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WHEN IS $(D+I, K+I)$ AN S-LASKERIAN PAIR?

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ABSTRACT. Let T be a strongly Laskerian domain such that T contains a field K as a subring. Let I be a non-zero proper ideal of T. Let D be a subring of K. Let S be a strongly multiplicatively closed subset of D. The aim of this article is to determine necessary and sufficient conditions in order that $(D+I, K+I)$ to be an S-Laskerian pair.

1. INTRODUCTION

The rings considered in this article are commutative with identity. Modules considered in this article are modules over commutative rings and are unitary. Subrings are assumed to contain the identity of the ring of which it is a subring. We use f.g. for finitely generated.

Let us first give a brief motivation for the research work carried out in this article. Let R be a ring and let S be a multiplicatively closed subset of R. We use m.c. subset for multiplicatively closed subset. Let M be a module over R. Recall that M is said to be *S-finite* if $sM \subseteq F$ for some $s \in S$ and some f.g. submodule F of M [1]. Also, recall that M is called *S-Noetherian* if each submodule of M is an S-finite module. We say that R is *S-Noetherian* if R regarded as an R-module is S-Noetherian [1]. Many interesting and inspiring theorems on S-Noetherian rings are contained in [1]. In [1], Anderson and Dumitrescu have proved S-Noetherian version of Cohen's Theorem, Eakin-Nagata Theorem, and Hilbert Basis Theorem (see [1, Corollaries 5, 7, and Proposition 9]). Motivated by the above mentioned research work on S-Noetherian rings, the concept of S-Laskerian rings is introduced and studied in [18].

It is useful to recall the following definitions from the literature before we mention a brief content of [18]. Let R be a ring. Let M be a module over R. Recall that M is said to be a *Laskerian R-module* if M is a f.g. R-module and every proper submodule of M is a finite intersection of primary submodules of M. We say that R is a *Laskerian ring* if R regarded as an R-module is Laskerian [6, Exercise 23, page 295]. A $\mathfrak{p}$-primary submodule N of M is said to be *strongly primary* if there exists a positive integer k such that $\mathfrak{p}^k M \subseteq N$. A f.g. R-module M is said to be a *strongly Laskerian R-module* if every proper submodule of M

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is a finite intersection of strongly primary submodules of M. We say that R is a *strongly Laskerian ring* if R regarded as an R-module is strongly Laskerian [6, Exercise 28, page 298]. Heinzer and Lantz have proved several interesting and inspiring theorems on Laskerian (respectively, strongly Laskerian) rings in [11].

Let S be a m.c. subset of a ring R. The concept of S-primary ideals of R, that is studied in [18] is inspired by the research work on S-prime ideals of a ring in [10]. Let $\mathfrak{q}$ be an ideal of R with $\mathfrak{q} \cap S = \emptyset$. Recall that $\mathfrak{q}$ is said to be an *S-primary ideal* of R if the following condition holds: there exists $s \in S$ such that for all $a, b \in R$ with $ab \in \mathfrak{q}$, either $sa \in \mathfrak{q}$ or $sb \in \sqrt{\mathfrak{q}}$ [18]. If in addition, there exist $s' \in S$ and $n \in \mathbb{N}$ such that $s'(\sqrt{\mathfrak{q}})^n \subseteq \mathfrak{q}$, then $\mathfrak{q}$ is said to be an *S-strongly primary* ideal of R. (In [18], an S-strongly primary ideal is referred to as a strongly S-primary ideal.) Some basic properties of S-primary (respectively, S-strongly primary) ideals of a ring are contained in [18]. Let I be an ideal of R such that $I \cap S = \emptyset$. Recall that I is said to be *S-decomposable* (respectively, *S-strongly decomposable*) if I is a finite intersection of S-primary (respectively, S-strongly primary) ideals of R [18, Introduction to Section 3]. (In [18], an S-strongly decomposable ideal is referred to as a strongly S-decomposable ideal.) Also, recall that R is said to be *S-Laskerian* (respectively, *S-strongly Laskerian*) if for each proper ideal I of R, either $I \cap S \neq \emptyset$ or there exists $s \in S$ such that $(I :_R s)$ is S-decomposable (respectively, S-strongly decomposable) [18]. (In, [18], an S-strongly Laskerian ring is referred to as a strongly S-Laskerian ring.) It is not hard to verify that any Laskerian (respectively, strongly Laskerian) ring is S-Laskerian (respectively, S-strongly Laskerian) (see [18, Introduction to Section 3]).

