathbb{N} \Leftrightarrow$$

Now we wonder whether the function

$$f(x) = \lambda x + \arctan x$$

is injective or not. Denote $f_1(x) = -\lambda x$ and $f_2(x) = \arctan x$. The function f_2 is an odd function and has the tangent $y = x$, in point 0. The function f_2 is under the tangent for $x > 0$ and above the tangent for $x < 0$, so the equation $f_1(x) = f_2(x)$ for $\lambda = -1$ has only trivial solution $x = 0$ and for $\lambda \in (-1, 0)$ there are also two nontrivial solutions. It means that $f(x) = 0$ has nontrivial solutions for $\lambda \in (-1, 0)$, so from

$$\lambda x_i + \arctan x_i = \lambda y_i + \arctan y_i \quad (i \in \mathbb{N}),$$

it does not have to follow that $x_i = y_i$.
We show that for $\lambda \in (-1, 0)$, from $(\lambda I - F)x = (\lambda I - F)y$ it does not follow $x = y$ and so the operator (2.12) is not injective in the case $\lambda \in (-1, 0)$. We just find out that

$$(-1, 0) \subset \sigma_R(F). \tag{2.13}$$

The superposition operator $(\lambda I - F)$ is generated by the function

$$f(s, u) = \lambda u - a(s) + \arctan u. \tag{2.14}$$

For fixed $s \in \mathbb{N}$ we can consider the function (2.14) as the function of one variable u, where $a(s)$ is a real constant. The first derivative of the function (2.14) is $f'_u(s, u) = \lambda + \frac{1}{1+u^2} = \frac{\lambda u^2+\lambda+1}{1+u^2}$.
For $\lambda < -1$ we have $f'_u(s, u) < 0$, so f is strictly decreasing function and $u \to \pm\infty \Rightarrow f(s, u) \to \mp\infty$ holds. Hence, f is a bijective function so as operator $\lambda I - F$ for $\lambda < -1$.
For $\lambda > 0$ we have $f'_u(s, u) > 0$, so f is strictly increasing function and $u \to \pm\infty \Rightarrow f(s, u) \to \pm\infty$ holds, so the function f is bijective. The operator $\lambda I - F$ is bijective for $\lambda > 0$.
For $\lambda = -1$ we have the function $f(s, u) = -u - a(s) + \arctan u$ and $f'_u(s, u) = -\frac{u^2}{1+u^2}$ is negative, except at $u = 0$ where $f'_u(s, 0) = 0$. The function f is strictly decreasing and $u = 0$ is a point of inflection; also $u \to \pm\infty \Rightarrow f(s, u) \to \mp\infty$ holds. Hence the function f as well as the operator $-I - F$ is bijective. We have shown that $\lambda I - F$ is bijective operator for $\lambda \leq -1$ and $\lambda > 0$ and now we have to find out if $(\lambda I - F)^{-1}$ is a continuous operator for these values of λ. For $\lambda \leq -1$ the function (2.14) is bijective, decreasing and continuous function, so there exists its inverse $f^{-1}(s, u)$ which is also bijective, decreasing and continuous

function (for every $s \in \mathbb{N}$). The operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(s,u)$, is a continuous operator and thus

$$(-\infty, -1] \subset \rho_R(F). \tag{2.15}$$

For $\lambda > 0$ the function (2.14) is bijective, increasing and continuous function, so there exists its inverse $f^{-1}(s,u)$ which is also bijective, increasing and continuous function (for every $s \in \mathbb{N}$). Again the operator $(\lambda I - F)^{-1}$ is continuous operator and hence

$$(0, +\infty) \subset \rho_R(F). \tag{2.16}$$

From (2.11), (2.13), (2.15) and (2.16) we get the Rhodius resolvent set of F is

$$\rho_R(F) = (-\infty, -1] \cup (0, \infty)$$

and the Rhodius spectrum of F is

$$\sigma_R(F) = (-1, 0].$$

□

In the sequel we need to investigate Fréchet differentiability of the superposition operator $F : l_p \to l_p$ generated by the function

$$f(s,u) = a(s) + \arctan u \quad \text{or} \quad f(s,u) = a(s) - \arctan u. \tag{2.17}$$

