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MULTIVALUED FIXED POINT THEOREMS IN CONE METRIC SPACES OVER BANACH ALGEBRAS

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Abstract. In this paper, we prove some fixed point results for generalized multivalued contraction type mappings in the setting of H-cone metric in cone metric spaces over Banach algebras by assuming a weaker condition on generalized contraction constants by means of spectral radius.

1. Introduction and preliminaries

Banach contraction principle plays an important role in metric fixed point theory, because of its importance this principle has been extended in many directions, especially the generalizations to multivalued cases. In 1969, Nadler [7] introduced the notion of multivalued contraction in the setting of metric spaces and extended Banach contraction principle to such contractions. Then many authors tried to generalize and to extend multivalued contractions [see for example [10], [8], [2]]. The concept of cone metric spaces was first introduced by Huang and Zhang [4]. Observe that cone metric space is a generalization of a metric space in which the domain of the distance function is replaced by a real Banach algebra. Recently Liu and Xu [5] introduced the concept of cone metric space over Banach algebra, replacing Banach space by Banach algebra as the underlying space of cone metric space. In this way they proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constant k by means of spectral radius $r(k)$. It is significant to introduce the concept of cone metric space over Banach algebra, since the cone metric space over Banach algebra is not equivalent to metric space in terms of fixed point theorems for generalized Lipschitz mappings. Here we need the following.

A real algebra $\mathcal{A}$ is a linear space whose vectors can be multiplied in such a way that $(xy)z = x(yz)$, $x(y+z) = xy+xz$ and $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every scalar α. A real *Banach algebra* $\mathcal{A}$ is a real Banach space which is also an algebra with identity e, and in which the multiplication structure is related to the norm by the requirements $\|xy\| \leq \|x\|\|y\|$ and $\|e\| = 1$. An element x in a real Banach algebra $\mathcal{A}$ is said to be invertible if there exists $y \in \mathcal{A}$ such that $xy = yx = e$, the inverse of x is denoted by x^{-1}.

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Let $\mathcal{A}$ be a real Banach algebra and $x \in \mathcal{A}$, then the spectrum of x is given by $\sigma(x)\colon = \{\lambda : x - \lambda e \text{ is singular}\}$, the spectral radius r of x is defined as $r(x) = sup\{|\lambda| : \lambda \in \sigma(x)\} = \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}}$. Let $\mathcal{A}$ be a Banach algebra with identity e, and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, then $e - x$ is invertible and $(e-x)^{-1} = \sum_{i=1}^{\infty} x^i$. See [3] for more details.

Let $\mathcal{A}$ be a real Banach algebra. A nonempty subset P of $\mathcal{A}$ is called a cone if, P is closed and $\{0_{\mathcal{A}}, e\} \subset P$, $\alpha P + \beta P \subset P$, for all non negative real numbers α, β, $P^2 \subset P$ and $P \cap -P = \{0_{\mathcal{A}}\}$, see [5].

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ on $\mathcal{A}$ with respect to P by

$$x \preceq y \text{ if and only if } y - x \in P.$$

The notation $x \precneqq y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes interior of P and if $int(P) \neq \phi$, then P is called a solid cone. The cone P is called a normal cone if there is a number $M > 0$ such that, for all $x, y \in \mathcal{A}$,

$$0_{\mathcal{A}} \preceq x \preceq y \Rightarrow \| x \| \leq M \| y \| .$$

The least positive number satisfying the above inequality is called the normal constant of P. Let X be a non-empty set, suppose that the mapping $d : X \times X \to \mathcal{A}$, satisfies; for all $x, y, z \in \mathcal{A}$, $0_{\mathcal{A}} \preceq d(x, y)$ and $d(x, y) = 0_{\mathcal{A}}$, if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, y) \preceq d(x, z) + d(z, y)$. Then d is called a cone metric on X and (X, d) is called a cone metric space with Banach algebra $\mathcal{A}$. Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X. We say that

(1) $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $0_{\mathcal{A}} \ll c$ there is $n \in \mathbb{N}$ such that $d(x_n, x) \ll c, \forall\ n \geq N$. We write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.
(2) $\{x_n\}$ is said to be a Cauchy sequence for each $c \in \mathcal{A}$ with $0_{\mathcal{A}} \ll c$ there is $k \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$, for all $n, m \geq k$.