$D + M$-constructions, a source of examples and counterexamples have been studied by several researchers in the literature (for example, refer [4, 5, 7]). This article is also motivated by the results proved by Barucci and Fontana in [3].

Let P be a property of rings. Let R be a subring of a ring T. We say that (R, T) is an *P pair* if A satisfies the property P for each intermediate ring A between R and T. Let S be a m.c. subset of R. We say that (R, T) is an *S-P pair* if A satisfies the property S-P for each intermediate ring A between R and T. The abbreviation LP (respectively, SLP) is used for Laskerian pair (respectively, for strongly Laskerian pair). We use the abbreviation S-LP (respectively, S-SLP) for S-Laskerian pair (respectively, for S-strongly Laskerian pair).

Let S be a m.c. subset of a ring R. Recall that S is said to be a *strongly multiplicatively closed subset* of R if $S \cap (\bigcap_{s \in S} Rs) \neq \emptyset$ [10]. We denote the group of units of R by $U(R)$. It is clear that $U(R)$ is a strongly m.c. subset of R.

Let T be a strongly Laskerian domain which contains a field K as a subring. Let I be a non-zero proper ideal of T. Let D be a subring of K. Let S be a strongly m.c. subset of D. The aim of this article is to determine necessary and sufficient conditions in order that $(D + I, K + I)$ to be an S-LP.

If a set A is a subset of a set B and $A \neq B$, we denote it by $A \subset B$ (or by $B \supset A$). Let R be a subring of a ring T. We denote the collection of all subrings A of T with $R \subseteq A$ by $[R, T]$.

Let S be a m.c. subset of a ring R. Let $f : R \to S^{-1}R$ denote the usual ring homomorphism given by $f(r) = \frac{r}{1}$. For any ideal I of R, $f^{-1}(S^{-1}I)$ is called the *saturation of I with respect to S* and is denoted by $Sat_S(I)$ or by $S(I)$.

We next give a brief account of the results that are proved in this article. This article consists of three sections including the introduction. In Section 2, we state and prove some preliminary results that are needed for proving the main result Theorem 3.1 of Section 3.

Let T be a strongly Laskerian domain such that T contains a field K as a subring. Let I be a non-zero proper ideal of T. Let D be a subring of K. Let S be a strongly m.c. subset of D. In Theorem 3.1, among other results, it is proved that $(D+I, K+I)$ is an S-LP if and only if $(D+I, K+I)$ is an S-SLP if and only if $S^{-1}D$ is a field and K is algebraic over D.

2. Some preliminary results

The aim of this section is to state and prove some preliminary results which are needed for proving the main result of this article. We denote the set of all proper ideals of a ring R by $\mathbb{I}(R)$ and $\mathbb{I}(R)\backslash\{(0)\}$ by $\mathbb{I}(R)^*$.

Motivated by the research work of Lu on rings and modules satisfying (accr) [13, 14], Hamed and Hizem have introduced and investigated the concept of rings and modules satisfying S-accr in [8]. Let M be a module over a ring R. Let S be a m.c. subset of R. Recall that an increasing sequence of submodules of M, $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ is said to be *S-stationary* if there exist $s \in S$ and $k \in \mathbb{N}$ such that $sN_n \subseteq N_k$ for all $n \geq k$ [8]. Recall that M is said to *satisfy S-accr* (respectively, *satisfy S-accr**) if for every submodule N of M and every f.g. (respectively, principal) ideal B of R, the increasing sequence of submodules of M, $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ is S-stationary. The ring R is said to *satisfy S-accr* (respectively, *satisfy S-accr**) if R regarded as an R-module satisfies S-accr (respectively, S-accr*) [8]. It is worth mentioning the following interesting results proved by Hamed and Hizem on rings and modules satisfying S-accr in [8]. For any R-module M, the properties S-accr and S-accr* are equivalent [8, Proposition 3.1]. Let N be a submodule of an R-module M. Then M satisfies S-accr if and only if N and $\frac{M}{N}$ satisfy S-accr [8, Theorem 3.2]. If R satisfies S-accr, then M satisfies S-accr for any f.g. R-module M [8, Theorem 3.3].

