

ON SOME TRANSFORMATION FORMULAS AND THEIR APPLICATIONS TO PARTIAL THETA FUNCTION IDENTITIES

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ABSTRACT. In this paper we derive three new transformation formulae for bilateral basic hypergeometric series. As applications of these transformation formulae, we deduce the well-known q -Guass summation formula and Sear's two ${}_3\phi_2$ transformation formulae. Further, we establish the reciprocity theorem of Ramanujan and its three variable generalization.

1. INTRODUCTION AND PRELIMINARIES

The basic hypergeometric series ${}_r\phi_r$ is defined by

$${}_r\phi_r \left[\begin{matrix} a_1, & a_2, & \dots & a_{r+1} \\ b_1, & b_2, & \dots & b_r \end{matrix} ; q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1})_n}{(q, b_1, b_2, \dots, b_r)_n} z^n,$$

where $|q| < 1$, $|z| < 1$,

$$(a)_{\infty} := (a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n),$$

$$(a)_n := (a; q)_n := \frac{(a)_{\infty}}{(aq^n)_{\infty}}$$

and

$$(a_1, a_2, a_3, \dots, a_m)_n = (a_1)_n (a_2)_n (a_3)_n \dots (a_m)_n, \quad n \text{ is an integer or } \infty.$$

An ${}_r\phi_r$ series is called well-poised if $a_1q = a_2b_1 = \dots = a_{r+1}b_r$ and very well-poised if it is well-poised and $a_2 = -a_3 = qa_1^{1/2}$. The bilateral basic hypergeometric series ${}_r\psi_r$ is defined by

$${}_r\psi_r \left[\begin{matrix} a_1, & a_2, & \dots & a_r \\ b_1, & b_2, & \dots & b_r \end{matrix} ; q; z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r)_n}{(b_1, b_2, \dots, b_r)_n} z^n,$$

where $\left| \frac{a_1 a_2 \dots a_r}{b_1 b_2 \dots b_r} \right| < |z| < 1$.

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In the recent past, many authors derived transformation and summation formulae for bilateral basic hypergeometric series from a unilateral series. Jackson [14], Bailey [5] [6], Jackson [15], Slater [21], Schlosser [20], Chen and Fu [10], Jouhet [16], Chen, Chen and Gu [11], Somashekara and Narasimha Murthy [25], Bayed, Somashekara Narasimha Murthy [7] are among those who contributed to this part of the literature.

In this paper, we use the following modified version of Cauchy's method for obtaining bilateral series identities. For the sequence $\Omega_k(n)$ with $0 \leq k \leq +\infty$, suppose that the following conditions hold:

- (a) $\Omega_k(n) \rightarrow 0$ as $n \rightarrow \infty$,
- (b) $\lim_{n \rightarrow \infty} \Omega_{2n}(n)$ exists but does not vanish and
- (c) $\lim_{n \rightarrow \infty} \Omega_{k+2n}(n)$ exists for each $k \in \mathbb{N}$.

Then we can reformulate the corresponding nonterminating series as follows:

$$\sum_{k=0}^{\infty} \Omega_k(n) = \sum_{k=0}^{n-1} \Omega_k(n) + \sum_{k=-n}^{\infty} \Omega_{k+2n}(n)$$

We can let $n \rightarrow \infty$ in the last equation to get the bilateral series identity subjected to suitable convergence of each series. Using this technique, Zhang and Zhang [28] have derived two new transformation formulae for bilateral ${}_3\psi_3$ and ${}_4\psi_4$ series using Bailey's nonterminating extension of Jackson's ${}_8\phi_7$ summation and Bailey's four-term nonterminating ${}_{10}\phi_9$ transformation respectively. As special cases, they have obtained nonterminating q -Saalschütz summation, Bailey's very well-poised ${}_6\psi_6$ summation and the nonterminating Watson's ${}_8\phi_7$ transformation. On these lines, we use some very well-poised series to obtain our main results. We then use our results to get as special cases, the q -Gauss summation formula [12, equation (II.8), p.354] and two Sear's transformation formulae [12, equation (III.10), p.359] and [12, equation (III.33), p.364]. We then use the three term Sear's transformation formula [12, equation (III.33), p.364] to obtain an identity which is analogous to the general identity of Andrews [3, Theorem 1] [4]. Finally we use this identity to derive the reciprocity theorem of Ramanujan and its three variable generalization.

In section 2, we present some standard identities which we employ to prove our main results. In section 3, we obtain three new transformation formulae for bilateral ${}_2\psi_2$ and ${}_3\psi_3$ series. We deduce from the transformation formulae, the well-known q -Gauss summation formula, Sear's two term ${}_3\phi_2$ transformation formula and Sear's three term ${}_3\phi_2$ transformation formula. In section 4, we deduce the reciprocity theorem of Ramanujan and its three variable generalization by using Sear's three term ${}_3\phi_2$ transformation formula.

2. SOME STANDARD IDENTITIES FOR BASIC HYPERGEOMETRIC SERIES

In this section, we list some standard identities for q -shifted factorials and basic hypergeometric series which will be used to prove our main results.