A sequence $\{x_n\}$ in P is said to be a c-sequence if for each $c \gg 0_{\mathcal{A}}$ there exists $n_0 \in \mathbb{N}$ such that $x_n \ll c$ for all $n \geq n_0$. Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ with cone P. The following properties will be used very often, [see [11]]

(1) If $\{x_n\}$ and $\{y_n\}$ are two c-sequences in P, then $\{\alpha x_n + \beta y_n\}$ is also a sequence for all real α and β. If $k \in P$, then $\{k x_n\}$ is also a c-sequence in P.
(2) If $\{x_n\}$ converges to $x \in X$, then $\{d(x_n, x)\}$ and $\{d(x_n, x_{n+m})\}$ are also c-sequences.
(3) If $h \in P$ with $r(h) < 1$, then the sequence $\{x_n\}$ defined by $x_n = h^n, n \in \mathbb{N}$ is a c-sequence.

Lemma 1.1 (see [11]). *Let x, y be vectors in $\mathcal{A}$. If x and y commute, then the spectral radius r satisfies the following properties:*

(1) $r(xy) \leq r(x)r(y)$
(2) $r(x + y) \leq r(x) + r(y)$
(3) $\| r(x) - r(y) \| \leq r(x - y)$

(4) $0 \leq r(x) \leq 1$, *then* $e - x$ *is invertible and* $r((e-x)^{-1}) \leq (1 - r(x))^{-1}$.

Lemma 1.2 (see [11])**.** *Let $\mathcal{A}$ be a Banach algebra, let $\{x_n\}$ be a sequence in $\mathcal{A}$. Suppose that $\{x_n\}$ converges to x and $\{x_n\}$ and x commutes for all n, then we have $\{r(x_n)\}$ converges to $r(x)$ as $n \to \infty$.*

Definition 1.3 (see [10])**.** Let (X, d) be a cone metric space over a Banach space E with solid cone P and let $\mathcal{N}(X)$ be a collection of nonempty subsets of X. A mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \to E$ is called an $H-$cone metric over a Banach space E with respect to (X, d) if for any $A_1, A_2 \in \mathcal{N}(X)$ the following conditions hold:

(H1) $H(A_1, A_2) = 0 \Rightarrow A_1 = A_2$.
(H2) $H(A_1, A_2) = H(A_2, A_1)$.
(H3) for all $\varepsilon \in E$ with $\varepsilon \gg 0$ and for all $x \in A_1$, there exists at least one $y \in A_2$ such that $d(x, y) \preceq H(A_2, A_1) + \varepsilon$.
(H4) One of the following holds:
  (1) for all $\varepsilon \in E$ with $\varepsilon \gg 0$ there is at least one $x \in A_1$, such that $H(A_1, A_2) \preceq d(x, y) + \varepsilon$ for all $y \in A_2$.
  (2) for all $\varepsilon \in E$ with $\varepsilon \gg 0$ there is at least one $x \in A_2$, such that $H(A_1, A_2) \preceq d(x, y) + \varepsilon$ for all $y \in A_1$.

2. Main results

In this section we use $H-$ cone metric over a Banach algebra $\mathcal{A}$ with respect to the cone metric space (X, ρ) over $\mathcal{A}$.

Definition 2.1. Let (X, ρ) be a cone metric space over $\mathcal{A}$. A mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \to \mathcal{A}$ is called an $H-$ cone metric over $\mathcal{A}$ if it satisfies the conditions (H1-H4) in the definition 1.3.

Example 2.2. Let $\mathcal{A} = \mathbb{R}^2$, with $\| (x_1, x_2) \| = |x_1| + |x_2|$. The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_1 + x_1 y_2).$$

Then $\mathcal{A}$ is a Banach algebra with unit $(1, 0)$.

Let $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$, then P is a normal cone with normal constant $M = 1$.

Let $X = \mathbb{R}^2$, cosider the metric ρ defined by

$$\rho((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, \alpha|x_2 - y_2|),\ \alpha \in \mathbb{R}, \alpha > 0.$$

Then (X, ρ) is a complete cone metric space with Banach algebra.

Let $u \otimes v = \{(x, y) \in X : 0 \leq x \leq u, 0 \leq y \leq v\}$. Let $\mathcal{N}(X) = \{u \otimes v : u, v \geq 0\}$, then the mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \to \mathcal{A}$ defined by $H(u_1 \otimes v_1, u_2 \otimes v_2) = (|u_1 - u_2|, \alpha|v_1 - v_2|)$, $\alpha \in \mathbb{R}, \alpha > 0$, is an H-cone metric over $\mathcal{A}$ with respect to (X, ρ).