It is known that a Laskerian module satisfies (accr) [13, Proposition 3.3]. Let S be a m.c. subset of a ring R. If R is S-Laskerian, then R satisfies S-accr* [18, Corollary 3.9(1)] and hence, S-accr by [8, Proposition 3.1]. Let T, K, I, D, S be as mentioned in the abstract of this article. In Theorem 3.1, we prove that $(D+I, K+I)$ is S-LP if and only if $(D+I, K+I)$ is an S-ACCR*P, where we use the abbreviation S-ACCR*P for S-accr* pair. Lemma 2.1 is used in the proof of Theorem 3.1.

Lemma 2.1. *Let T be an integral domain which contains a field K as a subring. Let $I \in \mathbb{I}(T)^*$. Let D be a subring of K and let S be a m.c. subset of D. Let $T_1 = D+I$. If T_1 satisfies S-accr*, then for each $d \in D\backslash\{0\}$, $S \cap (\bigcap_{n=1}^{\infty} Dd^n) \neq \emptyset$ and $S^{-1}D$ is a field.*

Proof. Assume that T_1 satisfies S-accr*. Let $d \in D\backslash\{0\}$. Let $a \in I\backslash\{0\}$. Since $K \subset T$ and $I \in \mathbb{I}(T)^*$, it follows that $\frac{a}{d^n} \in I \subset T_1$ for all $n \in \mathbb{N}$. Let J be the ideal of T_1 given by $J = T_1\frac{a}{d}$. Since T_1 satisfies S-accr* by assumption, the increasing sequence of ideals of T_1, $(J :_{T_1} d) \subseteq (J :_{T_1} d^2) \subseteq (J :_{T_1} d^3) \subseteq \cdots$ is S-stationary. Hence, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(J :_{T_1} d^j) \subseteq (J :_{T_1} d^k)$ for all $j \geq k$. Let $j \geq k$. Observe that $\frac{a}{d^{j+1}} \in (J :_{T_1} d^j)$. From $s(J :_{T_1} d^j) \subseteq (J :_{T_1} d^k)$, we get that $s\frac{a}{d^{j+1}}d^k \in J = T_1\frac{a}{d}$. Hence, there exist $d_j \in D$ and $a_j \in I$ such that $s\frac{a}{d^{j+1}}d^k = (d_j + a_j)\frac{a}{d}$. Since $K \cap I = (0)$, we obtain that $s = d_jd^{j-k}$. Thus given $d \in D\backslash\{0\}$, there exist $s \in S$, $k \in \mathbb{N}$, and $d_j \in D$ such that $s = d_jd^{j-k}$ for all $j \geq k$. This proves that $S \cap (\bigcap_{n=1}^{\infty} Dd^n) \neq \emptyset$.

We next verify that $S^{-1}D$ is a field. Let $\frac{d}{t}$ $(d \in D, t \in S)$ be a non-zero element of $S^{-1}D$. It is clear that $d \neq 0$. Since there exist $s \in S$ and $d' \in D$ such that $s = dd'$, it follows that $\frac{d'}{1}\frac{d}{t} = \frac{s}{t} \in U(S^{-1}D)$ and so, $\frac{d}{t} \in U(S^{-1}D)$. This proves that $S^{-1}D$ is a field. □

Let M be a module over a ring R. Let $n \in \mathbb{N}$. Recall that M is said to *satisfy n-acc* if every ascending sequence of submodules of M, each of which is generated by n elements stabilizes [12]. Also, recall that M is said to *satisfy pan-acc* if M satisfies n-acc for all $n \geq 1$ [12]. We say that R *satisfies n-acc* (respectively, *satisfies pan-acc*) if R regarded as an R-module satisfies n-acc (respectively, pan-acc).

