Gulf Journal of Mathematics ISSN: 2309-4966
Vol 14, Issue 1 (2023) 139-147 https://doi.org/10.56947/gjom.v14i1.946

INEQUALITIES FOR A CLASS OF COMPOSITE POLYNOMIALS

MOHD YOUSF MIR[1*] AND WALI MOHAMMAD SHAH[2]

Abstract. In this paper, we establish some inequalities for a more general class of polynomials $P(S(z))$, where $P(z) := \sum_{j=0}^{n} a_j z^j$ and $S(z) := \sum_{j=0}^{m} b_j z^j$ are polynomials of degree n and m respectively. These inequalities not only generalize the results due to Dubinin[J. Math. Sci., **143** (2007), 3069-3076] but also refine a result due to Shah and Liman [NFAA, 9(2004), 223-232].

1. Introduction and preliminaries

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree atmost n over the field $\mathbb{C}$ of complex numbers. If $P \in \mathcal{P}_n$ and P' be its derivative, then concerning the estimate of $\max_{|z|=1} |P'(z)|$ in terms of $\max_{|z|=1} |P(z)|$, we have the following well known result due to Bernstein [2].

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|, \text{ for all } z \in \mathbb{C}. \tag{1.1}$$

Equality holds in (1.1) if and only if all the zeros of P are at the origin. It is natural to ask what happens to inequality (1.1), if we impose restrictions on the location of zeros of P. In this connection the following inequalities are the earliest belonging to this domain of ideas which have a clear impact on the subsequent work carried forward since then.

If $P \in \mathcal{P}_n$ has all zeros in $|z| \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

and in case if $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Date: Received: Jul 30, 2022; Accepted: Oct 17, 2022.
* Corresponding author.
2010 *Mathematics Subject Classification.* 30A10, 30C10, 30C15.
Key words and phrases. Composite Polynomial, Inequalities, Zeros, Disk.

Inequality (1.2) was conjectured by Erdös and latter proved by Lax [7], whereas inequality (1.3) is due to Turán [12]. Inequality (1.3) was further refined by Dubinin [5], who with the same hypothesis for the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ obtained.

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right) \max_{|z|=1} |P(z)|. \tag{1.4}$$

If $P \in \mathcal{P}_n$ and $S \in \mathcal{P}_m$, then we define the composition of polynomials P and S by $(P \circ S)(z) = P(S(z))$, for all $z \in \mathbb{C}$. Clearly $P \circ S \in \mathcal{P}_{mn}$. For such class of polynomials Shah and Liman [11] proved several results. In particular they proved:

If $P \circ S \in \mathcal{P}_{mn}$ is such that $(P \circ S)(z) \neq 0$ for $|z| > 1$, then for any z with $|z| \leq 1$, we have

$$|P'(S(z))| \geq \frac{n}{2M'} |z|^{mn-m} \max_{|z|=1} |P(S(z))|,$$

where $M' = \max_{|z|=1} |S(z)|$.

2. Lemmas

Lemma 2.1. *If $P \in \mathcal{P}_n$ be such that $P(z) \neq 0$ in $|z| > 1$, then*

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)|.$$

The above lemma is due to Aziz and Dawood [1].

Lemma 2.2. *If $P \circ S \in \mathcal{P}_{mn}$, then for $|z| \geq 1$,*

$$(P \circ S)'(z) + (Q \circ S)'(z) \leq nm \max_{|z|=1} \big|(P \circ S)(z)\big|,$$

where $(Q \circ S)(z) = z^{mn} \overline{P\big(S\big(\frac{1}{\bar{z}}\big)\big)}$.

The above lemma follows from a result due to Govil and Rehman [6].

Lemma 2.3. *Let $f : E \to E$ be a holomorphic function,where $E = \{z : |z|<1\}$, with $|f(z)|<1$ for all $z \in E$. Assume that there is $b \in \partial E$, (a boundary of E), so that f extends continuously to b with $|f(b)| = 1$ and $f'(b)$ exists.Then*

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}. \tag{2.1}$$

The above lemma is due to Osserman [9] (see also [4], [10]).

3. Main results

In this paper, we first prove the following generalization of a result due to Dubinin [5].