Remark 2.3 (see[10]). If a mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \to \mathcal{A}$ is an $H-$ cone metric over $\mathcal{A}$, then $(\mathcal{N}(X), H)$ is a cone metric space over $\mathcal{A}$.

In this sequel we prove some multivalued fixed point results for mappings satisfying some generalized multivalued contractive type conditions in cone metric spaces over Banach algebras.

Theorem 2.4. *Let (X, ρ) be a complete cone metric space with Banach algebra $\mathcal{A}$, P be a cone in $\mathcal{A}$, let $C(X)$ be a non empty collection of nonempty closed subsets of X. Let $A : X \to C(X)$ be a multivalued mapping satisfying,*

$$H(Ax, Ay) \preceq \lambda[\rho(x, Ax) + \rho(y, Ay)], \text{ for all } x, y \in X,$$

where $\lambda \in P$ with $r(\lambda) \in [0, \frac{1}{2})$. Then A has fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in Ax_0$. Observe that if $x_0 = x_1$, then x_0 is a fixed point. Assume that $x_0 \neq x_1$, then by $(H3)$, for $0 \ll \eta_n$ with $\eta_n \preceq L^{2n}$, where $L = (e - \lambda)^{-1}\lambda$, we have $x_2 \in Ax_1$ such that

$$\begin{aligned} \rho(x_1, x_2) &\preceq H(Ax_1, Ax_0) + \eta_1 \\ &\preceq \lambda[\rho(x_1, Ax_1) + \rho(x_0, Ax_0)] + \eta_1. \end{aligned} \tag{2.1}$$

Again by (H3) we can choose $x_3 \in Ax_2$, such that

$$\begin{aligned} \rho(x_2, x_3) &\preceq H(Ax_2, Ax_1) + \eta_2 \\ &\preceq \lambda[\rho(x_1, Ax_1) + \rho(x_2, Ax_2)] + \eta_2. \end{aligned} \tag{2.2}$$

Continuing this process we obtain a sequence $\{x_n\} \in X$ satisfying,

$$\rho(x_n, x_{n+1}) \preceq \lambda[\rho(x_n, Ax_n) + \rho(x_{n-1}, Ax_{n-1})] + \eta_n, \tag{2.3}$$

where $x_{n+1} \in Ax_n$ for $n = 0, 1, 2, \dots$. Then it follows that,

$$\begin{aligned} \rho(x_n, x_{n+1}) &\preceq \lambda[\rho(x_n, x_{n+1}) + \rho(x_{n-1}, x_n)] + \eta_n \\ &= \lambda(e - \lambda)^{-1}\rho(x_{n-1}, x_n)] + (e - \lambda)^{-1}\eta_n \\ &\preceq L^2\rho(x_{n-2}, x_{n-1}) + L(e - \lambda)^{-1}\eta_{n-1} + (e - \lambda)^{-1}\eta_n \\ &\vdots \\ &\preceq L^n\rho(x_1, x_0) + (e - \lambda)^{-1}\sum_{i=1}^{n} L^{n-i}\eta_i \\ &\preceq L^n\rho(x_1, x_0) + (e - \lambda)^{-1}\sum_{i=1}^{n} L^{n-i}L^{2i} \\ &\preceq L^n\rho(x_1, x_0) + (e - \lambda)^{-1}L^{n+1}(e - L)^{-1}. \end{aligned} \tag{2.4}$$

Letting $\beta = \rho(x_1, x_0) + (e - \lambda)^{-1}L^1(e - L)^{-1}$, then equation (2.4) can be written as

$$\rho(x_n, x_{n+1}) \preceq L^n\beta, \tag{2.5}$$

where $L = (e - \lambda)^{-1}\lambda$, and $r(L) = r((e - \lambda)^{-1}\lambda) \leq \frac{r(\lambda)}{1 - r(\lambda)} < 1$. Now for $m > n$, (2.5) gives

$$\rho(x_n, x_m) \preceq L^n(e - L)^{-1}\beta. \tag{2.6}$$

Since L^n converges to $0_{\mathcal{A}}$ as $n \to \infty$, we have $L^n(e - L)^{-1}\beta \to 0_{\mathcal{A}}$ as $n \to \infty$. Thus we have for every $c \in P$ with $0_{\mathcal{A}} \ll c$, there is $n_1 \in \mathbb{N}$ such that $\rho(x_n, x_m) \ll c$ for all $m, n \geq n_1$. Thus $\{x_n\}$ is a Cauchy sequence in a complete cone metric space with Banach algebra, hence $\{x_n\}$ is convergent in X, and let $x = \lim_{n\to\infty} x_n$. Hence for every $c \in P$ with $0_{\mathcal{A}} \ll c$, there is $n_2 \in \mathbb{N}$ such that $\rho(x_n, x) \ll c$ for all $n \geq n_2$. So let $k = max\{n_1, n_2\}$, then for every $c \in P$ with $0_{\mathcal{A}} \ll c$, we have,

