Gulf Journal of Mathematics ISSN: 2309-4966
Vol 14, Issue 2 (2023) 77-88 https://doi.org/10.56947/gjom.v14i2.967

GEGENBAUER POLYNOMIALS FOR CERTAIN SUBCLASSES OF BAZILEVIČ FUNCTIONS ASSOCIATED WITH A GENERALIZED OPERATOR DEFINED BY CONVOLUTION

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Abstract. In this paper, a class $\mathcal{G}^{\beta}_{\eta_1,\eta_2}(\alpha,t)$, consisting of Bazilević functions of type α and involving a certain generalized differential operator is defined by means of Gegenbauer polynomials. Initial coefficient bounds and Fekete-Szegó estimates for functions belonging to this class are obtained. Furthermore, upon varying the involving parameters in our main results, a number of known and new results are stated as corollaries.

1. Introduction and preliminaries

Let $U=\{z\in\mathbb{C}:|z|<1\}$ be an open unit disk in $\mathbb{C}$ (Complex plane) and let $G(U)$ be the space of analytic functions in U. For a fixed number m, let $G[b,d,m]$ be the subclass of $G(U)$ of functions of the form

$$g(z)=b+b_{dm}\,z^{dm}+b_{dm+1}\,z^{dm+1}+\cdots, \qquad z\in U, \tag{1.1}$$

where $b\in\mathbb{C}$ and $d\in\mathbb{N}$ with $G\equiv G[0,1]$ and $G\equiv G[1,1]$. We also let $A_1=A$, for $d=1$ and a fixed number $m=1$, denote the usual class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}b_n\,z^n. \tag{1.2}$$

Also, let

$$A_{dm}=\left\{g\in G(U):g(z)=z+b_{dm+1}\,z^{dm+1}+\cdots,\ z\in U\right\}$$

and let S denote the class of all functions in A_{dm} which are univalent in U. Furthermore, let S^* and K respectively denote the class of star-like functions and convex functions in U such that

$$S^*=\left\{g\in S,\ Re\left(\frac{zg'(z)}{g(z)}\right)>0,\ z\in U\right\}$$

Date: Received: Aug 17, 2022; Accepted: Dec 22, 2022.
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2010 *Mathematics Subject Classification.* Primary: 30C45; Secondary: 30C50.
Key words and phrases. Univalent functions, Gegenbauer polynomials, subordination, Fekete-Szegó inequalities, Bazilević functions.

and

$$K = \left\{ g \in S,\ Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0,\ z \in U \right\}.$$

Let g and h be analytic functions in U (that is $g, h \in G(U)$). Then the g is said to be subordinate to h in U, written as $g \prec h$, if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$ such that $g(z) = h(w(z))$. In particular, if h is univalent in U, then we get

$$g(z) \prec h(z) \iff g(0) = h(0) \quad and \quad g(U) \subset h(U).$$

For more insight on subordination refer to [21].

Definition 1.1. [6] For function $g(z) \in A_{dm}$ given by

$$g(z) = z + \sum_{n=dm+1}^{\infty} b_n\, z^n, \quad z \in U, \tag{1.3}$$

the operator $\Im^{\delta}_{\eta_1,\eta_2} : A_{dm} \to A_{dm}$ is defined by

$$\Im^{\delta}_{\eta_1,\eta_2}\, g(z) = z + \sum_{n=dm+1}^{\infty} \left(\frac{1 + (\eta_1 + \eta_2)(l-1)}{1 + \eta_2(l-1)}\right)^{\delta} b_n\, z^n, \quad z \in U \tag{1.4}$$

for $g \in A_{dm}$, m a fixed number, $0 \le \eta_1 \le \eta_2$, $\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $d \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Definition 1.2. [7] Let $g \in A_{dm}$, $d \in \mathbb{N}$, $\delta \in \mathbb{N} \cup \{0\}$ and a fixed number m, the operator R^{δ} is given by

$$R^{\delta}\, g(z) = z + \sum_{n=dm+1}^{\infty} B^{\delta}_{\delta+n-1}\, b_n\, z^n, \quad z \in U. \tag{1.5}$$

where

$$\begin{aligned} B^{\delta}_{\delta+n-1} = B_n(\delta) = B(\delta, n) &= \binom{\delta + n - 1}{\delta} \\ &= \frac{(\delta+1)(\delta+2)\cdots(\delta+n-1)}{(n-1)!} \\ &= \frac{(\delta+1)_{n-1}}{(1)_{n-1}}. \end{aligned}$$