A q -shifted factorial identity is

$$\frac{(xq^{-2n})_\infty}{(xq^{-2n})_n} = (-1)^n x^n q^{-(n^2+n)/2} (q/x)_n (x)_\infty. \quad (2.1)$$

Heine's transformation [12, equation (III.2), p.359] is given by

$$\sum_{k=0}^{\infty} \frac{(A, B)_k}{(q, C)_k} z^k = \frac{(C/B, BZ)_\infty}{(C, Z)_\infty} \sum_{k=0}^{\infty} \frac{(ABZ/C, B)_k}{(q, BZ)_k} (C/B)^k. \quad (2.2)$$

Jackson's transformation [12, equation (III.4), p.359] is given by

$$\sum_{k=0}^{\infty} \frac{(A, B)_k}{(q, C)_k} Z^k = \frac{(AZ)_\infty}{(Z)_\infty} \sum_{k=0}^{\infty} \frac{(C/B, A)_k}{(q, C, AZ)_k} (-BZ)^k q^{\binom{k}{2}}. \quad (2.3)$$

The very well-poised ${}_6\phi_5$ summation formula [12, equation (II.20), p.356] is given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(a, qa^{1/2}, -qa^{1/2}, b, c, d)_k}{(q, a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d)_k} \left(\frac{aq}{bcd} \right)^k \\ = \frac{(aq, aq/bc, aq/bd, aq/cd)_\infty}{(aq/b, aq/c, aq/d, aq/bcd)_\infty}. \end{aligned} \quad (2.4)$$

Bailey's very well-poised-balanced ${}_8\phi_7$ series in terms of two balanced ${}_4\phi_3$ series [12, equation (III.36), p.364] is given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(a, qa^{1/2}, -qa^{1/2}, b, c, d, e, f)_k}{(q, a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\ = \frac{(aq, aq/de, aq/df, aq/ef)_\infty}{(aq/d, aq/e, aq/f, aq/def)_\infty} \sum_{k=0}^{\infty} \frac{(aq/bc, d, e, f)_k}{(q, aq/b, aq/c, def/a)_k} q^k \\ + \frac{(aq, aq/bc, d, e, f, a^2 q^2/bdef, a^2 q^2/cdef)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2 q^2/bcdef, def/aq)_\infty} \\ \times \sum_{k=0}^{\infty} \frac{(aq/de, aq/df, aq/ef, a^2 q^2/bcdef)_k}{(q, a^2 q^2/bdef, a^2 q^2/cdef, aq^2/def)_k} q^k. \end{aligned} \quad (2.5)$$

Bailey's very well-poised-balanced ${}_8\phi_7$ series in terms of two balanced ${}_8\phi_7$ series [12, equation (III.37), p.364-365] is given by

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(a, qa^{1/2}, -qa^{1/2}, b, c, d, e, f)_k}{(q, a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\
&= \frac{(aq, aq/de, aq/df, aq/ef, eq/c, fq/c, b/a, bef/a)_{\infty}}{(aq/d, aq/e, aq/f, aq/def, q/c, efq/c, be/a, bf/a)_{\infty}} \\
& \sum_{k=0}^{\infty} \frac{(ef/c, q(ef/c)^{1/2}, -q(ef/c)^{1/2}, aq/bc, aq/cd, ef/a, e, f)_k}{(q, (ef/c)^{1/2}, -(ef/c)^{1/2}, bef/a, def/a, aq/c, fq/c, eq/c)_k} \left(\frac{bd}{a} \right) \\
&+ \frac{b}{a} \frac{(aq, bq/a, bq/c, bq/d, bq/e, bq/f, d, e, f, aq/bc, bdef/a^2, a^2 q/bdef)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, aq/f, bd/a, be/a, bf/a, def/a, aq/def, q/c, b^2 q/a)_{\infty}} \\
& \times \sum_{k=0}^{\infty} \frac{(b^2/a, bqa^{-1/2}, -bqa^{-1/2}, b, bc/a, bd/a, be/a, bf/a)_k}{(q, ba^{-1/2}, -ba^{-1/2}, bq/a, bq/c, bq/d, bq/e, bq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k. \quad (2.6)
\end{aligned}$$

3. MAIN RESULTS

In this section, we derive the three new transformation formulae for bilateral ${}_2\psi_2$ and ${}_3\psi_3$ series.

Theorem 3.1. *We have, for $|\frac{aq}{bcd}| < 1$,*

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} \frac{(a, d)_k}{(aq/b, aq/c)_k} \left(\frac{aq}{bcd} \right)^k \\
&= a \frac{(q, b/a, c/a, aq/d)_{\infty}}{(b, c, q/a, q/d)_{\infty}} \sum_{k=0}^{\infty} \frac{(b, c)_k}{(q, aq/d)_k} q^k - a \frac{(q, aq, qa/bc, aq/bd, aq/cd, 1/a, bc/a)_{\infty}}{(b, c, q/a, q/d, aq/b, aq/c, aq/bcd)_{\infty}}. \quad (3.1)
\end{aligned}$$

Proof: Replacing a by aq^{-2n} and d by dq^{-2n} in (2.4) we obtain

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(b, c, aq^{-2n}, dq^{-2n})_k}{(q, aq^{1-2n}/b, aq^{1-2n}/c, aq/d)_k} \left(\frac{aq}{bcd} \right)^k \\
&= \frac{(aq^{1-2n}, aq^{1-2n}/bc, aq/bd, aq/cd)_{\infty}}{(aq^{1-2n}/b, aq^{1-2n}/c, aq/d, aq/bcd)_{\infty}}. \quad (3.2)
\end{aligned}$$