$$(e-\lambda)^{-1}\rho(x_n, x) \ll \tfrac{c}{3},\ \lambda(e-\lambda)^{-1}\rho(x_n, x_{n-1}) \ll \tfrac{c}{3}, \text{ for all } n \geq k.$$

Now we show that $x \in Ax$. Let $x_n \in Ax_{n-1}$, then by (H3), there is $y_n \in Ax$ satisfying the following,

$$\begin{aligned}\rho(x_n, y_n) &\preceq H(Ax_{n-1}, Ax) + \eta_n \\ &\preceq \lambda[\rho(x_{n-1}, Ax_{n-1}) + \rho(x, Ax)] + \eta_n \\ &\preceq \lambda[\rho(x_{n-1}, x_n) + \rho(x, y_n)] + \eta_n. \end{aligned} \tag{2.7}$$

Then by inequality (2.7), and by the properties of a c-sequence, for all $n \geq k$ we have,

$$\begin{aligned}\rho(x, y_n) &\preceq \rho(x, x_n) + \rho(x_n, y_n) \\ &\preceq \rho(x, x_n) + \lambda[\rho(x_{n-1}, x_n) + \rho(x, y_n)] + \eta_n \\ &\preceq (e-\lambda)^{-1}\rho(x, x_n) + \lambda(e-\lambda)^{-1}[\rho(x_{n-1}, x_n)] + (e-\lambda)^{-1}\eta_n \ll c. \end{aligned} \tag{2.8}$$

Hence $\lim_{n\to\infty} y_n = x$. Since Ax is closed we get $x \in Ax$. Thus A has a fixed point. □

Now we consider a mapping with a slightly different generalized contractive condition.

Theorem 2.5. *Let (X, ρ) be a complete cone metric space with Banach algebra $\mathcal{A}$, P be a cone in $\mathcal{A}$, let $C(X)$ be a non empty collection of nonempty closed subsets of X. Let $A : X \to C(X)$ be a multivalued mapping satisfying,*

$$H(Ax, Ay) \preceq \lambda[\rho(x, Ay) + \rho(y, Ax)], \textit{ for all } x, y \in X,$$

where $\lambda \in P$ with $r(\lambda) \in [0, \frac{1}{2})$. Then A has fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in Ax_0$, if $x_0 = x_1$, then x_0 is a fixed point, so let $x_0 \neq x_1$. Then by $(H3)$, for $0 \ll \eta_n$ with $\eta_n \preceq L^{2n}$, where $L = (e-\lambda)^{-1}\lambda$, we have $x_2 \in Ax_1$ such that,

$$\begin{aligned}\rho(x_1, x_2) &\preceq H(Ax_1, Ax_0) + \eta_1 \\ &\preceq \lambda[\rho(x_1, Ax_0) + \rho(x_0, Ax_1)] + \eta_1. \end{aligned} \tag{2.9}$$

Again by (H3) we can choose $x_3 \in Ax_2$, such that,

$$\begin{aligned}\rho(x_2, x_3) &\preceq H(Ax_2, Ax_1) + \eta_2 \\ &\preceq \lambda[\rho(x_1, Ax_2) + \rho(x_2, Ax_1)] + \eta_2. \end{aligned} \tag{2.10}$$

Continuing this process we obtain a sequence $\{x_n\} \in X$ satisfying,

$$\rho(x_n, x_{n+1}) \preceq \lambda[\rho(x_n, Ax_{n-1}) + \rho(x_{n-1}, Ax_n)] + \eta_n, \tag{2.11}$$

where $x_{n+1} \in Ax_n$ for $n = 0, 1, 2, \ldots$. Then it follows that

$$\begin{aligned}
\rho(x_n, x_{n+1}) &\preceq \lambda[\rho(x_n, x_n) + \rho(x_{n-1}, x_{n+1})] + \eta_n \\
&= \lambda(e-\lambda)^{-1}\rho(x_{n-1}, x_n)] + (e-\lambda)^{-1}\eta_n \\
&\preceq L^2\rho(x_{n-2}, x_{n-1}) + L(e-\lambda)^{-1}\eta_{n-1} + (e-\lambda)^{-1}\eta_n \\
&\vdots \\
&\preceq L^n\rho(x_1, x_0) + (e-\lambda)^{-1}L^{n+1}(e-L)^{-1}. \qquad (2.12)
\end{aligned}$$

