LOWER BOUNDS FOR BLOW-UP TIME OF COUPLED NONLINEAR KLEIN-GORDON EQUATIONS

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ABSTRACT. This paper deals with the blow up of coupled nonlinear Klein-Gordon equations. Lower bounds for the time of blow up is derived if the solutions blow up.

1. INTRODUCTION

In this paper, we consider the following coupled nonlinear Klein-Gordon equations with nonlinear damping terms

\[ \begin{align*}
    u_{tt} - \Delta u + m_1^2 u + |u_t|^{p-1} u_t &= f_1(u,v), & (x,t) &\in \Omega \times (0,T), \\
    v_{tt} - \Delta v + m_2^2 v + |v_t|^{q-1} v_t &= f_2(u,v), & (x,t) &\in \Omega \times (0,T), \\
    u(x,0) &= u_0(x), & u_t(x,0) &= u_1(x), & x &\in \Omega, \\
    v(x,0) &= v_0(x), & v_t(x,0) &= v_1(x), & x &\in \Omega, \\
    u(x,t) &= v(x,t) = 0, & x &\in \partial\Omega,
\end{align*} \]

where \( \Omega \) is a bounded domain with smooth boundary \( \partial\Omega \) in \( \mathbb{R}^n \) (\( n = 1, 2, 3 \)), \( m_1, m_2 > 0 \) and \( p, q \geq 1 \) are constants.

The coupled nonlinear Klein-Gordon equation which models the motion of charged mesons in an electromagnetic field is investigated [15].

In the absence of the \( m_1^2 u \) and \( m_2^2 v \) terms (\( m_1 = m_2 = 0 \)) the problem (1.1) reduces to the following form

\[ \begin{align*}
    u_{tt} - \Delta u + |u_t|^{p-1} u_t &= f_1(u,v), \\
    v_{tt} - \Delta v + |v_t|^{q-1} v_t &= f_2(u,v).
\end{align*} \]

Agre and Rammaha [2] studied the global existence and the blow up of solutions of the system (1.2) by using the same techniques as in [4]. For more related results, the reader is referred to [3], [5], [6].

Ye [16] proved the global existence and asymptotic stability of solutions of the system (1.1) for \( p = q \). For \( p = q = 1 \), the system was considered by many authors [7, 11, 17]. Wu [18] studied blow up of solutions of the system (1.1) for \( n = 3 \). Recently, in [13, 14] we studied blow up and decay of solutions of the system (1.1).

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In this work our aim is to find lower bound of the blow up time $T^*$ for solutions of (1.1).

This paper is organized as follows: In the next section, we present some lemmas and the local existence theorem. In section 3, we establish lower bounds for the blow up time when the blow up occurs.

2. Preliminaries

In this section, we give some assumptions and lemmas which will be used throughout this work. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We denote by $C$ various positive constants which may be different at different occurrences.

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take

$$f_1(u,v) = (r + 1) \left[ a |u + v|^{r-1} (u + v) + b |u|^{r+1} |v|^{r+1} \right],$$

$$f_2(u,v) = (r + 1) \left[ a |u + v|^{r-1} (u + v) + b |u|^{r+1} |v|^{r+1} \right],$$

where $a, b > 0$ are constants and $r$ satisfies

$$1 < r \quad \text{if} \quad n \leq 2,$$

$$1 < r \leq \frac{4-n}{n-2} \quad \text{if} \quad n > 2. \quad (2.1)$$

We can easily verify that

$$u f_1(u,v) + v f_2(u,v) = (r + 1) F(u,v), \quad \forall (u,v) \in R^2, \quad (2.2)$$

where

$$F(u,v) = \left[ a |u + v|^{r+1} + 2b |uv|^{r+1} \right]. \quad (2.3)$$

We define

$$J(t) = \frac{1}{2} (\|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2) - \int_\Omega F(u,v) \, dx, \quad (2.4)$$

and

$$I(t) = \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 - (r + 1) \int_\Omega F(u,v) \, dx. \quad (2.5)$$

We also define the energy functional as follows

$$E(t) = \frac{1}{2} (\|u_t\|^2 + \|v_t\|^2) + \frac{1}{2} (\|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2) - \int_\Omega F(u,v) \, dx. \quad (2.6)$$

We also define

$$W_- = \left\{ (u,v) : (u,v) \in H^1_0(\Omega) \times H^1_0(\Omega), \quad I(u,v) < 0 \right\}. \quad (2.7)$$

The next lemma shows that our energy functional (2.6) is a nonincreasing function along the solution of (1.1).