The function $f'_u(s,u) = \frac{1}{1+u^2}$ is certainly continuous for all $s \in \mathbb{N}$ and at all $x_0 \in l_p$, so according to the Theorem 1.4, this operator is differentiable at each point $x_0 \in l_p$. It is also known the following Theorem (see [4], [12]):

Theorem 2.3. *Let $f(s,u)$ be a Carathéodory function and operator F generated by the function $f(s,u)$ acts from l_p to l_q. If operator F is differentiable in $x_0 \in l_p$, then its (Fréchet) derivative in x_0 has the form*

$$F'(x_0)\, h(s) = a(s)\, h(s) \tag{2.18}$$

where $a \in l_q/l_p$ is given by

$$a(s) = \lim_{u \to 0} \frac{f(s, x_0(s) + u) - f(s, x_0(s))}{u} \tag{2.19}$$

If superposition operator G, generated by the function

$$g(s,u) = \begin{cases} \frac{1}{u}\left[f(s, x(s) + u) - f(s, x(s))\right]; & u \neq 0 \\ a(s); & u = 0, \end{cases}$$

acts from l_p to l_q/l_p, and it is continuous in 0, then F is differentiable in x_0 and it values (2.18).

The space l_q/l_p is the set of all multipliers $(a(s))$ from l_p to l_q. It is a Banach space of sequences, defined by

$$l_q/l_p = \begin{cases} l_{pq(p-q)^{-1}} & \text{for } p > q \\ l_\infty & \text{for } p \leq q. \end{cases}$$

From (2.18) and (2.19) we see that the Fréchet derivative of our operator $F : l_p \to l_p$ (2.17) at $x_0 = (x_1, x_2, ...)$ along $h = (h_1, h_2, ...)$ is a linear multiplication operator given with

$$F'(x_0)h = \left(\frac{1}{1+x_1^2}\, h_1, \frac{1}{1+x_2^2}\, h_2, \frac{1}{1+x_3^2}\, h_3, ... \right).$$

Here the multiplier is $a(s) = \left(\frac{1}{1+x_1^2}, \frac{1}{1+x_2^2}, \frac{1}{1+x_3^2}, ... \right)$ and this is a bounded sequence for every $x_0 \in l_p$, i.e. $a \in l_\infty$. The map $x \mapsto F'(x)$ is continuous, so the operator F is a continuously differentiable operator, $F \in \mathfrak{C}^1(l_p)$.

Theorem 2.4. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s, u) = a(s) + \arctan u$, where $(a(s))_s$ is a sequence from the space l_q $(1 \le q \le p \le \infty)$. Then the Neuberger spectrum of F is $\sigma_N(F) = [0, 1]$.*

Proof. Since $\sigma_R(F) \subseteq \sigma_N(F)$, from the Theorem 2.1 we get $[0, 1) \subseteq \sigma_N(F)$. For $\lambda = 1$ we consider the operator $I - F$ which is given with

$$(I - F)x = (x_1 - a_1 - \arctan x_1, x_2 - a_2 - \arctan x_2, \ldots).$$

It is generated by the function

$$f(s, u) = u - a(s) - \arctan u. \tag{2.20}$$

Its first derivative (with respect to u) is $f'_u(s, u) = 1 - \frac{1}{1+u^2} = \frac{u^2}{1+u^2}$ and this is a continuous function of one variable u, positive for $u \neq 0$ ((2.20) is an increasing function for fixed s), but $f'_u(u) = 0$. That is why for $u \neq 0$ we have

$$(f^{-1})'(u) = \frac{1}{f'(u)} = \frac{1+u^2}{u^2} = 1 + \frac{1}{u^2}$$

and the function $(f^{-1})'(u)$ is discontinuous one (the point $u = 0$ is the point of discontinuity) ([11]). Thus we conclude (from Theorem 1.4 and Theorem 2.3) that the operator $(I - F)^{-1}$ is not continuously differentiable operator, so

$$1 \in \sigma_N(F).$$

For $\lambda \neq 0$ the operator $\lambda I - F$ is generated by the function $f(s, u) = \lambda u - a(s) - \arctan u$ and its first derivative (with respect to u) is

$$f'_u(s, u) = \lambda - \frac{1}{1+u^2} = \frac{\lambda u^2 + \lambda - 1}{1+u^2}. \tag{2.21}$$