Definition 1.3. [15] Let $g(z)$ given by (1.3) be in A_{dm}, then

$$\begin{aligned} \Im R^{\delta}_{\eta_1,\eta_2}\, g(z) &= (S^{\delta}_{\eta_1,\eta_2} * R^{\delta})\, g(z) \\ &= z + \sum_{n=dm+1}^{\infty} B^{\delta}_{\delta+n-1} \left(\frac{1 + (\eta_1 + \eta_2)(l-1)}{1 + \eta_2(l-1)}\right)^{\delta} b_n\, z^n, \quad z \in U \end{aligned} \tag{1.6}$$

for $0 \le \eta_1 \le \eta_2$, $\delta \in \mathbb{N}_0$, $d \in \mathbb{N}$ and a fixed number m. The symbol * represents the Hadamard product (or convolution). In this case, we have the convolution of the Sălăgean operator $\Im^{n}_{\eta_1,\eta_2}$ and the Ruscheweyh operator R^{δ}.

Definition 1.4. [8] A function $f \in A$ of the form (1.2) belongs to the class $\mathcal{G}(\alpha, t)$, if it satisfies the subordination condition

$$\mathcal{G}(\alpha,t) = \left\{ f \in A : \frac{(D^n f(z))' \, (D^n f(z))^{\alpha-1}}{z^{\alpha-1}} \prec H(z,t) = \frac{1}{1+2tz+z^2} \right\}, \quad (1.7)$$

where $\alpha \in [0,1]$, $\frac{1}{2} < t \leq 1$, $z \in U$ and $H(z,t)$ is the Chebyshev polynomials of the second kind. Chebyshev polynomials are generally of four kinds. They are special functions belonging to the family of orthogonal polynomials and their significance in numerical analysis has increased in both theoretical and practical point of view, see [8, 20]. The Chebyshev polynomials of first and second kinds are denoted by $T_n(t)$ and $U_n(t)$ and respectively defined by

$$T_n(t) = \cos n\theta \qquad \text{and} \qquad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}$$

for all $t \in [-1,1]$, $t = \cos\theta$ and n is the degree of the polynomial. For some of the works that are related to Chebyshev polynomials of the second kind, interested readers are referred to [6, 14, 16, 25, 26, 30]. Another special case of orthogonal polynomials is Gegenbauer polynomials. They are representatively related to the class T_R of typically real functions, discovered in [17]. Real-valued functions play an important role in geometric function theory because of the relation $T_R = \overline{Co}S_R$ and its role in estimating coefficient bounds where S_R denotes the class of univalent functions in the unit disk with real coefficients and $T_R = \overline{Co}S_R$ denotes the closed convex ball of S_R. Orthogonal polynomials generally have been studied extensively as early as they were discovered by Legendre in 1784, see [13, 19]. Recently in [5], Amourah considered the generating function of Gegenbauer polynomials $H_\beta(x,z)$, which is given by the recurrence relation

$$H_\beta(x,z) = \frac{1}{(1-2xz+z^2)^\beta}, \quad z \in U, \quad (1.8)$$

for non-zero real constant β and $x \in [-1,1]$. For fixed β, the function H_β is analytic in U and it is expressible in Taylor series as

$$H_\beta(x,z) = \sum_{n=0}^{\infty} C_n^\beta(x)\, z^n \quad (1.9)$$

where $C_n^\beta(x)$ is Gegenbauer polynomial of degree n. It is obvious that H_β generates nothing when $\beta = 0$ and consequently, the generating function of the Gegenbauer polynomial is set to be

$$H_0(x,z) = 1 - \log(1-2xz+z^2) = \sum_{n=0}^{\infty} C_n^0(x)\; z^n \quad \mathit{for}\ \beta = 0. \quad (1.10)$$

Moreover, a normalization of α to be greater than $-\frac{1}{2}$ is desirable (see [9, 29]).