Let S be a m.c. subset of a ring R. Let M be a module over R. Let $n \geq 1$. Recall that M is said to *satisfy S-n-acc* if every ascending sequence of submodules of M, each of which is generated by n elements is S-stationary [19]. Recall that M is said to *satisfy S-pan-acc* if M satisfies S-n-acc for all $n \geq 1$ [19]. We say that R *satisfies S-n-acc* (respectively, *satisfies S-pan-acc*) if R regarded as an R-module satisfies S-n-acc (respectively, S-pan-acc). Some results on modules satisfying S-n-acc (respectively, S-pan-acc) are available in Section 2 of [19].

Let D be an integral domain and let S be a m.c. subset of D. Recall that *D satisfies S-ascending chain condition on principal ideals* (S-ACCP) if every increasing sequence of principal ideals of D is S-stationary [9, Definition 2.1]. Hamed and Kim have proved several interesting results on integral domains satisfying S-ACCP in [9]. Observe that the above definition can be extended to any ring R (which is not necessarily an integral domain) and for any m.c. subset S of R. It is clear that the concept S-ACCP agrees with S-1-acc.

In Theorem 3.1, we prove that $(D + I, K + I)$ is an S-LP if and only if $(D + I, K + I)$ is an S-1-acc pair, where T, K, I, D, S are as mentioned in the abstract of this article. We use Lemma 2.2 in the proof of Theorem 3.1.

Lemma 2.2. *Let T, K, I, D, S, T_1 be as in the statement of Lemma 2.1. If T_1 satisfies S-1-acc, then for each $d \in D\backslash\{0\}$, $S \cap (\bigcap_{n=1}^{\infty} Dd^n) \neq \emptyset$ and $S^{-1}D$ is a field.*

Proof. Assume that T_1 satisfies S-1-acc. Let $d \in D\backslash\{0\}$ and let $a \in I\backslash\{0\}$. Let J be the ideal of T_1 given by $J = T_1\frac{a}{d}$. We claim that $(J :_{T_1} d^n) = T_1\frac{a}{d^{n+1}}$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. It is clear that $\frac{a}{d^{n+1}}d^n = \frac{a}{d} \in J$ and so, $T_1\frac{a}{d^{n+1}} \subseteq (J :_{T_1} d^n)$. Let $d' \in D$ and $a' \in I$ be such that $d' + a' \in (J :_{T_1} d^n)$. This implies that

$(d'+a')d^n \in T_1\frac{a}{d}$. Hence, $d'+a' \in T_1\frac{a}{d^{n+1}}$. This shows that $(J :_{T_1} d^n) \subseteq T_1\frac{a}{d^{n+1}}$ and so, $(J :_{T_1} d^n) = T_1\frac{a}{d^{n+1}}$. Let us denote the ideal $(J :_{T_1} d^n)$ by J_n for each $n \geq 1$. Observe that $J_n = (J :_{T_1} d^n) \subseteq (J :_{T_1} d^{n+1}) = J_{n+1}$. As J_n is principal for each $n \geq 1$, we obtain that $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ is an increasing sequence of principal ideals of T_1. Since T_1 satisfies S-1-acc by assumption, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sJ_j \subseteq J_k$ for all $j \geq k$. Proceeding as in the proof of Lemma 2.1, it follows that $s \in \bigcap_{n=1}^{\infty} Dd^n$ and so, $S \cap (\bigcap_{n=1}^{\infty} Dd^n) \neq \emptyset$ for each $d \in D \backslash \{0\}$. Now, it follows as in the proof of Lemma 2.1 that $S^{-1}D$ is a field. □

Let T, K, I, D, S be as mentioned in the abstract of this article. We prove in Theorem 3.1 that $(D+I, K+I)$ is an S-LP if and only if $(D+I, K+I)$ is an S-n-acc pair for all $n \geq 1$. We use Lemma 2.3 in the proof of Theorem 3.1.