We have already shown in the proof of the Theorem 2.1 that $\lambda I - F$ is a bijective operator for $\lambda > 1$ and $\lambda < 0$. For $\lambda > 1$ the function (2.21) is a continuous, positive function (of one variable u), so there exists $(f^{-1})'(u)$

$$(f^{-1})'(u) = \frac{1}{f'(u)} = \frac{1+u^2}{\lambda u^2 + \lambda - 1}. \tag{2.22}$$

For $\lambda < 0$ the function (2.21) is a continuous, negative function (of one variable u), so again there exists $(f^{-1})'(u)$ which is given with (2.22). The function (2.22) is continuous for every $s \in \mathbb{N}$, $\lambda \neq 0$ and hence the operator $(\lambda I - F)^{-1}$, for

$\lambda > 1$ and $\lambda < 0$, is a continuously differentiable operator $((\lambda I - F)^{-1} \in \mathfrak{C}^1(l_p))$. Finally, we have found that the Neuberger resolvent set is

$$(-\infty, 0) \cup (1, +\infty) = \rho_N(F)$$

and the Neuberger spectrum is

$$\sigma_N(F) = [0, 1].$$

□

Theorem 2.5. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s,u) = a(s) - \arctan u$, where $(a(s))_s$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. Then the Neuberger spectrum of F is $\sigma_N(F) = [-1, 0]$.*

Proof. The Rhodius spectrum of this operator is $\sigma_R(F) = (-1, 0]$, so

$$(-1, 0] \subseteq \sigma_N(F).$$

For $\lambda = -1$ we consider the operator $-I - F$:

$$(-I - F)x = (-x_1 - a_1 + \arctan x_1, -x_2 - a_2 + \arctan x_2, \ldots).$$

It is generated by the function

$$f(s, u) = -u - a(s) + \arctan u. \tag{2.23}$$

Its first derivative with respect to u is $f'_u(s,u) = -1 + \frac{1}{1+u^2} = -\frac{u^2}{1+u^2}$ and this is a continuous function, negative for $u \neq 0$ ((2.23) is a decreasing function for fixed s), but $f'_u(u) = 0$, so for $u \neq 0$ we have

$$(f^{-1})'(u) = \frac{1}{f'(u)} = -\frac{1+u^2}{u^2} = -1 - \frac{1}{u^2}.$$

The function $(f^{-1})'(u)$ is discontinuous one, the point $u = 0$ is the point of discontinuity. From Theorem 1.4 and Theorem 2.3 it follows that the operator $(-I - F)^{-1}$ is not continuously differentiable operator so

$$-1 \in \sigma_N(F).$$

From the Theorem 2.2 we see that $\lambda I - F$ is a bijective operator for $\lambda < -1$ and $\lambda > 0$. The operator $\lambda I - F$ is generated by the function $f(s,u) = \lambda u - a(s) + \arctan u$. Its first derivative with respect to u is

$$f'_u(s,u) = \lambda + \frac{1}{1+u^2} = \frac{\lambda u^2 + \lambda + 1}{1+u^2}. \tag{2.24}$$

For $\lambda < -1$ the function (2.24) is a continuous, negative function (of one variable u), so there exists $(f^{-1})'(u)$

$$(f^{-1})'(u) = \frac{1}{f'(u)} = \frac{1+u^2}{\lambda u^2 + \lambda + 1}. \tag{2.25}$$

For $\lambda > 0$ the function (2.24) is continuous, positive function (of one variable u), so again there exists $(f^{-1})'(u)$ which is given with (2.25). The function (2.25) is continuous (for all $s \in \mathbb{N}$) and hence, the operator $(\lambda I - F)^{-1}$, for $\lambda < -1$ and

$\lambda > 0$, is a continuously differentiable operator (i.e. $(\lambda I - F)^{-1} \in \mathfrak{C}^1(l_p)$).
We have found that the Neuberger resolvent set is

$$(-\infty, -1) \cup (0, +\infty) = \rho_N(F)$$

and the Neuberger spectrum is

$$\sigma_N(F) = [-1, 0].$$

□

Corollary 2.6. *Let the superposition operator $F : l_p \to l_p$, be generated by the function $f(s, u) = a(s) + \arctan u$ or $f(s, u) = a(s) - \arctan u$, where $(a(s))_s$ is a sequence from the space l_q $(1 \leq q \leq p \leq \infty)$. Then the Rhodius and Neuberger spectral radius of operator F is $r_R(F) = r_N(F) = 1$.*