Another form of definition for the Gegenbauer polynomial of degree n is

$$C_n^\beta(x) = \frac{1}{n}\left[2x(n+\beta-1)\, C_{n-1}^\beta(x) - (n+2\beta-2)\, C_{n-1}^\beta(x)\right] \quad (1.11)$$

with the initial values

$$C_0^\beta(x) = 1, \quad C_1^\beta(x) = 2\beta x \quad and \quad C_2^\beta(x) = 2\beta(1+\beta)x^2 - \beta. \tag{1.12}$$

With respect to the Gegenbauer polynomials $C_n^\beta(x)$, we note the following special cases:

(i) when $\beta = 1$, we get the Chebyshev polynomials and
(ii) when $\beta = \frac{1}{2}$, we get the Legendre polynomials.

Interested readers may consult some further works involving Gegenbauer polynomials in [2, 3, 4, 23, 31, 32, 33, 34].

1.1. **Problem Statement.** The problem of finding the sharp bounds for the non-linear functional $|b_3 - \mu b_2^2|$ for Taylor-Maclaurin series popularly known as the famous Fekete-Szegó problem (or inequality) has a rich historical antecedent in geometric function theory. For its source see [10]. The Fekete-Szegó problem, no doubt, has since received great attention by many researchers especially in many subclasses of normalized univalent functions. Some of the papers that have appeared in the literature include [6, 12, 14, 16, 18, 24, 25, 26, 27, 28, 30].

The main goal of this present work is to explore univalent functions associated with Gegenbauer polynomials. Our motivation is the work of AbdulRahman et al. [1]. In particular, the aim of this work is to provide estimates for the initial coefficients of Bazilević functions of type α in the class $\mathcal{G}^\beta_{\eta_1,\eta_2}(\alpha,t)$, involving the operator $\Im R^\delta_{\eta_1,\eta_2} g(z)$ given by (1.6) defined by the Hadamard product of the Sălăgean operator $S^\delta_{\eta_1,\eta_2}$ and the Ruschewegh operator R^n. In addition, the problem of Fekete-Szegó in this class is also considered.

As a prelude to our main results, we shall need the following definition and lemma.

Definition 1.5. A function $g(z) \in A_{dm}$ of the form (1.3) belongs to the class $\mathcal{G}^\beta_{\eta_1,\eta_2}(\alpha,t)$, if it satisfies the subordination condition

$$\mathcal{G}^\beta_{\eta_1,\eta_2}(\alpha,t) = \left\{ g \in A_{dm} : \frac{(\Im R^\delta_{\eta_1,\eta_2} g(z))'(\Im R^\delta_{\eta_1,\eta_2} g(z))^{\alpha-1}}{z^{\alpha-1}} \prec H_\beta(t,z) = \frac{1}{(1-2tz+z^2)^\beta} \right\} \tag{1.13}$$

for $0 \le \alpha \le 1,\ 0 \le \eta_1 \le \eta_2,\ d \in \mathbb{N} = \{1,2,\cdots\},\ \delta \in \mathbb{N} = \mathbb{N} \cup \{0\},\ \beta, \alpha$ are real constants where $t \in [-1,1]$ *and* $z \in U$.

Let $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \in \Omega$, where Ω is the class of Schwarz functions.

Lemma 1.6. [12] *If $w \in \Omega$, then for any complex number μ,*

$$|w_2 - \mu w_1^2| \le \{1, |\mu|\}.$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

