Gulf Journal of Mathematics
Vol 5, Issue 2 (2017) 46-55

EDGE INJECTIVE DOMINATION OF GRAPHS

AKRAM ALQESMAH[1*], ANWAR ALWARDI[2] AND R. RANGARAJAN[3]

ABSTRACT. Let $G = (V, E)$ be a simple graph. A subset X of E is called edge injective dominating set (edge Inj-dominating set) if for every edge $f \in E - X$ there exists an edge $g \in X$ such that $|\Gamma(f, g)| \geq 1$, where $|\Gamma(f, g)|$ is the number of common edge neighbors between the edges f and g. The minimum cardinality of such edge dominating set denoted by $\gamma'_{in}(G)$ and is called edge injective domination number (edge Inj-domination number) of G. In this paper, we introduce the edge injective domination number, injective independence edge number (Inj-independence edge number) $\beta'_{in}(G)$ and edge injective domatic number (edge Inj-domatic number) $d'_{in}(G)$ of a graph G. Exact values for some standard graphs, bounds and some interesting results are established.

1. INTRODUCTION

In this paper, we concerned only about a finite, undirected graphs without loops and multiple edges. For a graph $G = (V, E)$, we denote the number of vertices and the number of edges by $p = |V|$ and $q = |E|$, respectively. we use $\langle Y \rangle$ to denote the subgraph of G induced by the set of vertices Y. $N(v)$ and $N[v]$ denote the open and closed neighborhood of a vertex v in G, respectively. The distance between two vertices u and v in G is the number of edges in the shortest path connecting them. The eccentricity of a vertex v is the greatest distance between v and any other vertex and denoted by $e(v)$. A set D of vertices in G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A line graph $L(G)$ of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. We denote to the smallest integer greater than or equal to x by $\lceil x \rceil$ and the greatest integer less than or equal to x by $\lfloor x \rfloor$. For terminology and notations not specifically defined here we refer the reader to [6]. For more details about domination number and its related parameters, we refer to [7, 8].

Date: Received: Dec 22, 2016; Accepted: Mar 3, 2017.
* Corresponding author.
2010 *Mathematics Subject Classification.* 05C69.
Key words and phrases. Edge injective domination number, Injective independence edge number, Edge injective domatic number.

The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with the vertex set $V(G)$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G, [2]. The concept of injective domination in graph has introduced in [5]. For a graph G, a subset D of $V(G)$ is called injective dominating set (Inj-dominating set) if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $|\Gamma(u, v)| \geq 1$, where $|\Gamma(u, v)|$ is the number of common neighborhood between the vertices u and v. The minimum cardinality of such dominating set denoted by $\gamma_{in}(G)$ and is called injective domination number (Inj-domination number) of G. Note that $\gamma_{in}(G) = \gamma(con(G))$. The Inj-neighborhood of a vertex $u \in V(G)$ denoted by $N_{in}(u)$ is defined as $N_{in}(u) = \{v \in V(G) : |\Gamma(u, v)| \geq 1\}$. The cardinality of $N_{in}(u)$ is called injective degree of the vertex u and denoted by $deg_{in}(u)$ in G, and $N_{in}[u] = N_{in}(u) \cup \{u\}$. The maximum and minimum injective degree of a vertex in G are denoted respectively by $\Delta_{in}(G)$ and $\delta_{in}(G)$. That is $\Delta_{in}(G) = \max_{u \in V} |N_{in}(u)|$, $\delta_{in}(G) = \min_{u \in V} |N_{in}(u)|$. The injective complement of G denoted by $\overline{G}^{inj}$ is the graph with same vertex set $V(G)$ and any two vertices u and v in $\overline{G}^{inj}$ are adjacent if and only if they are not Inj-adjacent in G. A subset S of V is called an injective independent set (Inj-independent set), if for every $u \in S, v \notin N_{in}(u)$ for all $v \in S - \{u\}$. An injective independent set S is called maximal if any vertex set properly containing S is not Inj-independent set, the maximum cardinality of Inj-independent set is denoted by β_{in}, and the lower Inj-independence number i_{in} is the minimum cardinality of the Inj-maximal independent set. For more details about the injective domination of graphs, we refer to [1, 4].