Let

$$\Omega_k(n) = \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(b, c, aq^{-2n}, dq^{-2n})_k}{(q, aq^{1-2n}/b, aq^{1-2n}/c, aq/d)_k} \left(\frac{aq}{bcd} \right)^k$$

so that we have

$$\Omega_n(n) = \frac{1 - a}{q^{2n} - a} \frac{(b, c, q^{n+1}/a, q^{n+1}/d)_n}{(q, bq^n/a, cq^n/a, aq/d)_n} q^n,$$

$$\Omega_{2n}(n) = \frac{1 - aq^{2n}}{q^{2n} - a} \frac{(b, c, q/a, q/d)_{2n}}{(q, b/a, c/a, aq/d)_{2n}}$$

and

$$\Omega_{k+2n}(n) = \frac{1 - aq^{2n+2k}}{q^{2n} - a} \frac{(b, c, q/a, q/d)_{2n}}{(q, b/a, c/a, aq/d)_{2n}} \frac{(bq^{2n}, cq^{2n}, a, d)_k}{(q^{1+2n}, aq/b, aq/c, aq^{1+2n}/d)_k} \left(\frac{aq}{bcd}\right)^k.$$

We can easily verify that

- (a) $\Omega_n(n) \rightarrow 0$ as $n \rightarrow \infty$,
- (b) $\lim_{n \rightarrow \infty} \Omega_{2n}(n)$ exists but does not vanish and
- (c) $\lim_{n \rightarrow \infty} \Omega_{k+2n}(n)$ exists for each $k \in \mathbb{N}$.

Therefore we can reformulate the infinite sum on the left side of (2.4) as

$$\sum_{k=0}^{n-1} \Omega_k(n) + \sum_{k=-n}^{\infty} \Omega_{k+2n}(n).$$

Then (3.2) becomes

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(b, c, aq^{-2n}, dq^{-2n})_k}{(q, aq^{1-2n}/b, aq^{1-2n}/c, aq/d)_k} \left(\frac{aq}{bcd}\right)^k \\ & + \sum_{k=-n}^{\infty} \frac{1 - aq^{2n+2k}}{q^{2n} - a} \frac{(b, c, q/a, q/d)_{2n}}{(q, b/a, c/a, aq/d)_{2n}} \frac{(bq^{2n}, cq^{2n}, a, d)_k}{(q^{1+2n}, aq/b, aq/c, aq^{1+2n}/d)_k} \left(\frac{aq}{bcd}\right)^k \\ & = \frac{(aq^{1-2n}, aq^{1-2n}/bc, aq/bd, aq/cd)_{\infty}}{(aq^{1-2n}, aq^{1-2n}/c, aq/d, aq/bcd)_{\infty}}. \end{aligned} \quad (3.3)$$

Letting $n \rightarrow \infty$ in (3.3) and using (2.1) we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(b, c)_k}{(q, aq/d)_k} q^k - \frac{1}{a} \frac{(b, c, q/a, q/d)_{\infty}}{(q, b/a, c/a, aq/d)_{\infty}} \sum_{k=-\infty}^{\infty} \frac{(a, d)_k}{(aq/b, aq/c)_k} \left(\frac{aq}{bcd}\right)^k \\ & = \frac{(aq, aq/bc, aq/bd, aq/cd, 1/a, bc/a)_{\infty}}{(aq/b, aq/c, aq/d, aq/bcd, b/a, c/a)_{\infty}}. \end{aligned} \quad (3.4)$$

On some simple manipulations, (3.4) yields (3.1).

Corollary 3.2. [12, equation (II.8), p.354],

$$\sum_{k=0}^{\infty} \frac{(A, B)_k}{(q, C)_k} \left(\frac{C}{AB}\right)^k = \frac{(C/A, C/B)_{\infty}}{(C, C/AB)_{\infty}}. \quad (3.5)$$

Proof: Putting $c = a$ in (3.1) and then changing $a \rightarrow A$, $d \rightarrow B$ and $b \rightarrow Aq/C$ in the resulting identity, we obtain (3.5).

Theorem 3.3. We have, for $\left| \frac{a^2 q^2}{bcdef} \right|, \left| \frac{cf}{a} \right| < 1$,

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} \frac{(a, b, f)_k}{(aq/c, aq/d, aq/e)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\
&= a \frac{(q, aq/b, c/a, d/a, e/a, aq/f)_{\infty}}{(c, d, e, q/a, q/b, q/f)_{\infty}} \sum_{k=0}^{\infty} \frac{(c, d, e)_k}{(q, aq/b, aq/f)_k} q^k \\
&- a \frac{(q, aq, 1/a, c/a, de/a, aq/b, aq/df, aq/ef, aq/de)_{\infty}}{(c, d, e, q/a, q/b, q/f, aq/d, aq/e, aq/def)_{\infty}} \sum_{k=0}^{\infty} \frac{(d, e, aq/bc)_k}{(q, aq/b, def/a)_k} \left(\frac{cf}{a} \right)^k \\
&- a \frac{(q, aq, 1/a, f, aq/bc, a^2 q^2/bdef, a^2 q^2/cdef, cdef/a^2 q)_{\infty}}{(c, q/a, q/b, aq/c, aq/d, aq/e, a^2 q^2/bcdef, def/aq)_{\infty}} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(aq/df, aq/ef, a^2 q^2/bcdef)_k}{(q, a^2 q^2/bdef, aq^2/def)_k} \left(\frac{cf}{a} \right)^k. \quad (3.6)
\end{aligned}$$