Lemma 2.3. *Let S be a m.c. subset of a ring R. If R is S-strongly Laskerian, then R satisfies S-pan-acc.*

Proof. Let $n \in \mathbb{N}$. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending sequence of ideals of R, each of which is generated by n elements. If $I_k \cap S \neq \emptyset$ for some $k \in \mathbb{N}$, then for any $s \in I_k \cap S$, $sI_j \subseteq Rs \subseteq I_k$ for all $j \geq k$. Hence, in proving that the above ascending sequence of n-generated ideals of R is S-stationary, we can assume that $I_k \cap S = \emptyset$ for all $k \in \mathbb{N}$. Since R is S-strongly Laskerian by hypothesis, we obtain from (1) $\Rightarrow$ (2) of [18, Proposition 3.2] that $S^{-1}R$ is strongly Laskerian and for any ideal I of R with $I \cap S = \emptyset$, $Sat_S(I) = (I :_R s)$ for some $s \in S$. Observe that $S^{-1}I_1 \subseteq S^{-1}I_2 \subseteq S^{-1}I_3 \subseteq \cdots$ is an ascending sequence of n-generated ideals of $S^{-1}R$. As $S^{-1}R$ is strongly Laskerian, we obtain from [12, Corollary 3.6(b)] that there exists $k \in \mathbb{N}$ such that $S^{-1}I_j = S^{-1}I_k$ for all $j \geq k$. Hence, $Sat_S(I_j) = Sat_S(I_k)$ for all $j \geq k$. It follows from (1) $\Rightarrow$ (2) of [18, Proposition 3.2] that there exists $s \in S$ such that $Sat_S(I_k) = (I_k :_R s)$. We claim that $sI_j \subseteq I_k$ for all $j \geq k$. Let $j \geq k$. Now, $I_j \subseteq Sat_S(I_j) = Sat_S(I_k) = (I_k :_R s)$ and so, $sI_j \subseteq I_k$. This shows that for any $n \in \mathbb{N}$, any increasing sequence of n-generated ideals of R is S-stationary. Therefore, R satisfies S-pan-acc. □

Lemma 2.4. *Let S be a strongly m.c. subset of a ring R. Then for any ideal I of R, $Sat_S(I) = (I :_R s')$ for any $s' \in S \cap (\bigcap_{s\in S} Rs)$.*

Proof. This is [19, Lemma 2.7]. □

Let R be a ring. Recall that R is said to *satisfy strong accr** if for any ideal I of R and any sequence $< a_n >$ of elements of R, the increasing sequence of residuals of the form $(I :_R a_1) \subseteq (I :_R a_1a_2) \subseteq (I :_R a_1a_2a_3) \subseteq \cdots$ terminates [19]. Let S be a m.c. subset of R. We recall that R is said to *satisfy S-strong accr** if for any ideal I of R and any sequence $< a_n >$ of elements of R, the increasing sequence of residuals of the form $(I :_R a_1) \subseteq (I :_R a_1a_2) \subseteq (I :_R a_1a_2a_3) \subseteq \cdots$ is S-stationary [19]. It is clear that if R satisfies S-strong accr*, then R satisfies S-accr*. If R is S-strongly Laskerian, then R satisfies S-strong accr* [18, Corollary 3.9(2)].

Let T, K, I, D, S be as mentioned in the abstract of this article. We prove in Theorem 3.1 , among other results, that $(D+I, K+I)$ is an S-LP if and only if $(D+I, K+I)$ is an S-SACCR*P where we use the abbreviation S-SACCR*P for S-strong accr* pair.