The concept of edge domination had introduced by Mitchell and Hedetniemi, [9]. Let $G = (V, E)$ be a graph. A subset X of E is called an edge dominating set of G if every edge in $E - X$ is adjacent to some edge in X. The CN-edge domination of graphs had introduced in [3]. Let f and e be any two edges in E. Then f and e are side to be common neighborhood adjacent (CN-adjacent) if f adjacent to e and there exist another edge g adjacent to both f and e. A set Y of edges is called common neighborhood edge dominating set (CN-edge dominating set) if every edge f not in Y is CN-adjacent to at least one edge $f \in Y$. The minimum cardinality of such CN-edge dominating set is denoted by $\gamma'_{cn}(G)$ and called CN-edge domination number of G. The CN-edge neighborhood of f denoted by $N_{cn}(f)$ is defined as $N_{cn}(f) = \{g \in E(G) : f \text{ and } g \text{ are CN-adjacent}\}$. The cardinality of $N_{cn}(f)$ is called the CN-edge degree of the edge f and denoted by $deg_{cn}(f)$.

Throughout this paper, we denote the path, cycle, complete and star graphs of p vertices by P_p, C_p, K_p and S_p, respectively. In this paper, we introduce the edge injective domination number, injective independence edge number and edge injective domatic number of a graph G. Exact values for some standard graphs, bounds and some interesting results are established.

2. EDGE INJECTIVE DOMINATING SETS

Definition 2.1. Let $G = (V, E)$ be simple graph and f, g be any two edges in E. Then f and g are injective adjacent (Inj-adjacent) if and only if $|\Gamma(f, g)| \geq 1$, where $|\Gamma(f, g)|$ is the number of common edge neighbors between the edges f and g.

Remark 2.2. By Definition 2.1, any two edges f and g in a graph G are Inj-adjacent if and only if either they lie in a subgraph of G isomorphic to C_3, S_p with $p \geq 4$ or they are pendent edges of a subgraph of G isomorphic to P_4.

Definition 2.3. A subset X of E is called edge injective dominating set (edge Inj-dominating set) if for every vertex $f \in E$ either $f \in X$ or there exists an edge $g \in X$ such that $|\Gamma(f, g)| \geq 1$. The minimum cardinality of edge Inj-dominating set of G denoted by $\gamma'_{in}(G)$ and is called edge injective domination number (edge Inj-domination number) of G.

For example consider a graph G in Figure 1. Then $\{a, d\}$ is a minimum edge dominating set, $\{a, b, e\}$ is a minimum CN-edge dominating set and $\{c\}$ is a minimum edge Inj-dominating set. Thus $\gamma'(G) = 2$, $\gamma'_{cn}(G) = 3$ and $\gamma'_{in}(G) = 1$.

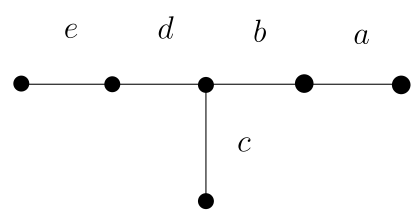

FIGURE 1. Graph G

An edge injective dominating set X is said to be minimal edge Inj-dominating set if no proper subset of X is edge Inj-dominating set. Clearly each minimum edge Inj-dominating set is minimal but the converse is not true in general, for example in the graph G in Figure 1, the set $\{d, e\}$ is a minimal edge Inj-dominating set but is not minimum edge Inj-dominating set.
The Inj-neighborhood of an edge f denoted by $N_{in}(f)$ is defined as $N_{in}(f) = \{g \in E(G) : |\Gamma(f, g)| \geq 1\}$. The cardinality of $N_{in}(f)$ is called edge Inj-degree of the edge f and denoted by $deg_{in}(f)$ in G, and $N_{in}[f] = N_{in}(f) \cup \{f\}$. If an edge $f \in E$ be such that $N_{in}(f) = \phi$, then f must be contained in any edge Inj-dominating set. Such edges are called Inj-isolated edges.

Remark 2.4. For a connected graph G, an edge f in G is an Inj-isolated edge if it has a full edge degree ($deg(f) = q - 1$) and is not contained in a subgraph of G isomorphic to C_3 or S_p with $p \geq 4$. Hence, G has an Inj-isolated edge if and only if G isomorphic to K_2, P_3 or P_4.