Proof: Replacing $a \rightarrow aq^{-2n}$, $b \rightarrow bq^{-2n}$ and $f \rightarrow fq^{-2n}$ in (2.5) we obtain

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(aq^{-2n}, bq^{-2n}, c, d, e, fq^{-2n})_k}{(q, aq/b, aq^{1-2n}/c, aq^{1-2n}/d, aq^{1-2n}/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\
&= \frac{(aq^{1-2n}, aq^{1-2n}/de, aq/df, aq/ef)_{\infty}}{(aq^{1-2n}/d, aq^{1-2n}/e, aq/f, aq/def)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq/bc, d, e, fq^{-2n})_k}{(q, aq/b, aq^{1-2n}/c, def/a)_k} q^k \\
&+ \frac{(aq^{1-2n}, aq/bc, d, e, fq^{-2n}, a^2 q^2/bdef, a^2 q^{2-2n}/cdef)_{\infty}}{(aq/b, aq^{1-2n}/c, aq^{1-2n}/d, aq^{1-2n}/e, aq/f, a^2 q^2/bcdef, def/aq)_{\infty}} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(aq^{1-2n}/de, aq/df, aq/ef, a^2 q^2/bcdef)_k}{(q, a^2 q^2/bdef, a^2 q^{2-2n}/cdef, aq^2/def)_k} q^k. \quad (3.7)
\end{aligned}$$

Let

$$\Omega_k(n) = \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(aq^{-2n}, bq^{-2n}, c, d, e, fq^{-2n})_k}{(q, aq/b, aq^{1-2n}/c, aq^{1-2n}/d, aq^{1-2n}/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k.$$

Then we have

$$\Omega_n(n) = \frac{1 - a}{q^{2n} - a} \frac{(c, d, e, q^{n+1}/a, q^{n+1}/b, q^{n+1}/f)_n}{(q, aq/b, aq/f, cq^n/a, dq^n/a, eq^n/a)_n} q^n,$$

$$\Omega_{2n}(n) = \frac{1 - aq^{2n}}{q^{2n} - a} \frac{(c, d, e, q/a, q/b, q/f)_{2n}}{(q, aq/b, aq/f, c/a, d/a, e/a)_{2n}}$$

and

$$\begin{aligned}\Omega_{k+2n}(n) &= \frac{1 - aq^{2n+2k}}{q^{2n} - a} \frac{(c, d, e, q/a, q/b, q/f)_{2n}}{(q, aq/b, aq/f, c/a, d/a, e/a)_{2n}} \\ &\quad \times \frac{(a, b, f, cq^{2n}, dq^{2n}, eq^{2n})_k}{(q^{1+2n}, aq^{1+2n}/b, aq/c, aq/d, aq/e, aq^{1+2n}/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k.\end{aligned}$$

We can easily verify that

- (a) $\Omega_n(n) \rightarrow 0$ as $n \rightarrow \infty$,
- (b) $\lim_{n \rightarrow \infty} \Omega_{2n}(n)$ exists but does not vanish and
- (c) $\lim_{n \rightarrow \infty} \Omega_{k+2n}(n)$ exists for each $k \in \mathbb{N}$.

Therefore we can reformulate the infinite sum on the left side of (2.5) as

$$\sum_{k=0}^{n-1} \Omega_k(n) + \sum_{k=-n}^{\infty} \Omega_{k+2n}(n).$$

Then (3.7) becomes

$$\begin{aligned}&\sum_{k=0}^{n-1} \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(aq^{-2n}, bq^{-2n}, c, d, e, fq^{-2n})_k}{(q, aq/b, aq^{1-2n}/c, aq^{1-2n}/d, aq^{1-2n}/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\ &\quad + \frac{(c, d, e, q/a, q/b, q/f)_{2n}}{(q, aq/b, aq/f, c/a, d/a, e/a)_{2n}} \\ &\quad \times \sum_{k=-n}^{\infty} \frac{1 - aq^{2n+2k}}{q^{2n} - a} \frac{(a, b, f, cq^{2n}, dq^{2n}, eq^{2n})_k}{(q^{1+2n}, aq^{1+2n}/b, aq/c, aq/d, aq/e, aq^{1+2n}/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\ &= \frac{(aq^{1-2n}, aq^{1-2n}/de, aq/df, aq/ef)_{\infty}}{(aq^{1-2n}/d, aq^{1-2n}/e, aq/f, aq/def)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq/bc, d, e, fq^{-2n})_k}{(q, aq/b, aq^{1-2n}/c, def/a)_k} q^k \\ &\quad + \frac{(aq^{1-2n}, aq/bc, d, e, fq^{-2n}, a^2 q^2/bdef, a^2 q^{2-2n}/cdef)_{\infty}}{(aq/b, aq^{1-2n}/c, aq^{1-2n}/d, aq^{1-2n}/e, aq/f, a^2 q^2/bcdef, def/aq)_{\infty}} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(aq^{1-2n}/de, aq/df, aq/ef, a^2 q^2/bcdef)_k}{(q, a^2 q^2/bdef, a^2 q^{2-2n}/cdef, aq^2/def)_k} q^k. \quad (3.8)\end{aligned}$$