Taking $\beta = \rho(x_1, x_0) + (e-\lambda)^{-1}L^1(e-L)^{-1}$, then the inequality (2.12) becomes:

$$\rho(x_n, x_{n+1}) \preceq L^n\beta. \qquad (2.13)$$

Then as in Theorem 2.4, we have $x_n \to x$ and for all $n \geq k$, we get,

$$(e-\lambda)^{-1}(e+\lambda)\rho(x_n, x) \ll \tfrac{c}{3},\ \lambda(e-\lambda)^{-1}\rho(x_n, x_{n-1}) \ll \tfrac{c}{3}.$$

Now we show that $x \in Ax$. Let $x_n \in Ax_{n-1}$, then by (H3), there is $y_n \in Ax$ satisfying the following,

$$\begin{aligned}
\rho(x_n, y_n) &\preceq H(Ax_{n-1}, Ax) + \eta_n \\
&\preceq \lambda[\rho(x_{n-1}, Ax) + \rho(x, Ax_{n-1})] + \eta_n \\
&\preceq \lambda[\rho(x_{n-1}, y_n) + \rho(x, x_n)] + \eta_n. \qquad (2.14)
\end{aligned}$$

Then by inequality (2.14), for all $n \geq k$, we have,

$$\begin{aligned}
\rho(x, y_n) &\preceq \rho(x, x_n) + \rho(x_n, y_n) \\
&\preceq \rho(x, x_n) + \lambda[\rho(x_{n-1}, y_n) + \rho(x, x_n)] + \eta_n \\
&\preceq (e+\lambda)\rho(x, x_n) + \lambda\rho(x_{n-1}, y_n) + \eta_n \\
&\preceq (e+\lambda)\rho(x, x_n) + \lambda[\rho(x_{n-1}, x) + \rho(y_n, x)] + \eta_n \\
&\preceq (e+\lambda)(e-\lambda)^{-1}\rho(x, x_n) + \lambda(e-\lambda)^{-1}\rho(x_{n-1}, x) + (e-\lambda)^{-1}\eta_n. \qquad (2.15)
\end{aligned}$$

Hence $\lim_{n\to\infty} y_n = x$. Since Ax is closed we get $x \in Ax$. Thus A has a fixed point. □

Example 2.6. Let $X = [0, 1]$ and let $\mathcal{A}$ be the set of all real valued functions on X which also have continuous derivative on X, with pointwise multiplication and

$$\| f \| = \sup_{t\in[0,1]} |f(t)|,$$

then $\mathcal{A}$ is a Banach algebra with this norm and $P = \{x : x(t) \geq 0\}$, is a non normal cone in $\mathcal{A}$. Consider the mapping $\rho : X \times X \to \mathcal{A}$ befined by,

$$\rho(x, y)(t) = |x-y|e^t, \text{ for all } x, y \in X \text{ and } t \in [0, 1].$$

Then (X, ρ) is a complete cone metric space with Banach algebra $\mathcal{A}$.

Let $C(X) = \{[0, x] : x \in X\}$ and define $H : C(X) \times C(X) \to \mathcal{A}$ by

$$H([0, x], [0, y]) = |x-y|e^t,$$

then $(C(X), H)$ is an H-cone metric space with Banach algebra $\mathcal{A}$.

Now let $A : X \to C(X)$ by

$$Ax = [0, \tfrac{x}{40}],\ x \in [0, 1].$$

Then

$$H(Ax, Ay) = \left|\frac{x}{40} - \frac{y}{40}\right| e^t = \frac{1}{40}|x-y|e^t. \tag{2.16}$$

Note that

$$\lambda(t)[\rho(x, Ax) + \rho(y, Ay)] = \lambda(t)\left[\frac{39x}{40} + \frac{39y}{40}\right] \succeq \lambda(t)\left[\frac{1}{ke}(\frac{x}{40} + \frac{y}{40})\right]$$

$$\succeq \frac{1}{40}(x+y)e^t \succ \frac{|x-y|}{40}e^t = H(Ax, Ay), \tag{2.17}$$

for all x, y in X and $\lambda(t) = ke^t$, $t \in [0,1]$ and $k \in [\frac{1}{8}, \frac{1}{2e}]$.
Thus A satisfies all the conditions of Theorem 2.4 to obtain a fixed point for A.