The maximum and minimum edge Inj-degree of an edge in $G = (V, E)$ are denoted respectively by $\Delta'_{in}(G)$ and $\delta'_{in}(G)$. That is $\Delta'_{in}(G) = \max_{f\in E}|N_{in}(f)|$, $\delta'_{in}(G) = \min_{f\in E}|N_{in}(f)|$. A subset Y of E is called an injective independent edge set (Inj-independent edge set), if for every $f \in Y, g \notin N_{in}(f)$ for all $g \in Y - \{f\}$. An injective independent edge set Y is called maximal if any edge set properly containing Y is not Inj-independent edge set, the maximum cardinality of an Inj-independent edge set is denoted by β'_{in} and is called the Inj-independence edge number of G. The distance between any two edges f and g is defined by the number of vertices in the shortest path containing them minus two, following of that the eccentricity of an edge f is the maximum distance between f and any other edge and denoted by $e(f)$.

Proposition 2.5. *For any graph G with at least one edge, $1 \leq \gamma'_{in}(G) \leq q$.*

Proposition 2.6. *For any graph G, $\gamma'_{in}(G) = \gamma_{in}(L(G))$.*

Proposition 2.7. *For any graph G, $\gamma'_{in}(G) \leq \gamma'_{cn}(G)$.*

Proof. From the definition of the CN-edge dominating set of a graph G. For any graph G any CN-edge dominating set is also edge injective dominating set. Hence, $\gamma'_{in}(G) \leq \gamma'_{cn}(G)$. □

We note that the edge Inj-domination number of a graph G may be greater than, smaller than or equal to the edge domination number of G.

Example 2.8.
(1) $\gamma'_{in}(P_3) = 2; \gamma'(P_3) = 1$.
(2) $\gamma'_{in}(C_5) = 2; \gamma'(C_5) = 2$.
(3) For any complete graph K_p with $p \geq 2$, $\gamma'_{in}(K_p) = 1; \gamma'(K_p) = \lfloor \frac{p}{2} \rfloor$.

Proposition 2.9. *Let G be a (p, q)-graph. Then $\gamma'_{in}(G) = q$ if and only if G is a forest with maximum edge degree $\Delta'(G) \leq 1$.*

Proof. Let G be a forest with $\Delta'(G) \leq 1$. Then we have two cases.
Case 1. If G is connected, then either $G \cong P_3$ or $G \cong K_2$. Hence, $\gamma'_{in}(G) = q$.
Case 2. If G is disconnected, then $G \cong n_1P_3 \cup n_2K_2 \cup n_3K_1$ for some n_1, n_2, and $n_3 \in \{0, 1, 2, \dots\}$, thus $\gamma'_{in}(G) = q$.
Conversely, if $\gamma'_{in}(G) = q$, then all the edges of G are Inj-isolated, which means (by Remark 2.2) G does not contain C_3, S_p with $p \geq 4$ and P_4 as subgraphs. Therefore G is isomorphic to K_2 or P_3 or to the disjoint union of K_2, P_3 and K_1, that is $G \cong n_1P_3 \cup n_2K_2 \cup n_3K_1$. Hence, G is a forest with $\Delta'(G) \leq 1$. □

Proposition 2.10. *Let G be a nontrivial connected graph. Then $\gamma'_{in}(G) = 1$ if and only if there exists an edge $f \in E(G)$ such that $N(f) = N_{cn}(f)$ and $e(f) \leq 2$.*

Proof. Let $f \in E(G)$ be any edge in G such that $N(f) = N_{cn}(f)$ and $e(f) \leq 2$. Then for any edge $g \in E(G) - \{f\}$, if g is adjacent to f, since $N(f) = N_{cn}(f)$, then $g \in N_{in}(f)$, and if g not adjacent to f, since $e(f) \leq 2$, then $|\Gamma(f, g)| \geq 1$. Thus for any edge $g \in E(G) - \{f\}$, $|\Gamma(u, v)| \geq 1$. Hence, $\gamma'_{in}(G) = 1$.
Conversely, if G is a graph with q edges and $\gamma'_{in}(G) = 1$, then there exist at least one edge $f \in E(G)$ such that $deg_{in}(f) = q - 1$, then by Remark 2.2, f and any

edge $g \in E(G) - \{f\}$ either lie in a subgraph of G isomorphic to C_3, S_p with $p \geq 4$ or they are pendent edges of a subgraph of G isomorphic to P_4. Hence, $N(f) = N_{cn}(f)$ and $e(f) \leq 2$. □

Corollary 2.11.