**Lemma 2.1.** $E(t)$ is a nonincreasing function for $t \geq 0$ and

$$E'(t) = - \left( \|u_t\|^{p+1}_{p+1} + \|v_t\|^{q+1}_{q+1} \right) \leq 0. \quad (2.8)$$
Proof. Multiplying the first equation of (1.1) by $u_t$ and the second equation by $v_t$, integrating over $\Omega$, using integrating by parts and summing up the product results, we get

$$E(t) - E(0) = -\int_0^t \left( \| u_\tau \|_{p+1}^{r+1} + \| v_\tau \|_{q+1}^{r+1} \right) d\tau \quad \text{for } t \geq 0. \quad (2.9)$$

□

Lemma 2.2. (Sobolev-Poincare inequality) [1]. Let $p$ be a number with $2 \leq p < \infty (n = 1, 2)$ or $2 \leq p \leq 2n/(n - 2) \ (n \geq 3)$, then there is a constant $C_* = C_*(\Omega, p)$ such that

$$\| u \|_p \leq C_* \| \nabla u \|, \ \forall u \in H^1_0(\Omega).$$

Lemma 2.3. There exist two positive constants $c_1$ and $c_2$ such that

$$\int_\Omega |f_1(u,v)|^2 \, dx \leq c_1 \left( \| \nabla u \|^2 + \| \nabla v \|^2 \right)^r$$

and

$$\int_\Omega |f_2(u,v)|^2 \, dx \leq c_2 \left( \| \nabla u \|^2 + \| \nabla v \|^2 \right)^r$$

are satisfied.

Proof. Our techniques of proof follows carefully the steps in [10, 12], with necessary modifications imposed by the nature of our problem.

$$\int_\Omega |f_1(u,v)|^2 \, dx \leq (r + 1)^2 \int_\Omega \left( |u + v|^r + |u|^{\frac{r+1}{2}} |v|^{\frac{r+1}{2}} \right)^2 \, dx$$

$$\leq (r + 1)^2 \int_\Omega \left( |u|^r + |v|^r + |u|^{\frac{r+1}{2}} |v|^{\frac{r+1}{2}} \right)^2 \, dx.$$

Now, we consider the following two cases.

Case 1. For $|u| \leq |v|$, we get

$$\int_\Omega |f_1(u,v)|^2 \, dx \leq (r + 1)^2 \int_\Omega \left( |v|^r + |v|^r + |v|^{\frac{r+1}{2}} |v|^{\frac{r+1}{2}} \right)^2 \, dx$$

$$\leq (r + 1)^2 \int_\Omega (3 |v|^r)^2 \, dx$$

$$\leq 9 (r + 1)^2 \| v \|_{2r}^{2r},$$

Thus, by the Sobolev-Poincare inequality, we have

$$\int_\Omega |f_1(u,v)|^2 \, dx \leq c_1 (r + 1)^2 \| \nabla v \|^{2r}$$

$$\leq c_1 (r + 1)^2 \left( \| \nabla u \|^2 + \| \nabla v \|^2 \right)^r.$$
Case 2. If $|u| > |v|$, we get

$$\int_{\Omega} |f_1(u, v)|^2 \, dx \leq (r + 1)^2 \int_{\Omega} \left( |u|^r + |u|^{\frac{r+1}{2}} |u|^{\frac{r+1}{2}} \right)^2 \, dx$$

$$\leq (r + 1)^2 \int_{\Omega} (3|u|^r) \, dx$$

$$\leq 9 (r + 1)^2 \|u\|_{2r}^2,$$

Thus, by the Sobolev-Poincare inequality, we have

$$\int_{\Omega} |f_1(u, v)|^2 \, dx \leq c_1 (r + 1)^2 \|\nabla u\|^{2r}$$

$$\leq c_1 (r + 1)^2 \left( \|\nabla u\|^2 + \|\nabla v\|^2 \right)^{\frac{r}{2}}.$$  \hfill $\square$

Next, we state the local existence theorem that can be established by combining arguments of \cite{4, 16}.

