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THE EXISTENCE OF A SOLUTION OF A NONLINEAR FREDHOLM INTEGRAL EQUATIONS OVER BICOMPLEX b-METRIC SPACES

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Abstract. In this manuscript, owing to the concept of bicomplex b-metric spaces, we prove common fixed point theorem in bicomplex b-metric spaces. In order to strengthen our main results, a suitable example is presented. More over, the results we obtained is supplement and improve on previous research findings. A fruitful application is also supplied to endorse our outcomes.

1. Introduction

The development of special algebra in commutative generalization of complex numbers as bicomplex numbers, bicomplex numbers, etc. was done by Serge [13]. Subsequently during 1930s many researchers contributed in this area [12, 14, 15]. Later, the bicomplex algebra and function theory was developed by Price [10]. Recently many interesting and significant applications were developed in this area. Also one can see the attempts in [3]. Luna-Elizaarrarás, Shapiro, Struppa and Vajiac [8] worked on elementary functions of bicomplex numbers. Then, Choi, Datta, Biswa and Islam [2] proved some common fixed point theorems on bicomplex valued metric spaces. In 2020, Datta, Pal, Sarkar and Saha [4] proved common fixed point theorems on bicomplex valued b-metric spaces. In 2022, Murthy et al obtained some results on common fixed points in bipolar metric space [9]. In 2021, Datta, Pal, Sarkarc and Manna [5] proved common fixed point theorem on bicomplex valued b-metric space. In 2021, Beg, Kumar Datta and Pal [1] proved the following fixed point theorems on bicomplex valued b-metric spaces. See [7, 6, 11, 16] for more information on b-metric spaces and applications.

Theorem 1.1. [1] *Let $(\mathit{\Lambda}, \rho)$ be a complete bicomplex valued metric space with degenerated $1+\rho(\zeta,\psi)$ and $||1+\rho(\zeta,\psi)|| \neq 0$ for all $\zeta,\psi \in \mathit{\Lambda}$ and let $\mathit{\Gamma}, \mathit{\Upsilon} : \mathit{\Lambda} \to \mathit{\Lambda}$ be mappings satisfying the condition*

$$\rho(\mathit{\Gamma}\zeta, \mathit{\Upsilon}\psi) \preceq_{i_2} \lambda\rho(\zeta,\psi) + \frac{\mu\rho(\zeta,\mathit{\Gamma}\zeta)\rho(\psi,\mathit{\Upsilon}\psi)}{1+\rho(\zeta,\psi)} \tag{1}$$

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for all $\zeta, \psi \in \Lambda$, *where* $\lambda, \mu \geq 0$ *with* $\lambda + \sqrt{2}\mu < 1$. *Then, there exists a unique common fixed point of* Γ *and* Υ.

In this paper, inspired by Theorem 1.1, we prove some common fixed point theorems on bicomplex b-metric spaces and provide an application.

2. Preliminaries

Throughout this paper, we denote the set of real, complex and bicomplex numbers respectively as $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$. Segre [13] defined the bicomplex number (briefly BIN) as:

$$\aleph = \beta_1 + \beta_2 i_1 + \beta_3 i_2 + \beta_4 i_1 i_2,$$

where $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{C}_0$, and independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We defined the set of BIN as:

$$\mathbb{C}_2 = \{\aleph : \aleph = \beta_1 + \beta_2 i_1 + \beta_3 i_2 + \beta_4 i_1 i_2, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\aleph : \aleph = \xi_1 + i_2 \xi_2, \xi_1, \xi_2 \in \mathbb{C}_1\},$$

where $\xi_1 = \beta_1 + \beta_2 i_1 \in \mathbb{C}_1$ and $\xi_2 = \beta_3 + \beta_4 i_1 \in \mathbb{C}_1$. If $\aleph = \xi_1 + i_2\xi_2$ and $\varpi = \alpha_1 + i_2\alpha_2$ are any two BIN's then the sum is defined by $\aleph \pm \varpi = (\xi_1 + i_2\xi_2) \pm (\alpha_1 + i_2\alpha_2) = \xi_1 \pm \alpha_1 + i_2(\xi_2 \pm \alpha_2)$ and the product is defined by $\aleph \cdot \varpi = (\xi_1 + i_2\xi_2)(\alpha_1 + i_2\alpha_2) = (\xi_1\alpha_1 - \xi_2\alpha_2) + i_2(\xi_1\alpha_2 + \xi_2\alpha_1)$.

A BIN $\aleph = \beta_1 + \beta_2 i_1 + \beta_3 i_2 + \beta_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$$

is degenerated. In that case $\aleph^{-1}$ exists and it is also degenerated.
The norm $||.||$ of $\mathbb{C}_2$ is a positive real valued function and $||.|| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} ||\aleph|| &= ||\xi_1 + i_2 z_2|| = \{|\xi|^2 + |\xi|^2\}^{\frac{1}{2}} \\ &= \left[\frac{|(\xi_1 - i_1\xi_2)|^2 + |(\xi_1 + i_1\xi_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= (\beta_1^2 + \beta_2^2 + \beta_1^2 + \beta_3^2 + \beta_4^2)^{\frac{1}{2}}, \end{aligned}$$

where $\aleph = \beta_1 + \beta_2 i_1 + \beta_3 i_2 + \beta_4 i_1 i_2 = \xi_1 + i_2\xi_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to the norm is a normed linear space, and also $\mathbb{C}_2$ is complete and so $\mathbb{C}_2$ is a Banach space. If $\aleph, \varpi \in \mathbb{C}_2$, then $||\aleph\varpi|| \leq \sqrt{2}||\aleph||||\varpi||$ holds instead of $||\aleph\varpi|| \leq ||\aleph||||\varpi||$ and therefore $\mathbb{C}_2$ is not a Banach algebra. The partial order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ is defined as: Let $\mathbb{C}_2$ be the set of BIN's and $\aleph = \xi_1 + i_2\xi_2$, $\varpi = \alpha_1 + i_2\alpha_2 \in \mathbb{C}_2$. Then $\aleph \preceq_{i_2} \varpi$ if and only if $\xi_1 \preceq \alpha_1$ and $\xi_2 \preceq \alpha_2$, i.e., $\aleph \preceq_{i_2} \varpi$ if one of the following assertions is fulfilled:

(1) $\xi_1 = \alpha_1$, $\xi_2 = \alpha_2$,
(2) $\xi_1 \prec \alpha_1$, $\xi_2 = \alpha_2$,
(3) $\xi_1 = \alpha_1$, $\xi_2 \prec \alpha_2$,

(4) $\xi_1 \prec \alpha_1$, $\xi_2 \prec \alpha_2$,

In particular, we can write $\aleph \precsim_{i_2} \varpi$ if $\aleph \preceq_{i_2} \varpi$ and $\aleph \neq \varpi$, i.e., one of 2, 3 and 4 is fulfilled and we will write $\aleph \prec_{i_2} \varpi$ if only 4 is fulfilled.

For any two BIN's $\aleph, \varpi \in \mathbb{C}_2$, such that

(1) $\aleph \preceq_{i_2} \varpi \Rightarrow ||\aleph|| \leq ||\varpi||$,
(2) $||\aleph + \varpi|| \leq ||\aleph|| + ||\varpi||$,
(3) $||\beta\aleph|| = \beta||\aleph||$, where $\beta \geq 0$,
(4) $||\aleph\varpi|| \leq \sqrt{2}||\aleph||||\varpi||$ and the equality holds only when at least one of $\aleph$ and ϖ is degenerated,
(5) $||\aleph^{-1}|| = ||\aleph||^{-1}$ if $\aleph$ is a degenerated BIN with $0 \prec \aleph$,
(6) $||\frac{\aleph}{\varpi}|| = \frac{||\aleph||}{||\varpi||}$ if ϖ is a degenerated BIN.