(1) *For any complete graph K_p, $\gamma'_{in}(K_p) = 1$.*
(2) *For any wheel graph W_p, $\gamma'_{in}(W_p) = 1$.*
(3) *For any complete bipartite graph $K_{r,m}$, $\gamma'_{in}(K_{r,m}) = \begin{cases} 2, & \text{if } r = 2 \text{ or } m = 2; \\ 1, & \text{otherwise.} \end{cases}$*
(4) *Let G_1 and G_2 be two graphs such that at least one of them is not the null graph. Then $\gamma'_{in}(G_1 + G_2) = 1$.*

Proposition 2.12. *For any connected (p, q)-graph G, $\gamma'_{in}(G) = q - 1$ if and only if $G \cong P_4$.*

Proof. Let $\gamma'_{in}(G) = q - 1$. Then by Remark 2.2, G does not contain a subgraph isomorphic to C_3 or S_p with $p \geq 4$. Thus any γ'_{in}-set of G must contain $q - 2$ Inj-isolated edges. Hence by Remark 2.4, $G \cong P_4$. The converse is obvious. □

Proposition 2.13. *For any connected (p, q)-graph G, $\gamma'_{in}(G) = q - 2$ if and only if G isomorphic to one of the following graphs C_3, C_4, S_4 and P_5.*

Proof. Let $\gamma'_{in}(G) = q - 2$. Then by Remark 2.4, G has no Inj-isolated edges. Suppose $q \geq 5$. Then there exists at least one edge f in G such that $|N_{in}(f)| \geq 2$. Since G has no Inj-isolated edges, then $\gamma'_{in}(G) < q - 2$. Suppose now $q \leq 4$. Then, if $q \leq 3$, G isomorphic to C_3 or S_4 and if $q = 4$, G only isomorphic to C_4 or P_5 (because all the other connected graphs of $q = 4$ have edge Inj-domination number equal to one). The converse is clear. □

Proposition 2.14 ([5]). *For any cycle C_p with $p \geq 3$,*

$$\gamma_{in}(C_p) = \begin{cases} \left\lceil \dfrac{p}{3} \right\rceil, & \text{if } p \text{ is odd;} \\ 2\left\lceil \dfrac{p}{6} \right\rceil, & \text{if } p \text{ is even.} \end{cases}$$

Proposition 2.15 ([5]). *For any path P_p with $p \geq 2$,*

$$\gamma_{in}(P_p) = \begin{cases} \left\lceil \dfrac{p+1}{6} \right\rceil + \left\lceil \dfrac{p-1}{6} \right\rceil, & \text{if } p \text{ is odd;} \\ 2\left\lceil \dfrac{p}{6} \right\rceil, & \text{if } p \text{ is even.} \end{cases}$$

According to Propositions 2.14, 2.15 and 2.6, and since $L(C_p) = C_p$, $L(P_p) = P_{p-1}$, then the following propositions are verified.

Proposition 2.16. *For any cycle C_p with $p \geq 3$,*

$$\gamma'_{in}(C_p) = \begin{cases} \left\lceil \dfrac{p}{3} \right\rceil, & \text{if } p \text{ is odd;} \\ 2\left\lceil \dfrac{p}{6} \right\rceil, & \text{if } p \text{ is even.} \end{cases}$$

Proposition 2.17. *For any path P_p with $p \geq 3$,*

$$\gamma'_{in}(P_p) = \begin{cases} \left\lceil \frac{p}{6} \right\rceil + \left\lceil \frac{p-2}{6} \right\rceil, & \text{if } p \text{ is odd;} \\ 2\left\lceil \frac{p-1}{6} \right\rceil, & \text{if } p \text{ is even.} \end{cases}$$