Theorem 3.1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *and* $S(z) := \sum_{j=0}^{m} b_j z^j$ *be two polynomials of degree* n *and* m *respectively, such that* $(P \circ S)(z) \neq 0$ *for* $|z| > 1$, *then for any* z *with* $|z| \leq 1$, *we have*

$$Re\left(\frac{z(P \circ S)'(z)}{(P \circ S)(z)}\right) \geq \frac{mn-1}{2} + \frac{|a_n b_m^n|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}. \tag{3.1}$$

The result is sharp for $S(z) = z$ *and equality holds for* $(P \circ S)(z) = z^n + \lambda$ *with* $|\lambda| = 1$.

Proof. For $P(z) = \sum_{j=0}^{n} a_j z^j$ and $S(z) = \sum_{j=0}^{m} b_j z^j$, we have

$$\begin{aligned}(P \circ S)(z) &= P(S(z))\\ &= a_n\big(b_m z^m + b_{m-1} z^{m-1} + ... + b_0\big)^n +\\ &\quad a_{n-1}\big(b_m z^m + b_{m-1} z^{m-1} + ... + b_0\big)^{n-1} + ... + a_0\\ &= a_n\left[\binom{n}{0} b_m^n z^{mn} + \left(\binom{n}{1} b_m^{n-1} b_{m-1}\right) z^{mn-1} + ... + b_0^n\right] + ... + a_1 b_0 + a_0\\ &= \sum_{j=0}^{mn} c_j z^j, \text{ where } c_0 = \sum_{j=0}^{n} a_j b_0^j, \ ..., \ c_{mn} = a_n b_m^n.\end{aligned}$$

Let $z_1, z_2, ..., z_{mn}$ be the zeros of $P \circ S \in \mathcal{P}_{mn}$, so that $(P \circ S)(z) = c_{mn} \prod_{j=1}^{mn}(z - z_j)$, $|z_j| \leq 1$.
If

$$Q(S(z)) = z^{mn}\overline{P\big(S(\frac{1}{\overline{z}})\big)},$$

then $Q(S(z))$ has all zeros in $|z| \geq 1$. Hence

$$F(z) = \frac{P(S(z))}{z^{mn-1}\overline{P\big(S(\frac{1}{\overline{z}})\big)}} = \frac{c_{mn}}{\overline{c_{mn}}} z \prod_{j=0}^{mn} \frac{z - z_j}{1 - \overline{z_j} z}, \tag{3.2}$$

is an analytic function in $|z| \leq 1$ with $F(0) = 0, |F(z)| = 1$ for $|z| = 1$. Thus by lemma 2.3, we have

$$|F'(z)| \geq \frac{2}{1 + |F'(0)|}.$$

It can be easily verified that

$$|F'(0)| = |\prod_{j=0}^{mn} z_j| = \frac{|\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n|}.$$

This gives,

$$|F'(z)| \geq \frac{2|a_n b_m^n|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}. \tag{3.3}$$

Now by using the fact that for $|z| = 1$, $|F'(z)| = \dfrac{zF'(z)}{F(z)}$, (see [8]), we have from (3.2), after differentiating both sides with respect to z

$$\frac{zF'(z)}{F(z)} = \left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right) + \overline{\left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right)} - (mn-1).$$

This implies

$$\frac{zF'(z)}{F(z)} = 2Re\left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right) - (mn-1).$$

That is

$$Re\left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right) = \frac{mn-1}{2} + \frac{|F'(z)|}{2}.$$

Equivalently for $|z| = 1$, we get by using (3.3)

$$Re\left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right) \geq \frac{mn-1}{2} + \frac{|a_n b_m^n|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}$$

This completely proves Theorem 3.1. □

Remark 3.2. For $S(z) = z$, Theorem 3.1 reduces to a result due to Dubinin [5, Theorem 6].

We next prove the following result, which besides generalizing another inequality proved by Dubinin [5] improves a result due to Shah and Liman [11].

Theorem 3.3. *Let $P(z)$ and $S(z)$ be as defined in Theorem 3.1, such that $(P\circ S)(z) \neq 0$ for $|z| > 1$, then for any z with $|z| \leq 1$, we have*

$$\max_{|z|=1} |P'(S(z))| \geq \frac{1}{2mM'}\left\{mn + \frac{|a_n b_m^n| - |\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}\right\} \max_{|z|=1} |P(S(z))|, \tag{3.4}$$

where $M' = \max_{|z|=1} |S(z)|$.