Definition 2.1. [4] *Let Λ be a nonempty set and let $s \geq 1$. Suppose that a mapping $\rho : \Lambda \times \Lambda \to \mathbb{C}_2$ such that*

(1) $0 \preceq_{i_2} \rho(\zeta, \psi)$ *for all* $\zeta, \psi \in \Lambda$ *and* $\rho(\zeta, \psi) = 0$ *if and only if* $\zeta = \psi$*;*
(2) $\rho(\zeta, \psi) = \rho(\psi, \zeta)$ *for all* $\zeta, \psi \in \Lambda$*;*
(3) $\rho(\zeta, \psi) \preceq_{i_2} s[\rho(\zeta, \eta) + \rho(\eta, \psi)]$ *for all* $\zeta, \psi, \eta \in \Lambda$.

Then ρ is called a bicomplex valued b-metric on Λ, and $(\Lambda, \rho_{\mathfrak{b}})$ is called a bicomplex valued b-metric space(briefly BICBMS).

Example 2.1. *Let $\Lambda = [0, 1]$ and $\rho : \Lambda \times \Lambda \to \mathbb{C}_2$ be defined by $\rho(\zeta, \psi) = |\zeta - \psi|e^{i_2 \frac{\pi}{6}}$. Then (Λ, ρ) is a BICBMS.*

Definition 2.2. [4] *Let (Λ, ρ) be a BICBMS. A sequence $\{\zeta_\ell\}$ in Λ is said to be convergent or converges to $\zeta \in \Lambda$ if for every $0 \prec_{i_2} \epsilon \in \mathbb{C}_2$ we can find $\ell_0 \in \mathbb{N}$ satisfying $\rho(\zeta_\ell, \zeta) \prec_{i_2} \mathfrak{r}$ for all $\ell \geq \ell_0$.*

Definition 2.3. [4] *Let (Λ, ρ) be a BICBMS. A sequence $\{\zeta_\ell\}$ in Λ is said to be a Cauchy sequence in (Λ, ρ) if for any $0 \prec_{i_2} \epsilon \in \mathbb{C}_2$, we can find $\mathfrak{h} \in \mathbb{N}$ satisfying $\rho(\zeta_\ell, \zeta_{\ell+\mathfrak{m}}) \prec_{i_2} \epsilon$ for all $\ell, \mathfrak{m} \in \mathbb{N}$ and $\ell, \mathfrak{m} \geq \mathfrak{h}$.*

Definition 2.4. [4] *Let (Λ, ρ) be a BICBMS. Let $\{\zeta_\ell\}$ be any sequence in Λ. Then, if every Cauchy sequence in Λ is convergent in Λ then (Λ, ρ) is said to be a complete BICBMS.*

Lemma 2.2. [1] *Let (Λ, ρ) be a BICBMS. A sequence $\{\zeta_\ell\} \in \Lambda$ converges to $\zeta \in \Lambda$ if and only if $\lim\limits_{\ell \to \infty} ||\rho(\zeta_\ell, \zeta)|| = 0$.*

Lemma 2.3. [1] *Let (Λ, ρ) be a BICBMS and $\{\zeta_\ell\}$ be a sequence in Λ. Then $\{\zeta_\ell\}$ is a Cauchy sequence in Λ if and only if $\lim\limits_{\ell \to \infty} ||\rho(\zeta_\ell, \zeta_{\ell+\mathfrak{m}})|| = 0$.*

3. Main results

Theorem 3.1. *Let (Λ, ρ) be a complete BICBMS with the coefficient $s \geq 1$. Let Γ and Υ be self-mappings on Λ satisfying the condition*

$$\rho(\Gamma\zeta, \Upsilon\psi) \preceq_{i_2} \lambda\rho(\zeta, \psi) + \frac{\mu\rho(\zeta, \Gamma\zeta)\rho(\psi, \Upsilon\psi) + \gamma\rho(\psi, \Gamma\zeta)\rho(\zeta, \Upsilon\psi)}{1 + \rho(\zeta, \psi)} \tag{2}$$

for all $\zeta, \psi \in \varLambda$ *where* $\lambda, \mu, \gamma \geq 0$ *with* $\lambda + \sqrt{2}\mu + \sqrt{2}\gamma < \frac{1}{s}$, *then* $\varGamma$ *and* $\varUpsilon$ *have a unique common fixed point.*

Proof. Let ζ_0 be an arbitrary point in $\varLambda$ and define $\zeta_{2\ell+1} = \varGamma\zeta_{2\ell}$, $\zeta_{2\ell+2} = \varUpsilon\zeta_{2\ell+1}$, $\ell = 0, 1, 2, \cdots$. Then

$$\begin{aligned}\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2}) =& \rho(\varGamma\zeta_{2\ell}, \varUpsilon\zeta_{2\ell+1}) \preceq_{i_2} \lambda\rho(\zeta_{2\ell}, \zeta_{2\ell+1}) \\ &+ \frac{\mu\rho(\zeta_{2\ell}, \varGamma\zeta_{2\ell})\rho(\zeta_{2\ell+1}, \varUpsilon\zeta_{2\ell+1}) + \gamma\rho(\zeta_{2\ell}, \varUpsilon\zeta_{2\ell+1})\rho(\zeta_{2\ell+1}, \varGamma\zeta_{2\ell})}{1 + \rho(\zeta_{2\ell}, \zeta_{2\ell+1})}.\end{aligned}$$

Since $\zeta_{2\ell+1} = \varGamma\zeta_{2\ell}$ implies $\rho(\zeta_{2\ell+1}, \varGamma\zeta_{2\ell}) = 0$,

$$\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2}) \preceq_{i_2} \lambda\rho(\zeta_{2\ell}, \zeta_{2\ell+1}) + \frac{\mu\rho(\zeta_{2\ell}, \zeta_{2\ell+1})\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})}{1 + \rho(\zeta_{2\ell}, \zeta_{2\ell+1})},$$

which implies that

$$||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})|| \leq \lambda||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})|| + \frac{\sqrt{2}\mu||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})||||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})||}{||1 + \rho(\zeta_{2\ell}, \zeta_{2\ell+1})||}.$$

Since $||1 + \rho(\zeta_{2\ell}, \zeta_{2\ell+1})|| > ||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})||$,

$$||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})|| \leq \lambda||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})|| + \sqrt{2}\mu||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})||,$$

and so

$$||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})|| \leq \frac{\lambda}{1 - \sqrt{2}\mu}||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})||.$$

Also,

$$\begin{aligned}\rho(\zeta_{2\ell+2}, \zeta_{2\ell+3}) &= \rho(\varUpsilon\zeta_{2\ell+1}, \varGamma\zeta_{2\ell+2}) \\ &= \rho(\varGamma\zeta_{2\ell+2}, \varUpsilon\zeta_{2\ell+1}) \\ &\preceq_{i_2} \lambda\rho(\zeta_{2\ell+2}, \zeta_{2\ell+1}) \\ &+ \frac{\mu\rho(\zeta_{2\ell+2}, \varGamma\zeta_{2\ell+2})\rho(\zeta_{2\ell+1}, \varUpsilon\zeta_{2\ell+1}) + \gamma\rho(\zeta_{2\ell+1}, \varGamma\zeta_{2\ell+2})\rho(\zeta_{2\ell+2}, \varUpsilon\zeta_{2\ell+1})}{1 + \rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})}.\end{aligned}$$