Theorem 2.18. *Let $G = (V, E)$ be a graph without Inj-isolated edges. If X is a minimal edge Inj-dominating set, then $E - X$ is an edge Inj-dominating set.*

Proof. Let X be a minimal edge Inj-dominating set of G. Suppose $E - X$ is not edge Inj-dominating set. Then there exists at least an edge f in X such that f is not edge Inj-dominated by any edge in $E - X$. Since G has no Inj-isolated edges, then f is edge Inj-dominated by at least one edge in $X - \{f\}$. Thus $X - \{f\}$ is edge Inj-dominating set of G, which contradicts the minimality of X. Thus every edge in X has common edge neighborhood with at least one edge in $E - X$. Hence, $E - X$ is an edge Inj-dominating set. □

Theorem 2.19. *Let G be a graph and X is an edge Inj-dominating set of G. Then X is minimal if and only if for every edge $f \in X$ one of the following condition holds*

(1) $N_{in}(f) \cap X = \phi$.
(2) *There exists an edge g in $E - X$ such that $N_{in}(g) \cap X = \{f\}$.*

Proof. Let X be a minimal edge Inj-dominating set of G. Suppose that conditions (1) and (2) do not hold. Then there exist an edge $g \in N_{in}(f) \cap X$ for some $f \in X$ and for any edge $e \in E - X$, $N_{in}(e) \cap X \neq \{f\}$. Hence, $X - \{f\}$ is an edge Inj-dominating set of G which contradicts the minimality of X. Therefore conditions (1) and (2) hold.
Conversely, suppose X is an edge Inj-dominating set of G and for each edge $f \in X$ one of the two conditions holds, we want to prove that X is a minimal edge Inj-dominating set of G. Suppose that X is not minimal. Then there exists at least one edge $f \in X$ such that $X - \{f\}$ is an edge Inj-dominating set of G. Then there is an edge $h \in X - \{f\}$ such that $h \in N_{in}(f)$. Hence, $N_{in}(f) \cap X \neq \phi$. Therefore condition (1) is not satisfied for f. Thus f must satisfy condition (2). That means there exists an edge $g \in E - X$ such that $N_{in}(g) \cap X = \{f\}$. But $X - \{f\}$ is an edge Inj-dominating set of G which means there is an edge $e \in X - \{f\}$ such that $e \in N_{in}(g) \cap X$. So, condition (2) also is not hold for f. Hence, X is a minimal edge Inj-dominating set of G. □

Theorem 2.20. *A graph G has a unique minimal edge Inj-dominating set if and only if the set of all Inj-isolated edges forms an edge Inj-dominating set.*

Proof. Let G be a graph with unique minimal edge Inj-dominating set X, and suppose $I'_{in} = \{f \in E : f \text{ is an Inj-isolated edge}\}$. Then $I'_{in} \subseteq X$. We want to prove that $X = I'_{in}$. For contrary, suppose that $X \neq I'_{in}$. Assume that $g \in X - I'_{in}$, since g is not Inj-isolated edge, then $E - \{g\}$ is edge Inj-dominating set of G. Hence, $E - \{g\}$ contains a minimal edge Inj-dominating set say F and clearly $F \neq X$, a contradiction to the fact that G has a unique minimal edge

Inj-dominating set. Therefore $X = I'_{in}$.
Conversely, if the set of all Inj-isolated edges in G forms an edge Inj-dominating set, then it is clear that G has a unique minimal edge Inj-dominating set. $\square$

Theorem 2.21. *For any (p, q)-graph G with maximum edge Inj-degree $\Delta'_{in}(G)$, $\gamma'_{in}(G) \leq q - \Delta'_{in}(G)$.*

Proof. Let $f \in E(G)$ be an edge such that $deg_{in}(f) = \Delta'_{in}(G)$. Then the set of edges $E(G) - N_{in}(f)$ is an edge Inj-dominating set of G. Hence, $\gamma'_{in}(G) \leq q - \Delta'_{in}(G)$. $\square$

Theorem 2.22. *Let $G = (V, E)$ be a graph and X is a γ'_{in}-set of G. Then $|E - X| \leq \sum_{f \in X} deg_{in}(f)$, where the equality holds if and only if the following conditions are satisfied*