3. When is $(D+I, K+I)$ an S-Laskerian pair?

Let T be a strongly Laskerian domain which contains a field K as a subring. Let $I \in \mathbb{I}(T)^*$. Let D be a subring of K and let S be a strongly m.c. subset of D. Let $T_1 = D + I$. In Theorem 3.1, we determine necessary and sufficient conditions in order that $(T_1, K+I)$ to be an S-LP.

Theorem 3.1. *Let T be a strongly Laskerian domain such that T contains a field K as a subring. Let $I \in \mathbb{I}(T)^*$. Let D be a subring of K. Let S be a strongly m.c. subset of D. Let $T_1 = D + I$. Then the following statements are equivalent:*
(1) $(T_1, K+I)$ is an S-SLP.
(2) $(T_1, K+I)$ is an S-LP.
*(3) $(T_1, K+I)$ is an S-ACCR*P.*
(4) $S^{-1}A$ is a field for each $A \in [D, K]$.
(5) $S^{-1}D$ is a field and K is algebraic over D.
*(6) $(T_1, K+I)$ is an S-SACCR*P.*
(7) $(T_1, K+I)$ is an S-n-acc pair for all $n \geq 1$.
(8) $(T_1, K+I)$ is an S-1-acc pair.

Proof. It is clear that S is a strongly m.c. subset of each $A \in [D, K]$ and hence, S is a strongly m.c. subset of $A + I$ for each $A \in [D, K]$.
(1) $\Rightarrow$ (2) This is clear, since any S-strongly Laskerian ring is S-Laskerian.
(2) $\Rightarrow$ (3) This is clear, since any S-Laskerian ring satisfies S-accr* by [18, Corollary 3.9(1)].
(3) $\Rightarrow$ (4) Let $A \in [D, K]$. Then $A + I \in [T_1, K+I]$. By (3), $A + I$ satisfies S-accr*. Hence, we obtain from Lemma 2.1 that $S^{-1}A$ is a field.
(4) $\Rightarrow$ (5) Assume that $S^{-1}A$ is a field for each $A \in [D, K]$. As $D \in [D, K]$, it follows that $S^{-1}D$ is a field. Let $\alpha \in K$. Note that $D[\alpha] \in [D, K]$. As $S^{-1}(D[\alpha]) = (S^{-1}D)[\alpha]$ is a field, we get that α is algebraic over $S^{-1}D$ and so, α is algebraic over D. Therefore, K is algebraic over D.
(5) $\Rightarrow$ (1) Assume that $S^{-1}D$ is a field and K is algebraic over D. Then K is algebraic over $S^{-1}D$ and so, K is integral over $S^{-1}D$. Let $A \in [D, K]$. Since $S^{-1}D \subseteq S^{-1}A$, we get that K is integral over the integral domain $S^{-1}A$. Hence, we obtain from [2, Proposition 5.7] that $S^{-1}A$ is a field.

Let $R \in [T_1 = D + I, K + I]$. Then $R = A + I$ for some $A \in [D, K]$. It is clear that $S^{-1}R = S^{-1}A + I$. Note that $S^{-1}R$ is a subring of T, I is an ideal common to both T and $S^{-1}R$, and T is strongly Laskerian by hypothesis. From $\frac{S^{-1}R}{I} \cong S^{-1}A$ is a field, we obtain from [16, Theorem 8 and Corollary 9] that $S^{-1}R$ is strongly Laskerian. Moreover, by hypothesis, S is a strongly m.c. subset of D and so, S is a strongly m.c. subset of R. Hence, we obtain from [19, Lemma 2.7] that there exists $s \in S$ such that $Sat_S(J) = (J :_R s)$ for any ideal J of R. It now follows from (2) $\Rightarrow$ (1) of [18, Proposition 3.2] that R is S-strongly Laskerian. Therefore, $(T_1, K+I)$ is an S-SLP.
(1) $\Rightarrow$ (6) This is clear, since any S-strongly Laskerian ring satisfies S-strong accr* by [18, Corollary 3.9(2)].
(6) $\Rightarrow$ (3) This is clear, since S-strong accr* implies S-accr*.