Since $\zeta_{2\ell+2} = \varUpsilon\zeta_{2\ell+1}$ implies $\rho(\zeta_{2\ell+2}, \varUpsilon\zeta_{2\ell+1}) = 0$,

$$\rho(\zeta_{2\ell+2}, \zeta_{2\ell+3}) \preceq_{i_2} \lambda\rho(\zeta_{2\ell+2}, \zeta_{2\ell+1}) + \frac{\mu\rho(\zeta_{2\ell+2}, \varGamma\zeta_{2\ell+2})\rho(\zeta_{2\ell+1}, \varUpsilon\zeta_{2\ell+1})}{1 + \rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})},$$

which implies that

$$||\rho(\zeta_{2\ell+2}, \zeta_{2\ell+3})|| \leq \lambda||\rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})|| + \frac{\sqrt{2}\mu||\rho(\zeta_{2\ell+2}, \zeta_{2\ell+3})||||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})||}{||1 + \rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})||}.$$

Since $||1 + \rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})|| > ||\rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})||$,

$$||\rho(\zeta_{2\ell+2}, \zeta_{2\ell+3})|| \leq \frac{\lambda}{1 - \sqrt{2}\mu}||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})||.$$

Puttig $\mathfrak{h} = \frac{\lambda}{1-\sqrt{2}\mu}$, we have (for all ℓ)

$$||\rho(\zeta_{\ell}, \zeta_{\ell+1})|| \leq \mathfrak{h}||\rho(\zeta_{\ell-1}, \zeta_{\ell})|| \leq \mathfrak{h}^2||\rho(\zeta_{\ell-2}, \zeta_{\ell-1})|| \leq \cdots \leq \mathfrak{h}^{\ell}||\rho(\zeta_0, \zeta_1)||.$$

Therefore, for any $\mathfrak{m} > \ell$, we have

$$\begin{aligned}||\rho(\zeta_\ell, \zeta_\mathfrak{m})|| &\le ||\rho(\zeta_\ell, \zeta_{\ell+1})|| + ||\rho(\zeta_{\ell+1}, \zeta_{\ell+2})|| + ||\rho(\zeta_{\ell+2}, \zeta_{\ell+3})|| \cdots + ||\rho(\zeta_{\mathfrak{m}-1}, \zeta_\mathfrak{m})|| \\ &\le [s\mathfrak{h}^\ell + s^2\mathfrak{h}^{\ell+1} + s^3\mathfrak{h}^{\ell+2} + \cdots + s^{\mathfrak{m}-\ell}\mathfrak{h}^{\mathfrak{m}-1}]||\rho(\zeta_0, \zeta_1)|| \\ &\le \left[\frac{(s\mathfrak{h})^\ell}{1 - s\mathfrak{h}}\right]||\rho(\zeta_0, \zeta_1)||,\end{aligned}$$

which implies that

$$||\rho(\zeta_\ell, \zeta_\mathfrak{m})|| \le \left[\frac{(s\mathfrak{h})^\ell}{1 - s\mathfrak{h}}\right]||\rho(\zeta_0, \zeta_1)|| \to 0 \text{ as } \ell \to \infty.$$

In view of Lemma 2.3, the sequence $\{\zeta_\ell\}$ is Cauchy. Since $\varLambda$ is complete, we can find $\delta \in \varLambda$ satisfying $\zeta_\ell \to \delta$ as $\ell \to \infty$. Suppose that $\delta \neq \varGamma\delta$ such that $0 \prec_{i_2} \eta = \rho(\delta, \varGamma\delta)$. Then we have

$$\begin{aligned}\eta = \rho(\delta, \varGamma\delta) &\preceq_{i_2} s\rho(\delta, \varUpsilon\zeta_{2\mathfrak{k}+1}) + s\rho(\varUpsilon\zeta_{2\mathfrak{k}+1}, \varGamma\delta) \\ &\preceq_{i_2} s\rho(\delta, \zeta_{2\mathfrak{k}+2}) + s\lambda\rho(\delta, \zeta_{2\mathfrak{k}+1}) + \frac{s\mu\rho(\delta, \varGamma\delta)\rho(\zeta_{2\mathfrak{k}+1}, \varUpsilon\zeta_{2\mathfrak{k}+1}) + s\gamma\rho(\zeta_{2\mathfrak{k}+1}, \varGamma\delta)\rho(\delta, \varUpsilon\zeta_{2\mathfrak{k}+1})}{1 + \rho(\delta, \zeta_{2\mathfrak{k}+1})} \\ &\preceq_{i_2} s\rho(\delta, \zeta_{2\mathfrak{k}+2}) + s\lambda\rho(\delta, \zeta_{2\mathfrak{k}+1}) + \frac{s\mu\eta\rho(\zeta_{2\mathfrak{k}+1}, \varUpsilon\zeta_{2\mathfrak{k}+1}) + s\gamma\rho(\zeta_{2\mathfrak{k}+1}, \varGamma\delta)\rho(\delta, \varUpsilon\zeta_{2\mathfrak{k}+1})}{1 + \rho(\delta, \zeta_{2\mathfrak{k}+1})}.\end{aligned}$$

Also, for every ℓ, we have

$$\begin{aligned}||\rho(\delta, \varGamma\delta)|| &\le ||s\rho(\delta, \zeta_{2\mathfrak{k}+2})|| + s\lambda||\rho(\delta, \zeta_{2\mathfrak{k}+1})|| \\ &\quad + \frac{s\mu\eta||\rho(\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}+2})|| + \sqrt{2}s\gamma||\rho(\zeta_{2\mathfrak{k}+1}, \varGamma\delta)||||\rho(\delta, \zeta_{2\mathfrak{k}+2})||}{||1 + \rho(\delta, \zeta_{2\mathfrak{k}+1})||}.\end{aligned}$$

Since $\ell \to \infty$, we get

$$||\rho(\delta, \varGamma\delta)|| = 0,$$

which is a absurd. So $\delta = \varGamma\delta$. Similarly, one can also show that $\delta = \varUpsilon\delta$. Let δ^*(in $\varLambda$) be another common fixed point of $\varGamma$ and $\varUpsilon$, i.e., $\delta^* = \varGamma\delta^* = \varUpsilon\delta^*$. Then

$$\begin{aligned}\rho(\delta, \delta^*) = \rho(\varGamma\delta, \varUpsilon\delta^*) &\preceq_{i_2} \lambda\rho(\delta, \delta^*) + \frac{\mu\rho(\delta, \varGamma\delta)\rho(\delta^*, \varUpsilon\delta^*) + \gamma\rho(\delta^*, \varGamma\delta)\rho(\delta, \varUpsilon\delta^*)}{1 + \rho(\delta, \delta^*)} \\ &= \lambda\rho(\delta, \delta^*) + \frac{\gamma\rho(\delta^*, \delta)\rho(\delta, \delta^*)}{1 + \rho(\delta, \delta^*)},\end{aligned}$$

which implies that

$$||\rho(\delta, \delta^*)|| \le \lambda||\rho(\delta, \delta^*)|| + \frac{\sqrt{2}\gamma||\rho(\delta^*, \delta)||||\rho(\delta, \delta^*)||}{||1 + \rho(\delta, \delta^*)||}.$$

Since $||1 + \rho(\delta, \delta^*)|| > ||\rho(\delta, \delta^*)||$,

$$||\rho(\delta, \delta^*)|| \le (\lambda + \sqrt{2}\gamma)||\rho(\delta, \delta^*)||,$$

which is a absurd. So $\delta = \delta^*$ (as $\lambda + \sqrt{2}\gamma < 1$). □

Setting $\varGamma = \varUpsilon$ in Theorem 3.1, one deduces the following.