Letting $n \rightarrow \infty$ in (3.8) and using (2.1) we obtain

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(c, d, e)_k}{(q, aq/b, aq/f)_k} q^k - \frac{1}{a} \frac{(c, d, e, q/a, q/b, q/f)_{\infty}}{(q, aq/b, aq/f, c/a, d/a, e/a)_{\infty}} \\
& \quad \times \sum_{k=-\infty}^{\infty} \frac{(a, b, f)_k}{(aq/c, aq/d, aq/e)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\
& = \frac{(aq, aq/de, aq/df, aq/ef, 1/a, de/a)_{\infty}}{(aq/d, aq/e, aq/f, d/a, e/a, aq/de f)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq/bc, d, e)_k}{(q, aq/b, de f/a)_k} \left(\frac{cf}{a} \right)^k \\
& \quad + \frac{(aq, aq/bc, d, e, f, a^2 q^2/bdef, a^2 q^2/cdef, 1/a, q/f, cdef/a^2 q)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2 q^2/bcdef, de f/aq, c/a, d/a, e/a)_{\infty}} \\
& \quad \times \sum_{k=0}^{\infty} \frac{(aq/df, aq/ef, a^2 q^2/bcdef)_k}{(q, a^2 q^2/bdef, aq^2/de f)_k} \left(\frac{cf}{a} \right)^k. \quad (3.9)
\end{aligned}$$

On some simple manipulations, (3.9) yields (3.6).

Corollary 3.4. [12, equation (III.10), p.359],

$$\sum_{k=0}^{\infty} \frac{(A, B, C)_k}{(q, D, E)_k} \left(\frac{DE}{ABC} \right)^k = \frac{(B, DE/BC, DE/AB)_{\infty}}{(D, E, DE/ABC)_{\infty}} \sum_{k=0}^{\infty} \frac{(D/B, E/B, DE/ABC)_k}{(q, DE/BC, DE/AB)_k} B^k. \quad (3.10)$$

Proof: Putting $c = a$ in (3.6) and then changing $a \rightarrow A$, $f \rightarrow B$, $b \rightarrow C$, $aq/d \rightarrow D$ and $aq/e \rightarrow E$ in the resulting identity, we obtain (3.10).

Theorem 3.5. We have, for $\left| \frac{a^2 q^2}{bcdef} \right|, \left| \frac{bd}{a} \right| < 1$,

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} \frac{(a, c, d)_k}{(aq/b, aq/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\
& = a \frac{(q, aq/c, b/a, e/a, f/a, aq/d)_{\infty}}{(b, e, f, q/a, q/c, q/d)_{\infty}} \sum_{k=0}^{\infty} \frac{(b, e, f)_k}{(q, aq/c, aq/d)_k} q^k \\
& \quad - a \frac{(q, aq/c, b/a, aq/de, aq/df, aq, aq/ef, 1/a, ef/a)_{\infty}}{(b, e, f, q/a, q/c, q/d, aq/de f, aq/e, aq/f)_{\infty}} \sum_{k=0}^{\infty} \frac{(e, f, aq/bc)_k}{(q, aq/c, de f/a)_k} \left(\frac{bd}{a} \right)^k \\
& \quad - a \frac{(q, b/a, bq/e, bq/f, aq/bc, aq, d, a^2 q/bdef, 1/a, bdef/a^2)_{\infty}}{(b, q/a, q/c, bd/a, a/b, aq/e, aq/f, bq/a, de f/a, aq/de f)_{\infty}} \sum_{k=0}^{\infty} \frac{(b, bc/a, bd/a)_k}{(q, bq/e, bq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k. \quad (3.11)
\end{aligned}$$

Proof: Replacing $a \rightarrow aq^{-2n}$, $c \rightarrow cq^{-2n}$ and $d \rightarrow dq^{-2n}$ in (2.6) we obtain