(1) $\Rightarrow$ (7) This follows immediately, since any S-strongly Laskerian ring satisfies S-n-acc for all $n \geq 1$ by Lemma 2.3.
(7) $\Rightarrow$ (8) This is clear, since if a ring satisfies S-n-acc for all $n \geq 1$, then it satisfies S-1-acc.
(8) $\Rightarrow$ (4) Let $A \in [D, K]$. As $A + I \in [T_1, K + I]$, by (8), $A + I$ satisfies S-1-acc. Hence, we obtain from Lemma 2.2 that $S^{-1}A$ is a field. □

We provide Example 3.2 to illustrate that (4) $\Rightarrow$ (3) (respectively, (4) $\Rightarrow$ (8)) of Theorem 3.1 can fail to hold if S is a strongly m.c. subset is omitted in the hypothesis of Theorem 3.1.

Example 3.2. Let $T = \mathbb{Q}[X]$ be the polynomial ring in one variable X over $\mathbb{Q}$. Let $I = XT$. Let $D = \mathbb{Z}$ and let $S = \mathbb{Z}\backslash\{0\}$. Let $T_1 = \mathbb{Z} + I$. Then $S^{-1}A$ is a field for each $A \in [\mathbb{Z}, \mathbb{Q}]$ but T_1 does not satisfy S-accr* and T_1 does not satisfy S-1-acc.

Proof. It is clear that $S = \mathbb{Z}\backslash\{0\}$ is a m.c. subset of $\mathbb{Z}$. Let $A \in [\mathbb{Z}, \mathbb{Q}]$. Then $S^{-1}A = \mathbb{Q}$ is a field. From $\bigcap_{n=1}^{\infty} \mathbb{Z}2^n = (0)$, it follows from Lemma 2.1 (respectively, Lemma 2.2) that T_1 does not satisfy S-accr* (respectively, T_1 does not satisfy S-1-acc). □

Corollary 3.3. *Let T, K, I, D, T_1 be as in the statement of Theorem 3.1. Then the following statements are equivalent:*
(1) *$(T_1, K + I)$ is an SLP.*
(2) *$(T_1, K + I)$ is an LP.*
(3) *$(T_1, K + I)$ is an ACCR*P.*
(4) *A is a field for each $A \in [D, K]$.*
(5) *D is a field and K is algebraic over D.*
(6) *$(T_1, K + I)$ is an SACCR*P.*
(7) *$(T_1, K + I)$ is an n-acc pair for all $n \geq 1$.*
(8) *$(T_1, K + I)$ is an 1-acc pair.*

Proof. For any ring R, any m.c. subset $S \subseteq U(R)$ is a strongly m.c. subset of R. Each one of the ring-theoretic property P: strongly Laskerian, Laskerian, strong accr*, accr*, n-acc ($n \in \mathbb{N}$) is such that R has P if and only if R has S-P. Hence, the proof of this corollary follows immediately from Theorem 3.1. □

Let T, K, I, D, T_1 be as in the statement of Theorem 3.1. If (T_1, T) is an SLP, then it is clear that $(T_1, K + I)$ is an SLP. We provide Example 3.4 to illustrate that $(T_1, K + I)$ is an SLP need not imply that (T_1, T) is an SLP.

Example 3.4. Let $K = \mathbb{Q}(\sqrt{2})$. Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over K. Let $I = (1 + XY)T$. Let $D = \mathbb{Q}$. Let $T_1 = \mathbb{Q} + I$. Then $(T_1, K + I)$ is an SLP but (T_1, T) is not an SLP.

Proof. Since the field K is algebraic over the field $\mathbb{Q}$, it follows from (5) $\Rightarrow$ (1) of Corollary 3.3 that $(T_1, K + I)$ is an SLP. Let $B = K[X] + I$. Since $X \notin U(K[X])$, it follows from the proof of [17, Proposition 1.3] that B does not satisfy (accr) and hence, B is not Laskerian by [13, Proposition 3]. As $B \in [T_1, T]$, it follows that (T_1, T) is not an SLP. □

We use the abbreviation NP for Noetherian pair and S-NP for S-Noetherian pair. Let T be an integral domain which contains a field K as a subring. Let D be a subring of K and let S be a m.c. subset of D. Let $I \in \mathbb{I}(T)^*$. Let $T_1 = D + I$. We prove in Theorem 3.5 that (T_1, T) is an S-NP if and only if (T_1, T) is an NP.