(1) *X is an Inj-independent edge set;*
(2) *for each $f \in E-X$ there exists only one edge $g \in X$ such that $N_{in}(f) \cap X = \{g\}$.*

Proof. Since X is an edge Inj-dominating set of G, then any edge in $E - X$ is Inj-adjacent to at least one edge in X. Therefore each edge in $E-X$ will be counted at least once in the edge Inj-degrees of edges in X. Hence, $|E-X| \leq \sum_{f \in X} deg_{in}(f)$. For the equality, suppose $|E - X| = \sum_{f \in X} deg_{in}(f)$. Then each edge in $E - X$ is counted in the sum exactly once. Thus, if X is not Inj-independent edge set in G, then the sum will exceed $|E - X|$ at least by two. Hence X must be Inj-independent edge set. Suppose now (2) is not satisfied. Then there exists an edge $f \in E - X$ such that $|N_{in}(f) \cap X| \geq 2$. This means that f will be counted in the edge Inj-degrees of edges in X at least twice, which implies that the sum will exceed $|E - X|$ at least by one. Hence, for the equality, conditions (1) and (2) must be hold. The converse is clear. $\square$

Theorem 2.23. *For any graph G, the set Y of Inj-independent edges in G is maximal Inj-independent edge set if and only if it is both Inj-independent edge and edge Inj-dominating set of G.*

Proof. Suppose Y is a maximal Inj-independent edge set of G. Then for each $f \in E - Y$, the set $Y \cup \{f\}$ is not Inj-independent edge set of G. Therefore for each $f \in E - Y$ there exists at least an edge $g \in Y$ such that f is Inj-adjacent to g. Hence Y is both Inj-independent edge and edge Inj-dominating set of G.
Conversely, let Y be both Inj-independent edge and edge Inj-dominating set of G. Suppose Y is not maximal Inj-independent edge set of G. Then there exists an edge $f \in E-Y$ such that $Y \cup \{f\}$ is an Inj-independent edge set, a contradiction, because Y is an edge Inj-dominating set of G. $\square$

Proposition 2.24. *Let $G = (V, E)$ be a graph and $Y \in E$ be both Inj-independent edge and edge Inj-dominating set of G. Then Y is a minimal edge Inj-dominating set of G.*

Proof. Suppose Y is not minimal edge Inj-dominating set of G. Then for some $g \in Y$ the set $Y - \{g\}$ is an edge Inj-dominating set of G, then there exists at least an edge g' in Y such that g is Inj-adjacent to g', a contradiction, because Y is Inj-independent edge set. $\square$

Corollary 2.25. *For any graph G, any maximal Inj-independent edge set is a minimal edge Inj-dominating set.*

Theorem 2.26. *For any graph G on q edges,*

$$\left\lceil \frac{q}{\Delta'_{in}(G)+1} \right\rceil \leq {\gamma'}_{in}(G) \leq q - \beta'_{in}(G) + q_0,$$

where q_0 is the number of Inj-isolated edges of G.

Proof. Let G be a graph with q edges and X be a ${\gamma'}_{in}$-set of G. From Theorem 2.22, we have $|E-X| \leq \sum_{f\in X} deg_{in}(f) \leq {\gamma'}_{in}(G)\Delta'_{in}(G)$. Therefore $q-{\gamma'}_{in}(G) \leq {\gamma'}_{in}(G)\Delta'_{in}(G)$. Hence, $\left\lceil \frac{q}{\Delta'_{in}(G)+1} \right\rceil \leq {\gamma'}_{in}(G)$.
For the upper bound, we characterize two cases:
Case 1. Assume that G has no P_4 component. Suppose $H = \langle E(G) - I'_{in}\rangle$, where I'_{in} is the set of Inj-isolated edges of G. Then H has no Inj-isolated edges. Suppose now, Y is a maximal Inj-independent edge of H. Then by Corollary 2.25, Y is a minimal edge Inj-dominating set of H. Since H has no Inj-isolated edges, then by Theorem 2.18, $E(H) - Y$ is an edge Inj-dominating set of H. Therefore ${\gamma'}_{in}(H) \leq |E(H) - Y| = q(H) - \beta'_{in}(H)$. But in this case, ${\gamma'}_{in}(H) = {\gamma'}_{in}(G) - q_0$, $q(H) = q - q_0$ and $\beta'_{in}(H) = \beta'_{in}(G) - q_0$. Hence, ${\gamma'}_{in}(G) \leq q - \beta'_{in}(G) + q_0$.
Case 2. Assume that G has a P_4 component. Suppose that $G = P_4 \cup G'$. Then $q = 3 + q(G')$, $\beta'_{in}(G) = 2 + \beta'_{in}(G')$, $q_0 = 1 + q_0(G')$ and ${\gamma'}_{in}(G) = {\gamma'}_{in}(P_4) + {\gamma'}_{in}(G') = 2 + {\gamma'}_{in}(G')$. Clearly that, G' has no P_4 component, then by Case 1, ${\gamma'}_{in}(G') \leq q(G') - \beta'_{in}(G') + q_0(G')$. Hence,