Corollary 3.2. *Let $(\varLambda, \rho)$ be a complete BICBMS with the coefficient $s \geq 1$. Let $\varUpsilon$ be a self-mapping in $\varLambda$ such that*

$$\rho(\varUpsilon\zeta, \varUpsilon\psi) \preceq_{i_2} \lambda\rho(\zeta, \psi) + \frac{\mu\rho(\zeta, \varUpsilon\zeta)\rho(\psi, \varUpsilon\psi) + \gamma\rho(\psi, \varUpsilon\zeta)\rho(\zeta, \varUpsilon\psi)}{1 + \rho(\zeta, \psi)} \tag{3}$$

for all $\zeta, \psi \in \varLambda$, where $\lambda, \mu, \gamma \geq 0$ with $\lambda + \sqrt{2}\mu + \sqrt{2}\gamma < 1$, then $\varUpsilon$ has a unique fixed point.

Theorem 3.3. *Let $(\varLambda, \rho)$ be a complete BICBMS with the coefficient $s \geq 1$. Let $\varGamma$ and $\varUpsilon$ be self-mappings in $\varLambda$ satisfying the condition*

$$\rho(\varGamma\zeta, \varUpsilon\psi) \preceq_{i_2} \begin{cases} \lambda\rho(\zeta, \psi) + \mu\frac{\rho(\zeta, \varGamma\zeta)\rho(\psi, \varUpsilon\psi) + \rho(\psi, \varGamma\zeta)\rho(\zeta, \varUpsilon\psi)}{\rho(\varGamma\zeta, \zeta) + \rho(\varUpsilon\psi, \psi)} & \\ +\gamma\frac{\rho(\zeta, \varGamma\zeta)\rho(\zeta, \varUpsilon\psi) + \rho(\psi, \varGamma\zeta)\rho(\psi, \varUpsilon\psi)}{\rho(\varGamma\zeta, \psi) + \rho(\varUpsilon\psi, \zeta)}, & \text{if } \mathcal{H} \neq 0,\ \mathcal{H}_1 \neq 0 \\ 0, & \text{if } \mathcal{H} = 0 \text{ or } \mathcal{H}_1 = 0 \end{cases} \tag{4}$$

for all $\zeta, \psi \in \varLambda$, where $\mathcal{H} = \rho(\varGamma\zeta, \zeta) + \rho(\varUpsilon\psi, \psi)$ and $\mathcal{H}_1 = \rho(\varGamma\zeta, \psi) + \rho(\varUpsilon\psi, \zeta)$ and $\lambda, \mu, \gamma \geq 0$ with $\lambda + \sqrt{2}\mu + \gamma < \frac{1}{s}$. Then $\varGamma$ and $\varUpsilon$ have a unique common fixed point.

Proof. Let ζ_0 be an arbitrary point in $\varLambda$. Define $\zeta_{2\ell+1} = \varGamma\zeta_{2\ell}$ and $\zeta_{2\ell+2} = \varUpsilon\zeta_{2\ell+1}$, $\ell = 0, 1, 2, \cdots$. Now, we distinguish two cases. First, if (for $\ell = 0, 1, 2, \cdots$)$\rho(\varGamma\zeta_{2\ell}, \zeta_{2\ell}) + \rho(\varUpsilon\zeta_{2\ell+1}, \zeta_{2\ell+1}) \neq 0$ and $\rho(\varGamma\zeta_{2\ell}, \zeta_{2\ell+1}) + \rho(\varUpsilon\zeta_{2\ell+1}, \zeta_{2\ell}) \neq 0$, then

$$\begin{aligned}\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2}) =& \rho(\varGamma\zeta_{2\ell}, \varUpsilon\zeta_{2\ell+1}) \preceq_{i_2} \lambda\rho(\zeta_{2\ell}, \zeta_{2\ell+1}) \\ &+ \mu\frac{\rho(\zeta_{2\ell}, \varGamma\zeta_{2\ell})\rho(\zeta_{2\ell+1}, \varUpsilon\zeta_{2\ell+1}) + \rho(\zeta_{2\ell+1}, \varGamma\zeta_{2\ell})\rho(\zeta_{2\ell}, \varUpsilon\zeta_{2\ell+1})}{\rho(\varGamma\zeta_{2\ell}, \zeta_{2\ell}) + \rho(\varUpsilon\zeta_{2\ell+1}, \zeta_{2\ell+1})} \\ &+ \gamma\frac{\rho(\zeta_{2\ell}, \varGamma\zeta_{2\ell})\rho(\zeta_{2\ell}, \varUpsilon\zeta_{2\ell+1}) + \rho(\zeta_{2\ell+1}, \varGamma\zeta_{2\ell})\rho(\zeta_{2\ell+1}, \varUpsilon\zeta_{2\ell+1})}{\rho(\varGamma\zeta_{2\ell}, \zeta_{2\ell+1}) + \rho(\varUpsilon\zeta_{2\ell+1}, \zeta_{2\ell})}.\end{aligned}$$

Since $\zeta_{2\ell+1} = \varGamma\zeta_{2\ell}$ and $\zeta_{2\ell+2} = \varUpsilon\zeta_{2\ell+1}$,

$$\begin{aligned}\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2}) \preceq_{i_2} & \lambda\rho(\zeta_{2\ell}, \zeta_{2\ell+1}) \\ &+ \mu\frac{\rho(\zeta_{2\ell}, \zeta_{2\ell+1})\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2}) + \rho(\zeta_{2\ell+1}, \zeta_{2\ell+1})\rho(\zeta_{2\ell}, \zeta_{2\ell+2})}{\rho(\zeta_{2\ell+1}, \zeta_{2\ell}) + \rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})} \\ &+ \gamma\frac{\rho(\zeta_{2\ell}, \zeta_{2\ell+1})\rho(\zeta_{2\ell}, \zeta_{2\ell+2}) + \rho(\zeta_{2\ell+1}, \zeta_{2\ell+1})\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})}{\rho(\zeta_{2\ell+1}, \zeta_{2\ell+1}) + \rho(\zeta_{2\ell+2}, \zeta_{2\ell})}\end{aligned}$$

or

$$\begin{aligned}\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2}) \preceq_{i_2} \lambda\rho(\zeta_{2\ell}, \zeta_{2\ell+1}) &+ \mu\frac{\rho(\zeta_{2\ell}, \zeta_{2\ell+1})\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})}{\rho(\zeta_{2\ell+1}, \zeta_{2\ell}) + \rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})} \\ &+ \gamma\frac{\rho(\zeta_{2\ell}, \zeta_{2\ell+1})\rho(\zeta_{2\ell}, \zeta_{2\ell+2})}{\rho(\zeta_{2\ell+2}, \zeta_{2\ell})},\end{aligned}$$

which implies that

$$\begin{aligned}||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})|| \leq \lambda||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})|| &+ \sqrt{2}\mu\frac{||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})||||\rho(\zeta_{2\ell+1}, \zeta_{2\ell+2})||}{||\rho(\zeta_{2\ell+1}, \zeta_{2\ell}) + \rho(\zeta_{2\ell+2}, \zeta_{2\ell+1})||} \\ &+ \gamma||\rho(\zeta_{2\ell}, \zeta_{2\ell+1})||.\end{aligned}$$