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(aq^{-2n}, b, cq^{-2n}, dq^{-2n}, e, f)_k}{(q, aq^{1-2n}/b, aq/c, aq/d, aq^{1-2n}/e, aq^{1-2n}/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\
&= \frac{(aq^{1-2n}, aq/de, aq/df, aq^{1-2n}/ef, eq^{1+2n}/c, fq^{1+2n}/c, bq^{2n}/a, befq^{2n}/a)_{\infty}}{(aq/d, aq^{1-2n}/e, aq^{1-2n}/f, aq/de, q^{1+2n}/c, eq^{1+2n}/c, beq^{2n}/a, bfq^{2n}/a)_{\infty}} \\
&\times \sum_{k=0}^{\infty} \frac{(efq^{2n}/c, q^{1+n}(ef/c)^{1/2}, -q^{1+n}(ef/c)^{1/2}, aq/bc, aq^{1+2n}/cd, efq^{2n}/a, e, f)_k}{(q, q^n(ef/c)^{1/2}, -q^n(ef/c)^{1/2}, befq^n/a, def/a, aq/c, fq^{1+2n}/c, eq^{1+2n}/c)_k} \left(\frac{bd}{a} \right)^k \\
&+ \frac{(aq^{1-2n}, bq^{2n}/a, bq^{1+2n}/c, bq^{1+2n}/d, bq/e, bq/f, dq^{-2n}, e, f, aq/bc)_{\infty}}{(aq^{-2n}/b, aq/c, aq/d, aq^{1-2n}/e, aq^{1-2n}/f, bd/a, beq^{2n}/a, bfq^{2n}/a, def/a, aq/def)_{\infty}} \\
&\times \frac{(bdefq^{2n}/a^2, a^2q^{1-2n}/bdef)_{\infty}}{(q^{1+2n}/c, b^2q^{1+2n}/a)_{\infty}} \\
&\times \sum_{k=0}^{\infty} \frac{(b^2q^{2n}/a, q^{1+n}b/a^{1/2}, -q^{1+n}b/a^{1/2}, b, bc/a, bd/a, beq^{2n}/a, bfq^{2n}/a)_k}{(q, q^n b/a^{1/2}, -q^n b/a^{1/2}, bq^{1+2n}/a, bq^{1+2n}/c, bq^{1+2n}/d, bq/e, bq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k. \tag{3.12}
\end{aligned}$$

Let

$$\Omega_k(n) = \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(aq^{-2n}, b, cq^{-2n}, dq^{-2n}, e, f)_k}{(q, aq^{1-2n}/b, aq/c, aq/d, aq^{1-2n}/e, aq^{1-2n}/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k$$

so that we have

$$\Omega_n(n) = \frac{1 - a}{q^{2n} - a} \frac{(b, e, f, q^{n+1}/a, q^{n+1}/c, q^{n+1}/d)_n}{(q, aq/c, aq/d, bq^n/a, eq^n/a, fq^n/a)_n} q^n,$$

$$\Omega_{2n}(n) = \frac{1 - aq^{2n}}{q^{2n} - a} \frac{(b, e, f, q/a, q/c, q/d)_{2n}}{(q, aq/c, aq/d, b/a, e/a, f/a)_{2n}}$$

and

$$\begin{aligned}
\Omega_{k+2n}(n) &= \frac{1 - aq^{2n+2k}}{q^{2n} - a} \frac{(b, e, f, q/a, q/c, q/d)_{2n}}{(q, aq/c, aq/d, b/a, e/a, f/a)_{2n}} \\
&\times \frac{(a, c, d, bq^{2n}, eq^{2n}, fq^{2n})_k}{(q^{1+2n}, aq/b, aq^{1+2n}/c, aq^{1+2n}/d, aq/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k.
\end{aligned}$$

We can easily verify that

- (a) $\Omega_n(n) \rightarrow 0$ as $n \rightarrow \infty$,
- (b) $\lim_{n \rightarrow \infty} \Omega_{2n}(n)$ exists but does not vanish and
- (c) $\lim_{n \rightarrow \infty} \Omega_{k+2n}(n)$ exists for each $k \in \mathbb{N}$.

Therefore we can reformulate the infinite sum on the left side of (2.6) as

$$\sum_{k=0}^{n-1} \Omega_k(n) + \sum_{k=-n}^{\infty} \Omega_{k+2n}(n).$$

Then (3.12) becomes

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{1 - aq^{-2n+2k}}{1 - aq^{-2n}} \frac{(aq^{-2n}, b, cq^{-2n}, dq^{-2n}, e, f)_k}{(q, aq^{1-2n}/b, aq/c, aq/d, aq^{1-2n}/e, aq^{1-2n}/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\ & \quad + \frac{(b, e, f, q/a, q/c, q/d)_{2n}}{(q, aq/c, aq/d, b/a, e/a, f/a)_{2n}} \\ & \quad \times \sum_{k=-n}^{\infty} \frac{1 - aq^{2n+2k}}{q^{2n} - a} \frac{(a, c, d, bq^{2n}, eq^{2n}, fq^{2n})_k}{(q^{1+2n}, aq/b, aq^{1+2n}/c, aq^{1+2n}/d, aq/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\ & = \frac{(aq^{1-2n}, aq/de, aq/df, aq^{1-2n}/ef, eq^{1+2n}/c, fq^{1+2n}/c, bq^{2n}/a, befq^{2n}/a)_{\infty}}{(aq/d, aq^{1-2n}/e, aq^{1-2n}/f, aq/def, q^{1+2n}/c, eq^{1+2n}/c, beq^{2n}/a, bfq^{2n}/a)_{\infty}} \\ & \quad \times \sum_{k=0}^{\infty} \frac{(efq^{2n}/c, q^{1+n}(ef/c)^{1/2}, -q^{1+n}(ef/c)^{1/2}, aq/bc, aq^{1+2n}/cd, efq^{2n}/a, e, f)_k}{(q, q^n(ef/c)^{1/2}, -q^n(ef/c)^{1/2}, befq^n/a, def/a, aq/c, fq^{1+2n}/c, eq^{1+2n}/c)_k} \left(\frac{bd}{a} \right)^k \\ & \quad + \frac{(aq^{1-2n}, bq^{2n}/a, bq^{1+2n}/c, bq^{1+2n}/d, bq/e, bq/f, dq^{-2n}, e, f, aq/bc)_{\infty}}{(aq^{-2n}/b, aq/c, aq/d, aq^{1-2n}/e, aq^{1-2n}/f, bd/a, beq^{2n}/a, bfq^{2n}/a, def/a, aq/def)_{\infty}} \\ & \quad \times \frac{(bdefq^{2n}/a^2, a^2 q^{1-2n}/bdef)_{\infty}}{(q^{1+2n}/c, b^2 q^{1+2n}/a)_{\infty}} \\ & \quad \times \sum_{k=0}^{\infty} \frac{(b^2 q^{2n}/a, q^{1+n}b/a^{1/2}, -q^{1+n}b/a^{1/2}, b, bc/a, bd/a, beq^{2n}/a, bfq^{2n}/a)_k}{(q, q^n b/a^{1/2}, -q^n b/a^{1/2}, bq^{1+2n}/a, bq^{1+2n}/c, bq^{1+2n}/d, bq/e, bq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k. \end{aligned} \tag{3.13}$$