The result is sharp for $S(z) = z$ and equality holds for $(P\circ S)(z) = z^n + \lambda$ with $|\lambda| = 1$.

Proof. Since $P\circ S \in \mathcal{P}_{mn}$ is such that $(P\circ S)(z) \neq 0$ for $|z| > 1$, therefore by Theorem 3.1, we have

$$Re\left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right) \geq \left(\frac{mn}{2} + \frac{1}{2}\frac{|a_n b_m^n| - |\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}\right). \tag{3.5}$$

Also for $|z| = 1$,

$$\left|\frac{(P\circ S)'(z)}{(P\circ S)(z)}\right| \geq Re\left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right). \tag{3.6}$$

Combining (3.5) and (3.6), we get for $|z|=1$

$$|(P(S(z))'| \geq \frac{1}{2}\left\{mn + \frac{|a_n b_m^n| - |\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}\right\}|P(S(z))|. \tag{3.7}$$

Since $S(z)$ is a polynomial of degree m and $|S(z)| \leq M'$ for $|z| \leq 1$, therefore by (1.1), we have $\max\limits_{|z|=1}|S'(z)| \leq mM'$, for all $z \in \mathbb{C}$. Hence from (3.7), we get for $|z|=1$

$$\max_{|z|=1}|P'(S(z))| \geq \frac{1}{2mM'}\left\{mn + \frac{|a_n b_m^n| - |\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}\right\}\max_{|z|=1}|P(S(z))|.$$

This completely proves Theorem 3.3. $\square$

Remark 3.4. If we put $S(z)=z$, then Theorem 3.3 reduces to a result of Dubinin [5].

Remark 3.5. Since all the zeros of $P(S(z))$ lie in $|z| \leq 1$, we have $|a_n b_m^n| \geq |\sum_{j=0}^{n} a_j b_0^j|$, showing that inequality (3.4) improves a result due to Shah and Liman [11].

We now prove some results for the class of polynomials having all zeros outside the unit disk. In this direction we first prove:

Theorem 3.6. *Let $P(z)$ and $S(z)$ be as defined in Theorem 3.1, such that $(P \circ S)(z) \neq 0$ for $|z| < 1$, then for any z with $|z| \geq 1$, we have*

$$Re\left(\frac{z(P\circ S)'(z)}{(P\circ S)(z)}\right) \leq \frac{mn+1}{2} - \frac{|a_n b_m^n|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}. \tag{3.8}$$

The result is sharp for $S(z)=z$ and equality holds for $(P\circ S)(z) = z^n + \lambda$ with $|\lambda| = 1$.

Proof. Let $z_1, z_2, ..., z_{mn}$ be the zeros of $(P\circ S)(z)$, then as in the proof of Theorem 3.1, we have $P(S(z)) = c_{mn}\prod_{j=1}^{mn}(z-z_j), |z_j| \geq 1$. Therefore, $Q(S(z)) = z^{mn}\overline{P\big(S(\frac{1}{\bar{z}})\big)}$ has all zeros in $|z| \leq 1$. Hence

$$T(z) = \frac{zQ(S(z))}{P(S(z))} = \frac{z^{mn+1}\overline{P\big(S(\frac{1}{\bar{z}})\big)}}{P(S(z))} = \frac{\overline{c_{mn}}}{c_{mn}}z\prod_{j=1}^{mn}\frac{1-\overline{z_j}z}{z-z_j}, \tag{3.9}$$

is an analytic function in $|z| \leq 1$, with $T(0)=0$ and $|T(z)| = 1$ for $|z|=1$. By using Lemma 2.3, we have

$$|T'(z)| \geq \frac{2}{1+|T'(0)|}, \tag{3.10}$$

where

$$|T'(0)| = \frac{1}{|\prod_{j=1}^{mn} z_j|} = \frac{|a_n b_m^n|}{|\sum_{j=0}^{n} a_j b_0^j|}.$$

This gives

$$|T'(z)| \geq \frac{2|\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}. \tag{3.11}$$