Since $||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}})+\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})||>||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}})||$,

$$||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||\leq\lambda||\rho(\zeta_{2\mathfrak{k}},\zeta_{2\mathfrak{k}+1})||+\sqrt{2}\mu||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||+\gamma||\rho(\zeta_{2\mathfrak{k}},\zeta_{2\mathfrak{k}+1})||,$$

and so

$$||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||\leq\frac{\lambda+\gamma}{1-\sqrt{2}\mu}||\rho(\zeta_{2\mathfrak{k}},\zeta_{2\mathfrak{k}+1})||.$$

Also

$$\begin{aligned}\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})=&\rho(\varGamma\zeta_{2\mathfrak{k}+2},\varUpsilon\zeta_{2\mathfrak{k}+1})\preceq_{i_2}\lambda\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})\\&+\mu\frac{\rho(\zeta_{2\mathfrak{k}+2},\varGamma\zeta_{2\mathfrak{k}+2})\rho(\zeta_{2\mathfrak{k}+1},\varUpsilon\zeta_{2\mathfrak{k}+1})+\rho(\zeta_{2\mathfrak{k}+1},\varGamma\zeta_{2\mathfrak{k}+2})\rho(\zeta_{2\mathfrak{k}+2},\varUpsilon\zeta_{2\mathfrak{k}+1})}{\rho(\varGamma\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+2})+\rho(\varUpsilon\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+1})}\\&+\gamma\frac{\rho(\zeta_{2\mathfrak{k}+2},\varGamma\zeta_{2\mathfrak{k}+2})\rho(\zeta_{2\mathfrak{k}+2},\varUpsilon\zeta_{2\mathfrak{k}+1})+\rho(\zeta_{2\mathfrak{k}+1},\varGamma\zeta_{2\mathfrak{k}+2})\rho(\zeta_{2\mathfrak{k}+1},\varUpsilon\zeta_{2\mathfrak{k}+1})}{\rho(\varGamma\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})+\rho(\varUpsilon\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})}.\end{aligned}$$

Since $\zeta_{2\mathfrak{k}+3}=\varGamma\zeta_{2\mathfrak{k}+2}$ and $\zeta_{2\mathfrak{k}+2}=\varUpsilon\zeta_{2\mathfrak{k}+1}$, we get

$$\begin{aligned}\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})\preceq_{i_2}&\lambda\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})\\&+\mu\frac{\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})+\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+3})\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+2})}{\rho(\zeta_{2\mathfrak{k}+3},\zeta_{2\mathfrak{k}+2})+\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})}\\&+\gamma\frac{\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+2})+\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+3})\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})}{\rho(\zeta_{2\mathfrak{k}+3},\zeta_{2\mathfrak{k}+1})+\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+2})}\end{aligned}$$

or

$$\begin{aligned}\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})\preceq_{i_2}&\lambda\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})+\mu\frac{\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})}{\rho(\zeta_{2\mathfrak{k}+3},\zeta_{2\mathfrak{k}+2})+\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})}\\&+\gamma\frac{\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+3})\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})}{\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+3})},\end{aligned}$$

which implies that

$$\begin{aligned}||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})||\leq&\lambda||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})||+\sqrt{2}\mu\frac{||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})||||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||}{||\rho(\zeta_{2\mathfrak{k}+3},\zeta_{2\mathfrak{k}+2})+\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})||}\\&+\gamma||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||.\end{aligned}$$

Since $||\rho(\zeta_{2\mathfrak{k}+3},\zeta_{2\mathfrak{k}+2})+\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})||>||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||$,

$$\begin{aligned}||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})||\leq&\lambda||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})||+\sqrt{2}\mu\frac{||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})||||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||}{||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+1})||}\\&+\gamma||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||\end{aligned}$$

or

$$||\rho(\zeta_{2\mathfrak{k}+2},\zeta_{2\mathfrak{k}+3})||\leq\frac{\lambda+\gamma}{1-\sqrt{2}\mu}||\rho(\zeta_{2\mathfrak{k}+1},\zeta_{2\mathfrak{k}+2})||.$$

Now, letting $\mathfrak{h}=\frac{\lambda+\gamma}{1-2\mu}$, we have (for all ℓ)

$$||\rho(\zeta_\ell,\zeta_{\ell+1})||\leq\mathfrak{h}||\rho(\zeta_{\ell-1},\zeta_\ell)||\leq\cdots\leq\mathfrak{h}^\ell||\rho(\zeta_0,\zeta_1)||.$$

So, for any $\mathfrak{m} > \ell$, we have

$$\begin{aligned}||\rho(\zeta_\ell, \zeta_\mathfrak{m})|| &\leq ||\rho(\zeta_\ell, \zeta_{\ell+1})|| + ||\rho(\zeta_{\ell+1}, \zeta_{\ell+2})|| + ||\rho(\zeta_{\ell+2}, \zeta_{\ell+3})|| + \cdots + ||\rho(\zeta_{\mathfrak{m}-1}, \zeta_\mathfrak{m})|| \\ &\leq [s\mathfrak{h}^\ell + s^2\mathfrak{h}^{\ell+1} + s^3\mathfrak{h}^{\ell+2} + \cdots + s^{\mathfrak{m}-\ell}\mathfrak{h}^{\mathfrak{m}-1}]||\rho(\zeta_0, \zeta_1)|| \\ &\leq \Big[\frac{(s\mathfrak{h})^\ell}{1 - s\mathfrak{h}}\Big]||\rho(\zeta_0, \zeta_1)||,\end{aligned}$$

which implies that

$$||\rho(\zeta_\ell, \zeta_\mathfrak{m})|| \leq \Big[\frac{(s\mathfrak{h})^\ell}{1 - s\mathfrak{h}}\Big]||\rho(\zeta_0, \zeta_1)|| \to 0 \text{ as } \ell \to \infty.$$

By Lemma 2.3, we conclude that $\{\zeta_\ell\}$ is a Cauchy sequence. Since Λ is a complete, we can find $\delta \in \Lambda$ satisfying $\zeta_\ell \to \delta$ as $\ell \to \infty$. Now, we assert that $\delta = \Gamma\delta$. Otherwise $0 \prec_{i_2} \eta = \rho(\delta, \Gamma\delta)$ and we have

$$\begin{aligned}\eta =& \rho(\delta, \Gamma\delta) \preceq_{i_2} s\rho(\delta, \Upsilon\zeta_{2\mathfrak{k}+1}) + s\rho(\Upsilon\zeta_{2\mathfrak{k}+1}, \Gamma\delta) \\ \preceq_{i_2}& s\rho(\delta, \zeta_{2\mathfrak{k}+2}) + s\lambda\rho(\delta, \zeta_{2\mathfrak{k}+1}) + s\mu\frac{\rho(\delta, \Gamma\delta)\rho(\zeta_{2\mathfrak{k}+1}, \Upsilon\zeta_{2\mathfrak{k}+1}) + \rho(\zeta_{2\mathfrak{k}+1}, \Gamma\delta)\rho(\delta, \Upsilon\zeta_{2\mathfrak{k}+1})}{\rho(\Gamma\delta, \delta) + \rho(\Upsilon\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}+1})} \\ &+ s\gamma\frac{\rho(\delta, \Gamma\delta)\rho(\delta, \Upsilon\zeta_{2\mathfrak{k}+1}) + \rho(\zeta_{2\mathfrak{k}+1}, \Gamma\delta)\rho(\zeta_{2\mathfrak{k}+1}, \Upsilon\zeta_{2\mathfrak{k}+1})}{\rho(\Gamma\delta, \zeta_{2\mathfrak{k}+1}) + \rho(\zeta_{2\mathfrak{k}+2}, \delta)},\end{aligned}$$