Theorem 3.5. *Let T be an integral domain such that T contains a field K as a subring. Let D be a subring of K and let S be a m.c. subset of D. Let $I \in \mathbb{I}(T)^*$. Let $T_1 = D + I$. Then the following statements are equivalent:*
(1) *(T_1, T) is an S-NP.*
(2) *T_1 is S-Noetherian.*
(3) *D is a field and T_1 is S-Noetherian.*
(4) *T_1 is Noetherian.*
(5) *(T_1, T) is an NP.*

Proof. (1) $\Rightarrow$ (2) This is clear.
(2) $\Rightarrow$ (3) Assume that T_1 is S-Noetherian. By hypothesis, $I \in \mathbb{I}(T)^*$. It is clear that $I \in \mathbb{I}(T_1)^*$. As T_1 is S-Noetherian, there exist $s \in S$ and a f.g. ideal J of T_1 such that $sI \subseteq J \subseteq I$. Since $s \in U(K) \subseteq U(T)$ and $I \in \mathbb{I}(T)^*$, it follows that $sI = I$. Hence, $I = J$ is a f.g. ideal of T_1. If $I = I^2$, then it follows from [15, Exercise 2.1, page 13] that $I = T_1 e$ for some idempotent element e of T_1. Since T_1 is an integral domain, we obtain that $e \in \{0, 1\}$ and this implies that $I \in \{(0), T_1\}$. This is impossible, since $I \in \mathbb{I}(T_1)^*$. Therefore, $I \neq I^2$. As I is a f.g. ideal of T_1, there exist $a_1, \ldots, a_k \in I$ such that $I = \sum_{i=1}^{k} T_1 a_i = \sum_{i=1}^{k}(D+I)a_i \subseteq (\sum_{i=1}^{k} Da_i)+I^2 \subseteq I$. Therefore, $I = (\sum_{i=1}^{k} Da_i)+I^2$. Hence, $\frac{I}{I^2} = \sum_{i=1}^{k} D(a_i+I^2)$ is a f.g. D-module. Observe that $\frac{I}{I^2} = \sum_{i=1}^{k} D(a_i + I^2) \subseteq \sum_{i=1}^{k} K(a_i + I^2) \subseteq \frac{I}{I^2}$. Hence, $\frac{I}{I^2}$ is a finite-dimensional vector space over K. As $\frac{I}{I^2}$ is isomorphic to a direct sum of a finite number of copies of K, we get that K is a f.g. D-module. It follows from $(iv) \Rightarrow (i)$ of [2, Proposition 5.1] that K is an integral extension of D. Therefore, we obtain from [2, Proposition 5.7] that D is a field.
(3) $\Rightarrow$ (4) Assume that D is a field and T_1 is S-Noetherian. As D is a field, it follows that $S \subseteq U(D) \subseteq U(T_1)$. Since T_1 is S-Noetherian, we obtain from [1, Proposition 2(e)] that T_1 is Noetherian.
(4) $\Rightarrow$ (5) Assume that T_1 is Noetherian. Let $A \in [T_1, T]$. Let $a \in I\backslash\{0\}$. Then $aA \subseteq I \subset T_1$. Hence, $A \subset \frac{1}{a}T_1$. Therefore, A is a submodule of a f.g. T_1-module. Since T_1 is Noetherian, we get that $\frac{1}{a}T_1$ is a Noetherian T_1-module and so, A is a Noetherian T_1-module. As any ideal of A is also an T_1-module, it follows that A satisfies a.c.c. on ideals of A. Hence, A is Noetherian. Therefore, (T_1, T) is an NP.
(5) $\Rightarrow$ (1) This is clear. $\square$

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