Also from (3.9), we have

$$\frac{zT'(z)}{T(z)} = mn + 1 - 2Re\left(\frac{z(P \circ S(z))'}{P \circ S(z)}\right). \tag{3.12}$$

Using the fact $|T'(z)| = \dfrac{zT'(z)}{T(z)}$ for $|z| = 1$, we get

$$Re\left(\frac{z(P \circ S)'(z)}{(P \circ S)(z)}\right) = \frac{mn+1}{2} - \frac{|T'(z)|}{2}.$$

Equivalently, for $|z| = 1$, we get by using (3.11)

$$Re\left(\frac{z(P \circ S)'(z)}{(P \circ S)(z)}\right) \leq \frac{mn+1}{2} - \frac{|\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n| + |\sum_{j=0}^{n} a_j b_0^j|}.$$

This completely proves Theorem 3.6. □

We also prove the following.

Theorem 3.7. *Let $P(z)$ and $S(z)$ be as defined in Theorem 3.1, such that $(P \circ S)(z) \neq 0$ for $|z| < 1$, then for any z, with $|z| \geq 1$, we have*

$$\max_{|z|=1} \big|P'(S(z))\big| \leq \frac{1}{2m'}\left\{\left[mn - \frac{|\sum\limits_{j=0}^{n} a_j b_0^j| - |a_n b_m^n|}{|\sum\limits_{j=0}^{n} a_j b_0^j| + |a_n b_m^n|} \frac{|P(S(z))|^2}{\left(\max\limits_{|z|=1}\big|\big(P(S(z))\big)\big|\right)^2}\right] \max_{|z|=1}\big|\big(P(S(z))\big)\big|\right\}.$$

Where $m' = \min\limits_{|z|=1}\big|S(z)\big|$.
The result is sharp for $S(z) = z$ and equality holds for $(P \circ S)(z) = z^n + \lambda$ with $|\lambda| = 1$.

Proof. We first assume that $P(S(z)) \neq 0$ for $|z| = 1$ and if

$$Q(S(z)) = z^{mn}\overline{P(S(\frac{1}{\bar{z}}))},$$

then it can be easily verified that for $|z| = 1$

$$\big|\big(Q(S(z))\big)'\big| = \left|mnP(S(z)) - z\big(P(S(z))\big)'\right|.$$

This gives by using inequality (3.8) for $|z|=1$

$$\left|\frac{z\big(Q(S(z))\big)'}{P(S(z))}\right|^2=\left|mn-\frac{z\big(P(S(z))\big)'}{P(S(z))}\right|^2$$

$$=(mn)^2+\left|\frac{z\big(P(S(z))\big)'}{P(S(z))}\right|^2-2mnRe\left\{\frac{z\big(P(S(z))\big)'}{P(S(z))}\right\}$$

$$\geq (mn)^2+\left|\frac{z\big(P(S(z))\big)'}{P(S(z))}\right|^2-mn\left\{mn-\frac{|\sum\limits_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum\limits_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\right\}$$

$$=\left|\frac{z\big(P(S(z))\big)'}{P(S(z))}\right|^2+\frac{|\sum\limits_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum\limits_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}mn.$$

This can further be written as

$$\big|\big(P(S(z))\big)'\big|^2+\frac{|\sum\limits_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum\limits_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}mn\big|\big(P(S(z))\big)\big|^2\leq\big|\big(Q(S(z))\big)'\big|^2.$$

This inequality is trivially true in case $P(S(z))=0$ on $|z|=1$. By using Lemma 2.2 in the above inequality, we get

$$\big|\big(P(S(z))\big)'\big|+\left(\big|\big(P(S(z))\big)'\big|^2+\frac{|\sum\limits_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum\limits_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}mn\big|\big(P(S(z))\big)\big|^2\right)^{\frac{1}{2}}\leq mn\left(\max_{|z|=1}\big|\big(P(S(z))\big)\big|\right).$$

This implies

$$\big|\big(P(S(z))\big)'\big|^2+\frac{|\sum\limits_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum\limits_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}mn\big|\big(P(S(z))\big)\big|^2$$

$$\leq (mn)^2\left(\max_{|z|=1}\big|\big(P(S(z))\big)\big|\right)^2+\big|\big(P(S(z))\big)'\big|^2-2mn\big|\big(P(S(z))\big)'\big|\left(\max_{|z|=1}\big|\big(P(S(z))\big)\big|\right).$$