$${\gamma'}_{in}(G) = 2 + {\gamma'}_{in}(G') \leq q(G') - \beta'_{in}(G') + q_0(G') + 2 = q - \beta'_{in}(G) + q_0.$$

□

3. EDGE INJECTIVE DOMATIC NUMBER OF GRAPHS

Definition 3.1. Let $G = (V, E)$ be a graph. The maximum partition of the edge set E into edge injective dominating sets is called edge injective domatic number (edge Inj-domatic number) of the graph G, and denoted by ${d'}_{in}(G)$.

Theorem 3.2.

(1) *For any (p, q)-connected graph G, ${d'}_{in}(G) = q$ if and only if for each edge $f \in E(G)$, $N(f) = N_{cn}(f)$ and $e(f) \leq 2$.*
(2) *${d'}_{in}(G) = 1$ if and only if G has at least one Inj-isolated edge.*
(3) *For any complete bipartite graph $K_{r,m}$,*

$${d'}_{in}(K_{r,m}) = \begin{cases} m, & \text{if } r = 2; \\ r, & \text{if } m = 2; \\ rm, & \text{otherwise.} \end{cases}$$

(4) *For any graph G, if $N_{in}(f) = N(f)$ for any edge f in $E(G)$, then*

$${d'}_{in}(G) = d'(G).$$

Proof.

(1) The proof is straightforward by Proposition 2.10.
(2) Suppose G be a graph with an Inj-isolated edge say f. Then f must be contained in each edge Inj-dominating set of G. Hence, $d'_{in}(G) = 1$. Conversely, suppose $d'_{in}(G) = 1$ and G has no Inj-isolated edge. Then by Theorem 2.18, for any minimal edge Inj-dominating set X the set $E - X$ is also edge Inj-dominating set of G. Thus $d'_{in}(G) \geq 2$, a contradiction. Hence, G must contain at least an Inj-isolated edge.

(3) and (4) the proof is obvious. □

Proposition 3.3. *For any graph G, $d'_{in}(G) \geq d'_{cn}(G)$.*

Proof. Let $G = (V, E)$ be a graph. Since any CN-edge dominating set of G is an edge Inj-dominating set. Then any partition of the edge set $E(G)$ into CN-edge dominating sets is also a partition of E into edge Inj-dominating sets. Hence, $d'_{in}(G) \geq d'_{cn}(G)$. □

Theorem 3.4. *For any (p, q)-graph G, $d'_{in}(G) \leq \dfrac{q}{\gamma'_{in}(G)}$.*

Proof. Suppose $d'_{in}(G) = d$ and $\{D_1, D_2, \ldots, D_d\}$ is a partition of $E(G)$ into d edge Inj-dominating sets. It is clear that $|D_i| \geq \gamma'_{in}(G)$, $\forall\, i = 1, 2, \ldots, d$ and we have $q = \sum_{i=1}^{d} |D_i| \geq d\gamma'_{in}(G)$. Hence, $d'_{in}(G) \leq \frac{q}{\gamma'_{in}(G)}$. □

Theorem 3.5. *For any (p, q)-graph G, $d'_{in}(G) \geq \left\lfloor \dfrac{q}{q - \delta'_{in}(G)} \right\rfloor$.*