2. Main results

Theorem 2.1. *Let $g \in A_{dm}$ belong to the class $\mathcal{G}^{\beta}_{\eta_1,\eta_2}(\alpha,t)$. Then*

$$|b_{m+1}| \leq \frac{2\beta t}{(\alpha+m)B_{\delta+m}(\delta)(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\lambda_2(l-1)})^{\delta}}$$

and

$$\begin{aligned}|b_{2m+1}| \leq & \frac{2\beta(1+\beta)t^2+2\beta t-\beta}{(\alpha+2m)B_{\delta+2m}(\delta)(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)})^{\delta}} \\ & + \frac{4\beta^2t^2}{(\alpha+m)(\alpha+2m)B_{\delta+2m}(\delta)(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\lambda_2(l-1)})^{\delta}} \\ & - \frac{4\alpha\beta^2t^2(m+1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+2m}(\delta)(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)})^{\delta}} \\ & - \frac{2\beta^2t^2\alpha(\alpha-1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+m}(\delta)B_{\delta+2m}(\delta)(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\lambda_2(l-1)})^{\delta}}.\end{aligned}$$

Proof. If $g \in \mathcal{G}^{\beta}_{\eta_1,\eta_2}(\alpha,t)$, then from (1.13), we have

$$\begin{aligned}&\frac{(\Im R^{\delta}_{\eta_1,\eta_2}\, g(z))'(\Im R^{\delta}_{\eta_1,\eta_2} g(z))^{\alpha-1}}{z^{\alpha-1}} \\ &\qquad = 1 + C_1^{\beta}(t)c_1 z + [C_1^{\beta}(t)c_2 + C_2^{\beta}(t)c_1^2]z^2 + \cdots, \quad (2.1)\end{aligned}$$

making use of the series expansion of $\Im R^{\delta}_{\eta_1,\eta_2}\, g(z)$ and $(\Im R^{n}_{\eta_1,\eta_2}\, g(z))'$ appropriately in (2.1) we get

$$\begin{aligned}&\frac{\left[z + \sum_{n=dm+1}^{\infty} nC^{\delta}_{\delta+n-1}\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta} b_n\, z^n\right]\left[1 + \sum_{n=dm+1}^{\infty} n\, C^{n}_{\delta+n-1}\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta} b_n(\alpha)\, z^{n-1}\right]}{z + \sum_{n=dm+1}^{\infty} n\, C^{\delta}_{\delta+n-1}\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta} b_n\, z^n} \\ &\qquad = 1 + C_1^{\beta}(t)c_1 z + [C_1^{\beta}(t)c_2 + C_2^{\beta}(t)c_1^2]z^2 + \cdots. \quad (2.2)\end{aligned}$$

Making use of binomial expansion and upon simplification, we get

$$1 + \sum_{n=dm+1}^{\infty} n\, C^{\delta}_{\delta+n-1}\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta} b_n\, z^{n-1},$$

and

$$\Bigg\{z + (m+1)\,B_{\delta+m}(\delta)\,b_{m+1}\,z^{m+1}\,\left(\frac{1+(\eta_1+\eta_2)(-1)}{1+\eta_2(l-1)}\right)^{\delta}$$
$$+ (2m+1)\,B_{\delta+2m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}\,b_{2m+1}\,z^{2m+1} + \cdots\Bigg\}$$
$$\times\Bigg\{1 + \alpha\,B_{\delta+m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}\,b_{m+1}\,z^{m}$$
$$+ \left[\alpha\,B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}\,b_{2m+1} + \frac{\alpha(\alpha-1)}{2!}\,B_{\delta+m}(\delta)b_{m+1}^{m+1}\right]z^{m+1} + \cdots\Bigg\}$$

$$= \Bigg\{1 + C_1^{\beta}(t)c_1 z + [C_1^{\beta}(t)c_2 + C_2^{\beta}(t)c_1^2]z^2 + \cdots\Bigg\}$$
$$\times\Bigg\{z + B_{\delta+m}(\delta)b_{m+1}z^{m+1}\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{n}$$
$$+ B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}\,b_{2m+1}z^{2m+1} + \cdots\Bigg\}. \quad (2.3)$$