Letting $n \rightarrow \infty$ in (3.13) and using (2.1) we obtain

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(b, e, f)_k}{(q, aq/c, aq/d)_k} q^k - \frac{1}{a} \frac{(b, e, f, q/a, q/c, q/d)_{\infty}}{(q, aq/c, aq/d, b/a, e/a, f/a)_{\infty}} \\
& \quad \times \sum_{k=-\infty}^{\infty} \frac{(a, c, d)_k}{(aq/b, aq/e, aq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k \\
& = \frac{(aq, aq/de, aq/df, aq/ef, 1/a, ef/a)_{\infty}}{(aq/d, aq/e, aq/f, e/a, f/a, aq/def)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq/bc, e, f)_k}{(q, aq/c, def/a)_k} \left(\frac{bd}{a} \right)^k \\
& \quad - \frac{(aq, aq/bc, d, e, f, 1/a, q/d, bq/e, bq/f, a^2 q/bdef, bdef/a^2)_{\infty}}{(aq/c, aq/d, aq/e, aq/f, bd/a, a/b, bq/a, def/a, e/a, f/a, aq/def)_{\infty}} \\
& \quad \times \sum_{k=0}^{\infty} \frac{(b, bc/a, bd/a)_k}{(q, bq/e, bq/f)_k} \left(\frac{a^2 q^2}{bcdef} \right)^k. \quad (3.14)
\end{aligned}$$

On some simple manipulations (3.14) yields (3.11).

Corollary 3.6. [12, equation (III.33), p.364],

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(A, B, C)_k}{(q, D, E)_k} \left(\frac{DE}{ABC} \right)^k = \frac{(Cq/A, q/D, E/B, E/C)_{\infty}}{(Cq/D, q/A, E/BC, E)_{\infty}} \sum_{k=0}^{\infty} \frac{(D/A, C, Cq/E)_k}{(q, BCq/E, Cq/A)_k} \left(\frac{Bq}{D} \right)^k \\
& - \frac{(q/D, Eq/D, D/A, B, C, DE/BCq, BCq^2/DE)_{\infty}}{(Cq/D, BCq/E, E/BC, q/A, Bq/D, D/q, E)_{\infty}} \sum_{k=0}^{\infty} \frac{(Aq/D, Bq/D, Cq/D)_k}{(q, q^2/D, Eq/D)_k} \left(\frac{DE}{ABC} \right)^k. \quad (3.15)
\end{aligned}$$

Proof: Putting $e = a$ in (3.11) and then changing $a \rightarrow C$, $c \rightarrow A$, $d \rightarrow B$, $aq/b \rightarrow D$ and $aq/f \rightarrow E$ in the resulting identity, we obtain (3.15).

4. PROOFS OF THE RECIPROCITY THEOREM OF RAMANUJAN AND ITS GENERALIZATION

We first establish the following theorem which is analogous to the general identity of Andrews [3, Theorem 1].

Theorem 4.1. *We have for $|A|, |a| < 1$*

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(c, b/aA)_k}{(-bq, -cq/a)_k} (Aq)^k - (c + a) \sum_{k=0}^{\infty} \frac{(-1/b)_{k+1} (-Acq/b)_k}{(-c/b)_{k+1} (aAq/b)_{k+1}} (-a)^k \\
& = \frac{(-Acq/b, q, c, a/b, aAq, bq/a)_{\infty}}{(aAq/b, -bq, -c/b, Aq, -cq/a, -a)_{\infty}}. \quad (4.1)
\end{aligned}$$

Proof: Changing $C \rightarrow q$ in (3.15) we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(A, B)_k}{(D, E)_k} \left(\frac{DE}{ABq} \right)^k &= \frac{(q^2/A, q/D, E/B, E/q)_{\infty}}{(q^2/D, q/A, E/Bq, E)_{\infty}} \sum_{k=0}^{\infty} \frac{(D/A, q^2/E)_k}{(Bq^2/E, q^2/A)_k} \left(\frac{Bq}{D} \right)^k \\ &- \frac{(q/D, Eq/D, D/A, B, q, DE/Bq^2, Bq^3/DE)_{\infty}}{(q^2/D, Bq^2/E, E/Bq, q/A, Bq/D, D/q, E)_{\infty}} \sum_{k=0}^{\infty} \frac{(Aq/D, Bq/D)_k}{(q, Eq/D)_k} \left(\frac{DE}{ABq} \right)^k \end{aligned}$$