which implies that

$$\begin{aligned}||\eta|| = ||\rho(\delta, \Gamma\delta)|| \leq& ||s\rho(\delta, \zeta_{2\mathfrak{k}+2})|| + \lambda s||\rho(\delta, \zeta_{2\mathfrak{k}+1})|| \\ &+ s\mu\frac{\eta||\rho(\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}+2})|| + \sqrt{2}||\rho(\zeta_{2\mathfrak{k}+1}, \Gamma\delta)||||\rho(\delta, \zeta_{2\mathfrak{k}+2})||}{||\rho(\Gamma\delta, \delta) + \rho(\zeta_{2\mathfrak{k}+2}, \zeta_{2\mathfrak{k}+1})||} \\ &+ s\gamma\frac{\eta||\rho(\delta, \zeta_{2\mathfrak{k}+2})|| + \sqrt{2}||\rho(\zeta_{2\mathfrak{k}+1}, \Gamma\delta)||||\rho(\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}+2})||}{||\rho(\Gamma\delta, \zeta_{2\mathfrak{k}+1}) + \rho(\zeta_{2\mathfrak{k}+2}, \delta)||},\end{aligned}$$

which is a absurd. Thus $||\eta|| = ||\rho(\delta, \Gamma\delta)|| = 0$, i.e., $\delta = \Gamma\delta$. It follows, similarly, that $\delta = \Upsilon\delta$. Assume that δ^* in Λ is another common fixed point of Γ and Υ. Then

$$\Gamma\delta^* = \Upsilon\delta^* = \delta^*.$$

Since $\mathcal{H} = \rho(\Gamma\delta, \delta) + \rho(\Upsilon\delta^*, \delta^*) = 0$, by definition of contraction condition

$$\rho(\delta, \delta^*) = \rho(\Gamma\delta, \Upsilon\delta^*) = 0,$$

and so $\delta = \delta^*$. Second, we consider the case:

$$\Big(\rho(\Gamma\zeta_{2\mathfrak{k}}, \zeta_{2\mathfrak{k}}) + \rho(\Upsilon\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}+1})\Big) \times \Big(\rho(\Gamma\zeta_{2\mathfrak{k}}, \zeta_{2\mathfrak{k}+1}) + \rho(\Upsilon\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}})\Big) = 0,$$

which implies $\rho(\Gamma\zeta_{2\mathfrak{k}}, \Upsilon\zeta_{2\mathfrak{k}+1}) = 0$. Now, if $\rho(\Gamma\zeta_{2\mathfrak{k}}, \zeta_{2\mathfrak{k}}) + \rho(\Upsilon\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}+1}) = 0$, then $\zeta_{2\mathfrak{k}} = \Gamma\zeta_{2\mathfrak{k}} = \zeta_{2\mathfrak{k}+1} = \Upsilon\zeta_{2\mathfrak{k}} = \zeta_{2\mathfrak{k}+2}$. Thus we have $\zeta_{2\mathfrak{k}+1} = \Gamma\zeta_{2\mathfrak{k}} = \zeta_{2\mathfrak{k}}$ and so there exist ℓ_1 and $\mathfrak{m}_1$ such that $\ell_1 = \Gamma\mathfrak{m}_1 = \mathfrak{m}_1$. Using foregoing arguments, one can also show that there exist ℓ_2 and $\mathfrak{m}_2$ such that $\ell_2 = \Upsilon\mathfrak{m}_2 = \mathfrak{m}_2$. Since $\rho(\Gamma\mathfrak{m}_1, \mathfrak{m}_1) + \rho(\Upsilon\mathfrak{m}_2, \mathfrak{m}_2) = 0$, $\rho(\Gamma\mathfrak{m}_1, \Upsilon\mathfrak{m}_2) = 0$ and so $\ell_1 = \Gamma\mathfrak{m}_1 = \Upsilon\mathfrak{m}_2 = \ell_2$ which in turn yields that $\ell_1 = \Gamma\mathfrak{m}_1 = \Gamma\ell_1$. Similarly, one can also have $\ell_2 = \Upsilon\ell_2$. Since $\ell_1 = \ell_2$, implies $\Gamma\ell_1 = \Upsilon\ell_1 = \ell_1$, $\ell_1 = \ell_2$, is a common fixed point of Γ and

Υ. Assume that ℓ_1^* in Λ is another common fixed point of Γ and Υ. Then we have

$$\Gamma\ell_1^* = \Upsilon\ell_1^* = \ell_1^*.$$

Since $\mathcal{H} = \rho(\Gamma\ell_1, \ell_1) + \rho(\Upsilon\ell_1^*, \ell_1^*) = 0$,

$$\rho(\ell_1, \ell_1^*) = \rho(\Gamma\ell_1, \Upsilon\ell_1^*) = 0.$$

This implies that $\ell_1^* = \ell_1$.

If $\rho(\Gamma\zeta_{2\mathfrak{k}}, \zeta_{2\mathfrak{k}+1}) + \rho(\Upsilon\zeta_{2\mathfrak{k}+1}, \zeta_{2\mathfrak{k}}) = 0$ implies that $\rho(\Gamma\zeta_{2\mathfrak{k}}, \Upsilon\zeta_{2\mathfrak{k}+1}) = 0$, the proof can be completed as in the preceding lines. □

By setting $\Gamma = \Upsilon$, we get the following.

Corollary 3.4. *Let (Λ, ρ) be a complete BICBMS with the coefficient $s \geq 1$. Let Υ be a self-mapping in Λ satisfying the condition*

$$\rho(\Upsilon\zeta, \Upsilon\psi) \preceq_{i_2} \begin{cases} \lambda\rho(\zeta,\psi) + \mu\frac{\rho(\zeta,\Upsilon\zeta)\rho(\psi,\Upsilon\psi)+\rho(\psi,\Upsilon\zeta)\rho(\zeta,\Upsilon\psi)}{\rho(\Upsilon\zeta,\zeta)+\rho(\Upsilon\psi,\psi)} \\ \quad +\gamma\frac{\rho(\zeta,\Upsilon\zeta)\rho(\zeta,\Upsilon\psi)+\rho(\psi,\Upsilon\zeta)\rho(\psi,\Upsilon\psi)}{\rho(\Upsilon\zeta,\psi)+\rho(\Upsilon\psi,\zeta)}, & \text{if } \mathcal{H} \neq 0,\ \mathcal{H}_1 \neq 0 \\ 0, & \text{if } \mathcal{H} = 0 \text{ or } \mathcal{H}_1 = 0 \end{cases} \tag{5}$$

for all $\zeta, \psi \in \Lambda$, where $\mathcal{H} = \rho(\Upsilon\zeta, \zeta) + \rho(\Upsilon\psi, \psi)$ and $\mathcal{H}_1 = \rho(\Upsilon\zeta, \psi) + \rho(\Upsilon\psi, \zeta)$ and $\lambda, \mu, \gamma \geq 0$ with $\lambda + \sqrt{2}\mu + \gamma < \frac{1}{s}$. Then Υ has a unique fixed point.