Proof. Suppose X be any subset of edges of G such that $|X| \geq q - \delta'_{in}(G)$. Since for any edge $f \in E - X$, $|N_{in}[f]| \geq 1 + \delta'_{in}(G)$, then $N_{in}(f) \cap X \neq \phi$. Therefore X is an edge Inj-dominating set of G. So, we can take any $\left\lfloor \frac{q}{q - \delta'_{in}(G)} \right\rfloor$ disjoint subsets each of cardinality $q - \delta'_{in}(G)$. Hence,

$$d_{in}(G) \geq \left\lfloor \frac{q}{q - \delta'_{in}(G)} \right\rfloor.$$

□

Theorem 3.6. *For any (p, q)-graph G, $d'_{in}(G) \leq \delta'_{in}(G) + 1$.*

Proof. For contrary, suppose $d'_{in}(G) > \delta'_{in}(G) + 1$. Then there exist at least $\delta'_{in}(G) + 2$ edge Inj-dominating sets which they are mutually disjoint. Let f be any edge in $E(G)$ such that $deg_{in}(f) = \delta'_{in}(G)$. Then there is at least one edge Inj-dominating set which has no intersection with $N_{in}[f]$. Hence, that edge Inj-dominating set can not dominate f, a contradiction. Therefore $d'_{in}(G) \leq \delta'_{in}(G) + 1$. □

Theorem 3.7. *For any (p, q)-graph G, $d'_{in}(G) + d'_{in}(\overline{G}^{inj}) \leq q + 1$.*

Proof. By Theorem 3.6, we have $d'_{in}(G) \leq \delta'_{in}(G) + 1$ and $d'_{in}(\overline{G}^{inj}) \leq \delta'_{in}(\overline{G}^{inj}) + 1$. Since $\delta'_{in}(\overline{G}^{inj}) = q - 1 - \Delta'_{in}(G)$. Hence,

$$d'_{in}(G) + d'_{in}(\overline{G}^{inj}) \leq \delta'_{in}(G) + q - \Delta'_{in}(G) + 1 \leq q + 1.$$

□

Theorem 3.8. *For any (p, q)-graph G, $d'_{in}(G) + \gamma'_{in}(G) \leq q + 1$.*

Proof. By Theorem 2.21, we have

$$\gamma'_{in}(G) \leq q - \Delta'_{in}(G) \leq q - \delta'_{in}(G),$$

and also from Theorem 3.6, $d'_{in}(G) \leq \delta'_{in}(G) + 1$. Hence,

$$d'_{in}(G) + \gamma'_{in}(G) \leq \delta'_{in}(G) + 1 + q - \delta'_{in}(G) = q + 1.$$

□

References

[1] Akram Alqesmah, Anwar Alwardi and R. Rangarajan, *Connected injective domination of graphs*, Bulletin of the International Mathematical Virtual Institute, **7 (1)** (2017), 73–83.
[2] A. Alwardi, B. Arsic, I. Gutman, N. D. Soner, *The common neighborhood graph and its energy*, Iran. J. Math. Sci. Inf., **7 (2)** (2012), 1–8.
[3] A. Alwardi and N. D. Soner, *CN-edge domination in graphs*, Vladikakazian Mathematical Journal, **15 (2)** (2013), 12–18.
[4] Anwar Alwardi, Akram Alqesmah and R. Rangarajan, *Independent injective domination of graphs*, Int. J. Adv. Appl. Math. and Mech. **3 (4)** (2016), 142-151.
[5] Anwar Alwardi, R. Rangarajan and Akram Alqesmah, *On the Injective domination of graphs*, Palestine Journal of Mathematics, In press.
[6] F. Harary, *Graph theory*, Addison-Wesley, Reading Mass. (1969).
[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc., New York (1998).
[8] S. M. Hedetneimi, S. T. Hedetneimi, R. C. Laskar, L. Markus and P. J. Slater, *Disjoint dominating sets in graphs*, Proc. Int. Conf. on Disc.Math., IMI-IISc, Bangalore (2006), 88–101.
[9] S. Mitchell and S. T. Hedetniemi, *Edge domination in trees*, Congr. Numer. **19** (1977), 489-509.

[1] Department of Studies in Mathematics, University of Mysore, Mysore 570 006, India.

E-mail address: aalqesmah@gmail.com, a_wardi@hotmail.com, rajra63@gmail.com