Proof. If the superposition operator $F : l_p \to l_p$ is generated by the function $f(s, u) = a(s) + \arctan u$, then from the Theorem 2.1 and Theorem 2.4 we have

$$r_R(F) = \{sup|\lambda| : \lambda \in \sigma_R(F)\} = \{sup|\lambda| : \lambda \in [0, 1)\} = 1,$$
$$r_N(F) = \{sup|\lambda| : \lambda \in \sigma_N(F)\} = \{sup|\lambda| : \lambda \in [0, 1]\} = 1.$$

If the superposition operator $F : l_p \to l_p$ is generated by the function
$f(s, u) = a(s) - \arctan u$, then from the Theorem 2.2 and Theorem 2.5 we have

$$r_R(F) = \{sup|\lambda| : \lambda \in \sigma_R(F)\} = \{sup|\lambda| : \lambda \in (-1, 0]\} = 1,$$
$$r_N(F) = \{sup|\lambda| : \lambda \in \sigma_N(F)\} = \{sup|\lambda| : \lambda \in [-1, 0])\} = 1.$$

□

The superposition operator F generated by the function

$$f(s, u) = t(s) \pm \operatorname{arccot} u,$$

where $t = (t(s))_{s\in\mathbb{N}}$ is a sequence from the space l_q $(1 \leq q \leq \infty)$, appears to be a special case of above mentioned operators.
Let us show that this operator acts from the space l_∞ to the space l_∞.
Let $x = (x_1, x_2, \cdots)$ be an arbitrary element of l_∞;
$t \in l_q \subseteq l_\infty \Rightarrow \sup\limits_{s\in\mathbb{N}} |t(s)| = B < \infty$. Now we have

$$\sup_{s\in\mathbb{N}}|Fx(s)| = \sup_{s\in\mathbb{N}}|t(s) \pm \operatorname{arccot} x_s| \leq \sup_{s\in\mathbb{N}}|t(s)| + \sup_{s\in\mathbb{N}}|\operatorname{arccot} x_s|$$
$$\leq B + \pi < \infty.$$

We see that for every $x \in l_\infty$ it holds $Fx \in l_\infty$, so indeed F acts from l_∞ to l_∞ (space of bounded sequences). Knowing that $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x\, (x \in \mathbb{R})$, we come up to the following conclusions.
$f(s, u) = t(s) + \operatorname{arccot} u = t(s) + \frac{\pi}{2} - \arctan x = a(s) - \arctan x$, where
$a(s) = t(s) + \frac{\pi}{2}$. It is clear that $a \in l_\infty$ and hence, for the operator $F_1 : l_\infty \to l_\infty$, generated by the function $f(s, u) = t(s) + \operatorname{arccot} u$, we can apply Theorem 2.2 and Theorem 2.5 and get

$$\sigma_R(F_1) = (-1, 0]; \quad \sigma_N(F_1) = [-1, 0].$$

$f(s, u) = t(s) - \operatorname{arccot} u = t(s) - \frac{\pi}{2} + \arctan x = a(s) + \arctan x$, where
$a(s) = t(s) - \frac{\pi}{2}$. Again $a \in l_\infty$ and so, for the operator $F_2 : l_\infty \to l_\infty$, generated

by the function $f(s,u) = t(s) - \operatorname{arccot} u$, we can apply Theorem 2.1 and Theorem 2.4 and get

$$\sigma_R(F_2) = [0,1); \quad \sigma_N(F_2) = [0,1].$$

Also, according to the Corollary 2.6 we have

$$r_R(F_1) = r_R(F_2) = r_N(F_1) = r_N(F_2) = 1.$$

For all the operators that we considered in this paper we can conclude:

- their Rhodius spectra are nonempty and bounded sets,
- their Neuberger spectra are nonempty, bounded and closed sets and
- their Rhodius and Neuberger spectral radius is always 1.

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