Now using Lemma 2.1, we get on simplification for $|z|=1$

$$\max_{|z|=1}\big|P'(S(z))\big|\leq\frac{1}{2m'}\left\{\left[mn-\frac{|\sum\limits_{j=0}^{n}a_jb_0^j|-|a_nb_m^n|}{|\sum\limits_{j=0}^{n}a_jb_0^j|+|a_nb_m^n|}\frac{|P(S(z))|^2}{\left(\max\limits_{|z|=1}\big|\big(P(S(z))\big)\big|\right)^2}\right]\max_{|z|=1}\big|\big(P(S(z))\big)\big|\right\}.$$

This completely proves Theorem 3.7. □

If we put $S(z) = z$, in Theorem 3.7, then we get the following result.

Corollary 3.8. *If $P \in \mathcal{P}_n$ be such that $P(z) \neq 0$ in $|z| < 1$, then for any z with $|z| = 1$,*

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{1}{2}\left\{\left[n - \frac{|a_0| - |a_n|}{|a_0| + |a_n|} \frac{|P(z)|^2}{\left(\max_{|z|=1}\left|P(z)\right|\right)^2}\right] \max_{|z|=1}\left|P(z)\right|\right\}.$$

Acknowledgment: The authors are highly grateful to the referee for his/her useful suggestions.
Conflict of Interest: The authors declare that they have no conflict of interest.
Funding The first author is highly thankful to the funding agency DST-INSPIRE for their financial support.
Availability of data: It is not applicable in our paper.

References

1. A. Aziz and Q. M. Dawood,*Inequalities for a polynomial and its derivative,* journal of appx. theory **54** (1988),306-313. https://doi.org/10.1016/0021-9045(88)90006-8
2. S. N. Bernstein, *Sur la limitation des dérivées des polynomes,* C. R. Acad. Sci. Paris., **190** (1930), 338-340.
3. P. Borwein, T. Erdélyi, *Polynomials and polynomial inequalities,* Texts in Math, **161,** Springer-verlag, New York, 1995. Zbl 0840.26002.
4. B. Catal and B. N. Órnak. *Some results for classes of holomorphic functions,* Gulf Journal of Mathematics. **6**(2018), 46-60. https://doi.org/10.1108/SL-05-2018-140
5. V. N. Dubinin, *Applications of the schwarz lemma to inequalities for entire functions with constraints on zeros,* Journal of Mathematical Sciences, **143** (2007), 3069-3076. https://doi.org/10.1007/s10958-007-0192-4
6. N. K. Govil and Q. I. Rahman, *Functions of exponential type not vanishing in a half plane and related polynomial,* Trans. Am. Math. Soe. 137 (1969) 501-517. https://doi.org/10.1090/S0002-9947-1969-0236385-6
7. P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial,* Bull. Amer. Math. Soc., **50** (1944), 509-513. https://doi.org/10.1090/S0002-9904-1944-08177-9
8. X. Li, R. N. Mohapatra and R. S. Rodriguez, *Bernstein-type inequalities for rational functions with prescribed poles,* J. London Math. Soc., **51** (1995), 523-531. https://doi.org/10.1112/jlms/51.3.523
9. R. Osserman. *A sharp schwarz inequality on the boundary for functions regular in disk,* Proceedings of American Mathematical society. **12**, 3513-3517.
10. B. N. Órnak. *Inequalities for angular derivatives at the boundary of the unit disk,* Gulf Journal of Mathematics. **4**(2016), 90-99. https://doi.org/10.4169/math.mag.90.2.99
11. W. M. Shah and A. Liman, *On some Bernstein type inequalities for polynomials,* Non linear functional analysis and applications, 9(2004), 223-232.
12. P. Turán, *Über die ableitung von polynomen,* Compos. Math., **7** (1939), 89-95.

[1] Department of Mathematics, Central University of Kashmir, Ganderbal-191201, India.
Email address: myousf@cukashmir.ac.in

[2] Department of Mathematics, Central University of Kashmir, Ganderbal-191201, India
Email address: wali@cukashmir.ac.in