Example 3.5. *Consider $\Lambda = \{0, \frac{1}{3}, 3\}$ and define a mapping $\rho : \Lambda \times \Lambda \to \mathbb{C}_2$ by $\rho(\zeta, \psi) = (1 + i_2)|\zeta - \psi|^2$ for all $\zeta, \psi \in \Lambda$, where $|.|$ is the usual real modulus. Then (Λ, ρ) is a complete BICBMS. Define $\Upsilon : \Lambda \to \Lambda$ by*

$$\Upsilon(0) = 0,\ \Upsilon\left(\frac{1}{3}\right) = 0 \text{ and } \Upsilon(3) = \frac{1}{3}.$$

Let $\lambda = \frac{1}{6}$, $\mu = \frac{1}{5}$ and $\lambda = \frac{1}{9}$. Then $s(\lambda + \sqrt{2}\mu + \sqrt{2}\gamma) = s(\frac{1}{6} + \frac{2}{5} + \frac{2}{9}) < 1$. Then, clearly

$$0 = \rho(\Upsilon\zeta, \Upsilon\psi) \preceq_{i_2} \lambda\rho(\zeta, \psi) + \frac{\mu\rho(\zeta, \Upsilon\zeta)\rho(\psi, \Upsilon\psi) + \gamma\rho(\psi, \Upsilon\zeta)\rho(\zeta, \Upsilon\psi)}{1 + \rho(\zeta, \psi)}.$$

Hence, all the assertions of Corollary 3.2 are fulfilled. Therefore, 0 is the unique fixed point of Υ.

Example 3.6. *Let $\Lambda = \mathcal{Q}(0, \mathfrak{q})$, $\mathfrak{q} > 1\ \forall\ \zeta, \psi \in \Lambda$. Define $\rho : \Lambda \times \Lambda \to \mathbb{C}_2$ by*

$$\rho(\zeta(\wp), \psi(\wp)) = \frac{i_2}{2\pi}\left|\int_{\mathcal{E}} \frac{\zeta(\wp)}{\wp} - \int_{\mathcal{E}} \frac{\psi(\wp)}{\wp}\right|^2$$

and let $\mathcal{E}$ be a closed path in Λ containing a zero. We prove that ρ is a BICBMS. For this,

$$\begin{aligned}
\rho(\zeta(\wp),\psi(\wp)) &= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|^2\\
&= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}+\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|^2\\
&\preceq_{i_2} \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}\bigg|^2+\frac{i_3}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|^2\\
&+2\frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}\bigg|\bigg|\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|\\
&\preceq_{i_2} \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}\bigg|^2+\frac{i_3}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|^2\\
&+\frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}\bigg|^2+\frac{i_3}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\eta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|^2\\
&= 2(\rho(\zeta(\wp),\eta(\wp))+\rho(\eta(\wp),\psi(\wp))).
\end{aligned}$$

Clearly, ρ is a complete BICBMS. Define $\varGamma,\varUpsilon:\Lambda\to\Lambda$ by

$$\varGamma\zeta(\wp)=\wp,\ \varUpsilon\psi(\wp)=e^{\wp}-1.$$

Using the Cauchy integral formula when the mappings $\varGamma$ and $\varUpsilon$ are analytic, we get

$$\rho(\varGamma\zeta(\wp),\varUpsilon\psi(\wp))=\frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\wp}{\wp}-\int_{\mathcal{E}}\frac{e^{\wp}-1}{\wp}\bigg|^2=0,$$

$$\rho(\zeta(\wp),\psi(\wp))=\frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|^2,$$

$$\begin{aligned}
\rho(\zeta(\wp),\varGamma\zeta(\wp)) &= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\wp}{\wp}\bigg|^2\\
&= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}\bigg|^2,
\end{aligned}$$

$$\begin{aligned}
\rho(\psi(\wp),\varUpsilon\psi(\wp)) &= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}-\int_{\mathcal{E}}\frac{e^{\wp}-1}{\wp}\bigg|^2\\
&= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\psi(\wp)}{\wp}\bigg|^2,
\end{aligned}$$

$$\begin{aligned}
\rho(\zeta(\wp),\varGamma\psi(\wp)) &= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}-\int_{\mathcal{E}}\frac{\wp}{\wp}\bigg|^2\\
&= \frac{i_2}{2\pi}\bigg|\int_{\mathcal{E}}\frac{\zeta(\wp)}{\wp}\bigg|^2,
\end{aligned}$$

$$\begin{aligned}\rho(\zeta(\wp), \Upsilon\psi(\wp)) &= \frac{i_2}{2\pi}\left|\int_{\mathcal{E}} \frac{\zeta(\wp)}{\wp} - \int_{\mathcal{E}} \frac{e^{\wp}-1}{\wp}\right|^2 \\ &= \frac{i_2}{2\pi}\left|\int_{\mathcal{E}} \frac{\zeta(\wp)}{\wp}\right|^2.\end{aligned}$$

Clearly,

$$\rho(\Gamma\zeta, \Upsilon\psi) \preceq_{i_2} \lambda\rho(\zeta,\psi) + \frac{\mu\rho(\zeta,\Gamma\zeta)\rho(\psi,\Upsilon\psi) + \gamma\rho(\psi,\Gamma\zeta)\rho(\zeta,\Upsilon\psi)}{1+\rho(\zeta,\psi)}$$

for all $\zeta, \psi \in \Lambda$, where $\lambda, \mu, \gamma \geq 0$ with $s(\lambda + \sqrt{2}\mu + \sqrt{2}\gamma) < 1$. Therefore, all the assertions of Theorem 3.1 are fulfilled and so the mappings Γ and Υ have a unique common fixed point in Λ.

4. Applications

In this section, we give an applications using the Theorem 3.1 and Corollary 3.2.

Let $\Lambda = C[\leftthreetimes_1, \leftthreetimes_2]$ be a set of all real continuous functions on $[\leftthreetimes_1, \leftthreetimes_2]$ equipped with metric $\rho(\zeta,\psi) = (1+i_2)(|\zeta(\sqcup) - \psi(\sqcup)|)$ for all $\zeta, \psi \in C[\leftthreetimes_1, \leftthreetimes_2]$ and $\sqcup \in [\leftthreetimes_1, \leftthreetimes_2]$, where $|.|$ is the usual real modulus. Then, (Λ, ρ) is a complete BICBMS. Now, we consider the system of non-linear Fredholm integral equation

$$\zeta(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \ell_1(\sqcup, \mathfrak{s}, \zeta(\mathfrak{s}))d\mathfrak{s} \tag{6}$$

and

$$\zeta(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \ell_2(\sqcup, \mathfrak{s}, \zeta(\mathfrak{s}))d\mathfrak{s}, \tag{7}$$

where $\sqcup, \mathfrak{s} \in [\leftthreetimes_1, \leftthreetimes_2]$. Assume that $\ell_1, \ell_2 : [\leftthreetimes_1, \leftthreetimes_2] \times [\leftthreetimes_1, \leftthreetimes_2] \times \Lambda \to \mathbb{R}$ and $\mathfrak{v} : [\leftthreetimes_1, \leftthreetimes_2] \to \mathbb{R}$ continuous, where $\mathfrak{v}(\sqcup)$ is a given function in Λ. We define a partial order $\preceq_{i_2}$ in $\mathbb{C}_2$ as $\zeta \preceq_{i_2} \psi$ iff $\zeta \leq \psi$.