**Theorem 2.4.** (Local existence). Suppose that (2.1) holds. Then there exist $p, q$ satisfying

$$\begin{cases} 
1 \leq p, q & \text{if } n \leq 2, \\
1 \leq p, q \leq \frac{n+2}{n-2} & \text{if } n > 2
\end{cases}$$

and further $(u_0, v_0) \in H^1_0(\Omega) \times H^1_0(\Omega), (u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$. Then problem (1.1) has a unique local solution

$$(u, v) \in C([0, T); H^1_0(\Omega)),$$

$$(u_t, v_t) \in C([0, T); L^2(\Omega)) \cap L^{p+1}(\Omega \times [0, T)) \text{ and } (v_t) \in C([0, T); L^2(\Omega)) \cap L^{q+1}(\Omega \times [0, T)) .$$

Moreover, at least one of the following statements holds true:

i) $T = \infty$,  

ii) $\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \rightarrow \infty$ as $t \rightarrow T^*$.

**Theorem 2.5.** (see \cite{13}) Assume $r > \max \{p, q\}$, the initial energy $E(0) < 0$. Then the solution of this system blows up in finite time $T^*$.

3. Lower bounds for the blow up time

In this section, we prove lower bounds for the blow up time $T^*$.

**Theorem 3.1.** Assume that (2.1) holds. Assume further that $(u_0, v_0) \in W_-, (u_0, v_0) \in H^1_0(\Omega) \times H^1_0(\Omega), (u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$ and $1 < p, q < r$. Let $u$ be the solution of the problem (1.1), which blows up at a finite time $T^*$. Then

$$\int_{\psi(0)}^{\infty} \frac{d\tau}{\psi(\tau) + E(0) + \frac{c_2 + c_3}{2} (r + 1)^r \psi^r(\tau)} \leq T^*.$$
Proof. We define
\[ \psi(t) = \int_{\Omega} F(u,v) \, dx \quad (3.1) \]
Therefore
\[ \psi'(t) = \int_{\Omega} (u_t F_u + v_t F_v) \, dx. \quad (3.2) \]
Using Young’s inequality, we have
\[ \psi'(t) \leq \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) \, dx + \frac{1}{2} \int_{\Omega} (F_u^2 + F_v^2) \, dx. \]
By the Lemma 2.3, we obtain
\[ \psi'(t) \leq \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) \, dx + \frac{c_2 + c_3}{2} (\|\nabla u\|^2 + \|\nabla v\|^2)^r. \quad (3.3) \]
Since \( I(t) < 0 \), we get
\[ \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 < (r + 1) \int_{\Omega} F(u,v) \, dx. \quad (3.4) \]
Inserting (3.4) into (3.3), to get
\[ \psi'(t) \leq \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) \, dx + \frac{c_2 + c_3}{2} \left( (r + 1) \int_{\Omega} F(u,v) \, dx \right)^r \]
\[ = \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) \, dx + \frac{c_2 + c_3}{2} (r + 1)^r \left( \int_{\Omega} F(u,v) \, dx \right)^r \]
\[ = \frac{1}{2} \int_{\Omega} (u_t^2 + v_t^2) \, dx + \frac{c_2 + c_3}{2} (r + 1)^r \psi^r(t). \quad (3.5) \]
By the definition \( E(t) \), we have
\[ (\|u_t\|^2 + \|v_t\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \]
\[ = 2 \int_{\Omega} F(u,v) \, dx + 2E(t) \]
\[ \leq 2 \psi(t) + 2E(0). \quad (3.6) \]
Combining (3.5)-(3.6), we get
\[ \psi'(t) \leq \psi(t) + E(0) + \frac{c_2 + c_3}{2} (r + 1)^r \psi^r(t). \quad (3.7) \]
Applying Theorem 2.5, we have
\[ \lim_{t \to T^*} \int_{\Omega} F(u,v) \, dx = \infty. \quad (3.8) \]
According to (3.7) and (3.8), we obtain
\[ \int_{\psi(0)}^{\psi(T^*)} \frac{d\tau}{\psi(\tau) + E(0) + \frac{c_2 + c_3}{2} (r + 1)^r \psi^r(\tau)} \leq T^*. \]
This completes the proof. \( \Box \)
References

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