Next we consider α- admissible multivalued mappings in cone metric spaces over Banach algebras and prove some fixed point theorems.

Definition 2.7. Let (X, ρ) be a complete cone metric space over Banach algebra with cone P and $A : X \to C(X)$ be a set valued mapping. Then A is said to be α^*-admissible if there is a function $\alpha : X \times X \to P$ such that, whenever $\alpha(x, y) \succeq e$ implies $\alpha^*(Ax, Ay) \succeq e$, where $\alpha^*(Ax, Ay) = inf\{\alpha(a, b) : a \in Ax, b \in Ay\}$.

Theorem 2.8. *Let (X, ρ) be a complete cone metric space with Banach algebra $\mathcal{A}$, P be a cone in $\mathcal{A}$, let $C(X)$ be a non empty collection of nonempty closed subsets of X. Let $A : X \to C(X)$ be an α^*-admissible mapping which satisfies:*

$\alpha^*(Ax, Ay)H(Ax, Ay) \preceq \lambda\rho(x, y)$, *for all* $x, y \in X$, *and* $\lambda \in P$ *with* $r(\lambda) < 1$.

Also suppose that

(1) *there exists* $x_0 \in X$ *and* $x_1 \in Ax_0$ *such that* $\alpha(x_0, x_1) \succeq e$.

(2) *if* $\{x_n\}$ *is a sequence in* X*such that* $\alpha(x_n, x_{n+1}) \succeq e$, *for all* n *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \succeq e$ *for all* n.

Then there exists a point $u \in X$ *such that* $u \in Au$.

Proof. If $x_1 = x_0$, then x_0 is a fixed point. So let $x_1 \neq x_0$, if $x_1 \in Ax_1$, then x_1 is a fixed point. So we take $x_1 \notin Ax_1$. Then

$$0_{\mathcal{A}} \prec d(x_1, Ax_1) \preceq \alpha^*(Ax_0, Ax_1)H(Ax_0, Ax_1). \tag{2.18}$$

Then we can choose $x_2 \in Ax_1$ such that, if $x_2 \neq x_1$,

$$0_{\mathcal{A}} \prec d(x_1, x_2) \preceq \lambda d(x_1, x_0).$$

Now since A is α^*-admissible, we have $\alpha(x_0, x_1) \succeq e$ implies $\alpha^*(Ax_0, Ax_1) \succeq e$, and hence $\alpha(x_1, x_2) \succeq e$, also $\alpha^*(Ax_1, Ax_2) \succeq e$. So if $x_2 \notin Ax_2$ we can choose $x_3 \in Ax_2$ such that,

$$0_{\mathcal{A}} \prec d(x_2, x_3) \preceq \alpha^*(Ax_1, Ax_2)H(Ax_1, Ax_2) \preceq \lambda d(x_1, x_2) \preceq \lambda^2 d(x_1, x_0).$$

Then by continuing this process we obtain a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n+1}, \alpha(x_n, x_{n+1}) \succeq e$ and $d(x_n, x_{n+1}) \preceq \lambda^n d(x_1, x_0)$. Since $\lambda \in P$ with $r(\lambda) < 1$, we have $\}x_n\}$ is a Cauchy sequence in X and since X is a complete cone metric space there is $u \in X$ such that $x_n \to u$.

Now to show that $u \in Au$. Let $x_n \in Ax_{n-1}$, then by (H3), for $0_{\mathcal{A}} \ll \eta_n \preceq \lambda^{2n}$ and by applying the condition (2) in the statement of the theorem, there is $y_n \in Au$ such that

$$\begin{aligned} d(x_n, y_n) &\preceq H(Ax_{n-1}, Au) + \eta_n \\ &\preceq \alpha(x_{n-1}, u)H(Ax_{n-1}, Au) + \eta_n \\ &\preceq \lambda d(x_{n-1}, u). \end{aligned} \tag{2.19}$$

Also

$$\begin{aligned} d(y_n, u) &\preceq d(x_n, u) + d(x_n, y_n) \\ &\preceq d(x_n, u) + \lambda d(x_{n-1}, u) + \eta_n. \end{aligned} \tag{2.20}$$

Then from the inequality (2.20), we have as $n \to \infty$ the sequence y_n converges to u, since Au is a closed subset of X, we get $u \in Au$ and hence A has a fixed point. □

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