Theorem 4.1. *Suppose that (Λ, ρ) is a complete BICBMS equipped with metric $\rho(\zeta,\psi) = (1+i_2)(|\zeta(\sqcup) - \psi(\sqcup)|)$ for all $\zeta, \psi \in \Lambda$, $\sqcup \in [\leftthreetimes_1, \leftthreetimes_2]$ and $\Gamma, \Upsilon : \Lambda \to \Lambda$ be continuous operator on Λ defined by*

$$\Gamma\zeta(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \ell_1(\sqcup, \mathfrak{s}, \zeta(\mathfrak{s}))d\mathfrak{s}. \tag{8}$$

and

$$\Upsilon\zeta(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \ell_2(\sqcup, \mathfrak{s}, \zeta(\mathfrak{s}))d\mathfrak{s}. \tag{9}$$

If there exists $\lambda < 1$ such that for all $\zeta, \psi \in \Lambda$ with $\zeta \neq \psi$ and $\mathfrak{s}, \sqcup \in [\leftthreetimes_1, \leftthreetimes_2]$ such that

$$|\ell_1(\sqcup, \mathfrak{s}, \zeta(s)) - \ell_2(\sqcup, \mathfrak{s}, \psi(\mathfrak{s}))| \leq \lambda|\zeta(\sqcup) - \psi(\sqcup)|, \tag{10}$$

then the integral operators defined by (8) and (9) have a unique common solution.

Proof. Consider,

$$\begin{aligned}(1+i_2)(|\varGamma\zeta(\sqcup)-\varUpsilon\psi(\sqcup)|) &= \frac{(1+i_2)}{|\leftthreetimes_2-\leftthreetimes_1|}\Bigg(\Bigg|\int_{\leftthreetimes_1}^{\leftthreetimes_2}\ell_1(\sqcup,\mathfrak{s},\zeta(\mathfrak{s}))d\mathfrak{s} \\ &\quad - \int_{\leftthreetimes_1}^{\leftthreetimes_2}\ell_2(\sqcup,\mathfrak{s},\psi(\mathfrak{s}))d\mathfrak{s}\Bigg|\Bigg) \\ &\leq \frac{(1+i_2)}{|\leftthreetimes_2-\leftthreetimes_1|}\Bigg(\int_{\leftthreetimes_1}^{\leftthreetimes_2}|\ell_1(\sqcup,\mathfrak{s},\zeta(\mathfrak{s})) \\ &\quad - \ell_2(\sqcup,\mathfrak{s},\psi(\mathfrak{s}))|d\mathfrak{s}\Bigg) \\ &\leq \frac{(1+i_2)\lambda}{|\leftthreetimes_2-\leftthreetimes_1|}\Bigg(\int_{\leftthreetimes_1}^{\leftthreetimes_2}|\zeta(\sqcup)-\psi(\sqcup)|d\mathfrak{s}\Bigg) \\ &\leq \frac{\lambda}{|\leftthreetimes_2-\leftthreetimes_1|}\int_{\leftthreetimes_1}^{\leftthreetimes_2}(1+i_2)|\zeta(\sqcup)-\psi(\sqcup)|d\mathfrak{s} \\ &\leq \frac{\lambda(1+i_2)|\zeta(\sqcup)-\psi(\sqcup)|}{|\leftthreetimes_2-\leftthreetimes_1|}\int_{\leftthreetimes_1}^{\leftthreetimes_2}d\mathfrak{s}.\end{aligned}$$

Therefore,

$$\rho(\varGamma\zeta,\varUpsilon\psi)) \leq \lambda\rho(\zeta,\psi).$$

Hence, all the hypothesis of Theorem 3.1 are fulfilled with $\lambda < 1$, $\mu = \gamma = 0$ and so, the integral operators $\varGamma$ and $\varUpsilon$ defined by (8) and (9) have a unique common solution. □

Theorem 4.2. *Let $\varLambda = \mathbb{C}^\ell$ be a complete BICBMS with the metric*

$$\rho(\zeta,\psi) = \sum_{i=1}^{\ell}\big(|\zeta_i-\psi_i|^2 + i_2|\zeta_i-\psi_i|^2\big), \tag{11}$$

where $\zeta,\psi \in \varLambda$. If

$$\sum_{j=1}^{\ell}|\lambda_{ij}|^2 \preceq_{i_2} \lambda < 1,\ \forall\ i = 1,2,...\ell,$$

then the linear system

$$\begin{cases}\mathfrak{b}_1 = \beta_{11}\zeta_1+\beta_{12}\zeta_2+\cdots+\beta_{1\ell}\zeta_\ell \\ \mathfrak{b}_2 = \beta_{21}\zeta_1+\beta_{22}\zeta_2+\cdots+\beta_{2\ell}\zeta_\ell \\ \quad\vdots \\ \mathfrak{b}_\ell = \beta_{\ell 1}\zeta_1+\beta_{\ell 2}\zeta_2+\cdots+\beta_{\ell\ell}\zeta_\ell\end{cases}$$

of ℓ linear equations and ℓ unknowns has a unique solution.

Proof. Define $\varUpsilon : \varLambda \to \varLambda$ by

$$\varUpsilon(\zeta) = \mathcal{A}\zeta + \mathfrak{b},$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3, ...\zeta_\ell) \in \mathbb{C}^\ell$, $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2, ...\mathfrak{b}_\ell) \in \mathbb{C}^\ell$ and

$$\mathcal{A} = \begin{pmatrix} \beta_{11} & \beta_{11} & \cdots & \beta_{1\ell} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{\ell 1} & \beta_{\ell 2} & \cdots & \beta_{\ell\ell} \end{pmatrix}.$$

Now,

$$\begin{aligned} \rho(\Upsilon(\zeta), \Upsilon(\psi)) &= \sum_{j=1}^{\ell} (|\lambda_{ij}(\zeta_j - \psi_j)|^2 + i_2 |\lambda_{ij}(\zeta_j - \psi_j)|^2) \\ &\preceq_{i_2} \sum_{j=1}^{\ell} |\lambda_{ij}|^2 \bigg(\sum_{j=1}^{\ell} (|(\zeta_j - \psi_j)|^2 + i_2 |(\zeta_j - \psi_j)|^2) \bigg) \\ &\preceq_{i_2} \lambda \sum_{j=1}^{\ell} (|(\zeta_j - \psi_j)|^2 + i_2 |(\zeta_j - \psi_j)|^2) \\ &= \lambda \rho(\zeta, \psi) \\ &= \lambda \rho(\zeta, \psi) + \frac{\mu \rho(\zeta, \Upsilon\zeta)\rho(\psi, \Upsilon\psi) + \gamma \rho(\psi, \Upsilon\zeta)\rho(\zeta, \Upsilon\psi)}{1 + \rho(\zeta, \psi)}. \end{aligned}$$

Hence all the conditions of Corollary 3.2 are fulfilled with $\lambda = \frac{1}{6}, \mu = 0, \gamma = 0$, $s(\lambda + \sqrt{2}\mu + \sqrt{2}\gamma) < 1$ and so the linear system of equation has a unique solution. □

5. Conclusion

In this paper, we proved some common fixed point theorems in bicomplex valued b-metric space. An illustrative example and application in bicomplex valued b-metric space were given.

Declarations

Availablity of data and materials
Not applicable.

Competing interest
The authors declare that they have no competing interests.

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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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