For analytic functions

$$p(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots\,. \quad (2.4)$$

It is fairly well known that if $|p(z)| < 1$, $(z \in U)$, then

$$|c_j| \le 1 \quad for\ \ all\ \ j \in \mathbb{N}. \quad (2.5)$$

Thus upon comparing the corresponding coefficients in (2.3), we have

$$\alpha\,B_{\delta+m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}\,b_{m+1}$$
$$+ m\,B_{\delta+m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}\,b_{m+1} = C_1^{\beta}(t)c_1$$

which implies that

$$(\alpha+m)\,B_{\delta+m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}\,b_{m+1} = C_1^{\beta}(t)c_1$$

so that

$$b_{m+1} = \frac{C_1^{\beta}(t)c_1}{(\alpha+m)\,B_{\delta+m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} = \frac{2\beta t c_1}{(\alpha+m)\,B_{\delta+m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \quad (2.6)$$

since $C_1^{\beta}(t) = 2\beta t$. By applying (2.5) in (2.6), we have

$$|b_{m+1}| \le \frac{2\beta t}{(\alpha+m)\,B_{\delta+m}(\delta)\,\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(j-i)}\right)^{\delta}}.$$

Next, we now find the bound on $|b_{2m+1}|$ by using b_{m+1} after appropriately comparing the coefficients b_{2m+1}, we have that

$$\begin{aligned}
b_{2m+1} =& \frac{C_1^{\beta}(t)c_2 + C_2^{\beta}(t)c_1^2}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(j-i)}{1+\eta_2(l-1)}\right)^n} \\
&+ \frac{(C_1^{\beta}(t))^2c_1^2}{(\alpha+m)(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&- \frac{C_1^{\beta}(t))^2c_1^2\alpha(m+1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&- \frac{\alpha(\alpha-1)(C_1^{\beta}(t))^2c_1^2}{2(\alpha+m)^2(\alpha+2m)B_{\delta+m}(\delta)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}
\end{aligned}$$

and since $C_1^{\beta}(t) = 2\beta t$ and $C_2^{\beta}(t) = 2\beta(1+\beta)t^2 - \beta$ we get

$$\begin{aligned}
b_{2m+1} =& \frac{\big(2\beta(1+\beta)t^2-\beta\big)c_1^2 + 2\beta t c_2}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&+ \frac{(2\beta t)^2c_1^2}{(\alpha+m)(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&- \frac{(2\beta t)^2c_1^2\alpha(m+1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&- \frac{\alpha(\alpha-1)(2\beta t)^2c_1^2}{2(\alpha+m)^2(\alpha+2m)B_{\delta+m}(\delta)B_{n+2m}(n)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}.
\end{aligned} \tag{2.7}$$

By applying (2.5) on (2.7) we get

$$\begin{aligned}
|b_{2m+1}| \leq& \frac{2\beta(1+\beta)t^2 + 2\beta t - \beta}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&+ \frac{4\beta^2t^2}{(\alpha+m)(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&- \frac{4\alpha\beta^2t^2(m+1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\
&- \frac{2\beta^2t^2\alpha(\alpha-1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+m}(\delta)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}.
\end{aligned}$$

□

Theorem 2.2. *If $g(z)$ of the form (1.3) belongs to the class $\mathcal{G}^{\beta}_{\lambda_1,\lambda_2}(\alpha,t)$, then*

$$|b_{2m+1}-\mu b_{m+1}^2| \le \frac{2\beta t}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \max\left\{1, \left|\frac{2(1+\beta)t^2-\beta}{2\beta t}+\frac{2\beta t}{\alpha+m} - \frac{2\beta t\,\alpha(m+1)}{(\alpha+m)^2} - \frac{\alpha(\alpha-1)\beta t}{(\alpha+m)^2B_{\delta+m}(\delta)} - \mu\frac{2\beta t(\alpha+2m)B_{\delta+2m}(\delta)}{(\alpha+m)^2[B_{\delta+m}(\delta)]^2\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}\right|\right\}.$$

The result is sharp.