Using q -Guass summation formula, (3.5) on the right side of the above equation, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(A, B)_k}{(D, E)_k} \left(\frac{DE}{ABq} \right)^k &= \frac{(q^2/A, q/D, E/B, E/q)_{\infty}}{(q^2/D, q/A, E/Bq, E)_{\infty}} \sum_{k=0}^{\infty} \frac{(D/A, q^2/E)_k}{(Bq^2/E, q^2/A)_k} \left(\frac{Bq}{D} \right)^k \\ &- \frac{(q/D, D/A, B, q, DE/Bq^2, Bq^3/DE, E/B, E/A)_{\infty}}{(q^2/D, Bq^2/E, E/Bq, q/A, Bq/D, D/q, E, DE/ABq)_{\infty}} \end{aligned} \quad (4.2)$$

Changing $A \rightarrow -q/b$, $B \rightarrow -Acq/b$, $D \rightarrow -cq/b$ and $E \rightarrow Aaq^2/b$ in (4.2), we obtain (4.1) on some simplifications.

We now deduce the reciprocity theorem of Ramanujan [19] and its three variable generalization [17].

Corollary 4.2. (Ramanujan) We have, for $a, b \neq 0$, $-q^{-n}$, $n \in \mathbb{Z}^+$,

$$\rho(a, b) - \rho(b, a) = \left(\frac{1}{b} - \frac{1}{a} \right) \frac{(aq/b, bq/a, q)_{\infty}}{(-aq, -bq)_{\infty}}, \quad (4.3)$$

where

$$\begin{aligned} \rho(a, b) &:= \left(1 + \frac{1}{b} \right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n} \\ &:= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n-1}}{(-aq)_{n+1}}. \end{aligned} \quad (4.4)$$

Proof: Letting $A, c \rightarrow 0$ in (4.1) we obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k b^k a^{-k} q^{k(k+1)/2}}{(-bq)_k} - a \sum_{k=0}^{\infty} (-1/b)_{k+1} (-a)^k = \frac{(q, a/b, bq/a)_{\infty}}{(-bq, -a)_{\infty}}. \quad (4.5)$$

Setting $A = -q/b$, $B = q$, $Z = -a$ in (2.2) and then letting $C \rightarrow 0$ in the resulting identity, we obtain

$$\sum_{k=0}^{\infty} (-1/b)_{k+1} (-a)^k = (1 + 1/b) \sum_{k=0}^{\infty} \frac{(-1)^k a^k b^{-k} q^{k(k+1)/2}}{(-a)_{k+1}}. \quad (4.6)$$

Substituting (4.6) in (4.5), we obtain (4.3) on some simplifications.

Corollary 4.3. We have, for $|c| < |a| < 1$ and $|c| < |b| < 1$,

$$\rho_3(a, b; c) - \rho_3(b, a; c) = \left(\frac{1}{b} - \frac{1}{a} \right) \frac{(c, aq/b, bq/a, q)_\infty}{(-c/a, -c/b, -aq, -bq)_\infty}, \quad (4.7)$$

where

$$\rho_3(a, b; c) := \left(1 + \frac{1}{b} \right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1}}. \quad (4.8)$$

Proof: Letting $A \rightarrow 0$ in (4.1) we obtain

$$\sum_{k=0}^{\infty} \frac{(c)_k (-1)^k b^k a^{-k} q^{k(k+1)/2}}{(-bq, -cq/a)_k} - (c+a) \sum_{k=0}^{\infty} \frac{(-1/b)_{k+1}}{(-c/b)_{k+1}} (-a)^k = \frac{(q, c, a/b, bq/a)_\infty}{(-bq, -c/b, -cq/a, -a)_\infty}. \quad (4.9)$$

Setting $A = q, B = -q/b, C = -cq/b$ and $Z = -a$ in (2.3) we obtain

$$\sum_{k=0}^{\infty} \frac{(-1/b)_{k+1}}{(-c/b)_{k+1}} (-a)^k = \frac{(1 + 1/b)}{(1 + a)} \sum_{k=0}^{\infty} \frac{(c)_k (-1)^k a^k b^{-k} q^{k(k+1)/2}}{(-aq)_k (-c/b)_{k+1}}. \quad (4.10)$$

Substituting (4.10) in (4.9), we obtain (4.7) on some simplifications.

Remark. The reciprocity theorem (4.3) was first proved by Andrews [3]. Later, Somashekara and Fathima [22], Bhargava, Somashekara and Fathima [9], Kim, Somashekara and Fathima [18], Guruprasad and Pradeep [13], Adiga and Anitha [1], Berndt, Chan, Yeap and Yee [8], Kang [17], Somashekara and Narasimha murthy [24], Somashekara, Narasimha murthy and Shalini [26], [27] have contributed to the proof of (4.3). Recently, Somashekara and Narasimha murthy [24] have given a finite form of (4.3). For more details one may refer the book by Andrews and Berndt [4]. The three variable generalization (4.7) of (4.3) was established by Kang [17]. Later, Adiga and Guruprasad [2], Somashekara and Mamta [23], Somashekara, Narasimha murthy and Shalini [26] have contributed to the proof of (4.7). In [24], Somashekara and Narasimha murthy have given a finite form of (4.7).

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