Proof. From (2.6),

$$b_{m+1}^2 = \frac{(C_1^{\beta}(t))^2c_1^2}{(\alpha+m)^2[B_{\delta+m}(\delta)]^2\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{2\delta}} \tag{2.8}$$

By making use of (2.7) and (2.8) we have that

$$\begin{aligned} b_{2m+1}-\mu b_{m+1}^2 =& \frac{C_1^{\beta}(t)c_2+C_2^{\beta}(t)c_1^2}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\ &+ \frac{(C_1^{\beta}(t))^2c_1^2}{(\alpha+m)(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\ &- \frac{C_1^{\beta}(t))^2c_1^2\alpha(m+1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\ &- \frac{\alpha(\alpha-1)(C_1^{\beta}(t))^2c_1^2}{2(\alpha+m)^2(\alpha+2m)B_{\delta+m}(\delta)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\ &- \mu\frac{(C_1^{\beta}(t))^2c_1^2}{(\alpha+m)^2[B_{\delta+m}(\delta)]^2\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{2\delta}} \end{aligned}$$

so that

$$b_{2m+1}-\mu b_{m+1}^2 = \frac{C_1^{\beta}(t)}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}\left\{c_2+\left[\frac{C_2^{\beta}(t)}{C_1^{\beta}(t)}+\frac{C_1^{\beta}(t)}{\alpha+m} - \frac{\alpha(m+1)C_1^{\beta}(t)}{(\alpha+m)^2} - \frac{\alpha(\alpha-1)C_1^{\beta}(t)}{2(\alpha+m)^2B_{\delta+m}(\delta)} - \mu\frac{C_1^{\beta}(t)(\alpha+2m)B_{\delta+2m}(\delta)}{(\alpha+m)^2[B_{\delta+m}(\delta)]^2\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}\right]c_1^2\right\}$$

By applying (2.5) and in view of Lemma 1.6 we have that

$$|b_{2m+1}-\mu b_{m+1}^2|$$
$$\leq \frac{C_1^\beta(t)}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^\delta}\max\left\{1,\left|\frac{C_2^\beta(t)}{C_1^\beta(t)}+\frac{C_1^\beta(t)}{\alpha+m}\right.\right.$$
$$\left.\left.-\frac{\alpha(m+1)C_1^\beta(t)}{(\alpha+m)^2}-\frac{\alpha(\alpha-1)C_1^\beta(t)}{2(\alpha+m)^2B_{n+m}(n)}-\mu\frac{C_1^\beta(t)(\alpha+2m)B_{\delta+2m}(\delta)}{(\alpha+m)^2[B_{\delta+m}(\delta)]^2\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^\delta}\right|\right\},$$

which is equivalent to

$$|b_{2m+1}-\mu b_{m+1}^2|$$
$$\leq \frac{2\beta t}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\lambda_2(l-1)}\right)^\delta}\max\left\{1,\left|\frac{2(1+\beta)t^2-\beta}{2\beta t}+\frac{2\beta t}{\alpha+m}\right.\right.$$
$$\left.\left.-\frac{2\beta t\,\alpha(m+1)}{(\alpha+m)^2}-\frac{\alpha(\alpha-1)\beta t}{(\alpha+m)^2B_{\delta+m}(\delta)}-\mu\frac{2\beta t(\alpha+2m)B_{\delta+2m}(\delta)}{(\alpha+m)^2[B_{\delta+m}(\delta)]^2\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^\delta}\right|\right\}.$$

□

Putting $\beta=1,\ \eta_1=\eta_2=0$ and $m=1$ in Theorem 2.1 and Theorem 2.2, we have the following.

Corollary 2.3. *If f given by (1.2) belongs to the class $\mathcal{G}_{0,0}^1(\alpha,t)$ and taking $b=a$, then*

$$|a_2|\leq\frac{2t}{(\alpha+1)B_{\delta+m}(\delta)}$$

and

$$|a_3|\leq\frac{4t^2-2t-1}{(\alpha+2)B_{\delta+2m}(\delta)}+\frac{4t^2}{(\alpha+1)(\alpha+2)B_{\delta+2m}(\delta)}$$
$$-\frac{8\alpha t^2}{(\alpha+1)^2(\alpha+2)B_{\delta+2m}(\delta)}-\frac{4\alpha(\alpha-1)t^2}{2(\alpha+1)^2(\alpha+2)B_{\delta+1}(\delta)B_{\delta+2m}(\delta)}$$

where $B_{\delta+m}(\delta)=\binom{\delta+m}{\delta}\equiv\binom{n+1}{n}$, $B_{\delta+2m}(\delta)=\binom{\delta+2m}{\delta}\equiv\binom{n+2}{n}$ whenever $\delta=n$. This is the result in Theorem 2.1 of [1] for the class $\mathcal{G}(\alpha,t)$ for a fixed number m.

If the function f of the form (1.2) belongs to the class $\mathcal{G}_{0,0}^1(\alpha,t)$ and taking $b=a$, then we get

Corollary 2.4.

$$|a_3-\mu a_2^2|\leq\frac{2t}{(\alpha+2)B_{\delta+2m}(\delta)}\max\left\{1,\left|\frac{4t^2-1}{2t}+\frac{2t}{\alpha+1}-\frac{4\alpha t}{(\alpha+1)^2}\right.\right.$$
$$\left.\left.-\frac{\alpha(\alpha-1)t}{(\alpha+1)^2B_{\delta+2m}(\delta)}-\mu\frac{2(\alpha+2)tB_{\delta+2m}(\delta)}{(\alpha+1)^2[B_{\delta+m}(\delta)]^2}\right|\right\}$$

where $B_{\delta+m}(\delta)=\binom{\delta+m}{\delta}\equiv\binom{n+1}{n}$, $B_{\delta+2m}(\delta)=\binom{\delta+2m}{\delta}\equiv\binom{n+2}{n}$ whenever $\delta=n$. This is the result in Theorem 3.2 in [1] for the class $\mathcal{G}(\alpha,t)$.

Putting $\beta = 1$ in Theorem 2.1 and Theorem 2.2, we respectively have the following.

Corollary 2.5. *If the function $g(z)$ of the form (1.3) belong to the class $\mathcal{G}^1_{\eta_1,\eta_2}(\alpha,t)$, then*

$$|b_{m+1}| \leq \frac{2t}{(\alpha+m)B_{\delta+m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}$$

and

$$\begin{aligned} |b_{2m+1}| \leq & \frac{4t^2-2t-1}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(j-i)}\right)^{n}} \\ & + \frac{2t^2}{(\alpha+m)(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(j-i)}{1+\lambda_2(l-1)}\right)^{\delta}} \\ & - \frac{4\alpha t^2(m+1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \\ & - \frac{2t^2\alpha(\alpha-1)}{(\alpha+m)^2(\alpha+2m)B_{\delta+m}(\delta)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}. \end{aligned}$$

This is presumably a new result.

Corollary 2.6. *If the function $g(z)$ of the form (1.3) belongs to the class $\mathcal{G}^1_{\lambda_1,\lambda_2}(\alpha,t)$, then*

$$\begin{aligned} & |b_{2m+1}-\mu b^2_{m+1}| \\ & \leq \frac{2t}{(\alpha+2m)B_{\delta+2m}(\delta)\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}} \max\left\{1, \left|\frac{4t^2-2t-1}{2t}+\frac{2t}{\alpha+m}\right.\right. \\ & \left.\left. -\frac{2t\,\alpha(m+1)}{(\alpha+m)^2}-\frac{\alpha(\alpha-1)t}{(\alpha+m)^2B_{\delta+m}(\delta)}-\mu\frac{2t(\alpha+2m)B_{\delta+2m}(\delta)}{(\alpha+m)^2[B_{\delta+m}(\delta)]^2\left(\frac{1+(\eta_1+\eta_2)(l-1)}{1+\eta_2(l-1)}\right)^{\delta}}\right|\right\} \end{aligned}$$

This is presumably a new result.

3. Conclusion

In this work, we defined a new class consisting of Bazilević functions of type α. The new class involved the Gegenbauer polynomials, a generalized operator and the subordination principle. Some of the obtained results are the initial coefficient bounds and the Fekete-Szegó estimates for functions belonging to the new class. Upon varying various involving parameters, the results presented in this work lead to some known results. Also, some presumably new results were exhibited as corollaries.

Acknowledgement. The author would like to thank the editor and reviewers for their valuable suggestions.

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