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DEGENERATE THIRD-ORDER EQUATION: DIRECT AND INVERSE PROBLEMS

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Abstract. For a third-order equation with a discontinuous coefficient and power degeneracy, a direct problem with a nonlocal lateral condition and an inverse problem for a factor of the right-hand side are studied in a rectangle. A uniqueness criterion is obtained: the principal determinant splits into three factors of different exponential orders, so the small denominator accompanies only the even part of the right-hand side. Hence for an odd right-hand side the problem is classically solvable with finitely smooth data and without restricting the domain; otherwise a Gevrey-type condition is needed. The inverse problem is uniquely solvable for an odd factor.

Keywords. Third order, discontinuous coefficient, degenerate coefficient, Gevrey class, inverse problem.
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1. Introduction and preliminaries

The equation studied in this paper is mathematically related to the viscous transonic equation, which arises in studying the structure of compressible flows near a sonic surface, taking into account viscosity and thermal conductivity [23]; boundary value problems for the viscous transonic equation itself have recently been treated by separation of variables in [2]. The simplest linear analogue is the Cattabriga equation $u_{xxx} - u_{yy} = 0$ [6]; the introduction of a discontinuous coefficient $\operatorname{sgn}(x)$ generates a class of equations with changing direction of evolution [17, 20, 27]. The equation is parabolic with multiple characteristics in the terminology of Cattabriga [6]: the principal part of the operator is ∂_x^3, its symbol ξ^3 vanishes on the conormal to the lines $y = \text{const}$ with multiplicity three. The difference from known settings is twofold: the coefficient changes sign abruptly, and the lateral conditions are nonlocal: the derivatives of the solution at the boundaries $x = \pm a$ are related by the parameter γ.

Equations of this type also arise in the theory of nonlinear waves with dispersion: the same operator ∂_x^3 appears in the Korteweg-de Vries equation, and the fundamental solution of its linearization is expressed in terms of the Airy

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function, which also appears in the kernels of the Cattabriga potentials [6]; the Kawahara equation [13] is a special case of an odd-order equation.

A systematic study of boundary value problems for odd-order equations with multiple characteristics was begun in [4], where a fundamental solution was constructed; for the operator $D_x^{2n+1} + (-1)^n D_y^2$, analogs of the single- and double-layer potentials, a uniqueness criterion, and existence theorems were obtained in [7]. Equations with discontinuous coefficients were studied in [8], problems in a rectangle with and without degeneracy in [1, 12, 14, 15, 18], nonlocal problems for equations with discontinuous coefficients in [16], inverse problems in [3], and even-order equations in [10, 11, 24, 25, 26].

This paper extends the results of [6, 7] in two directions: it introduces a discontinuous coefficient $\operatorname{sgn}(x)$, generating conjugation conditions on the line $x = 0$, and a power-law degeneracy of y^m, $0 \le m < 1$, under which the spectral problem with respect to y is solved in Bessel functions of order $1/(2-m)$. The main effect to which the paper is devoted is as follows: after splitting the solution by parity, the problem decomposes into two three-point problems with determinants of different exponential orders, and small denominators arise only in one of them, accompanying only the even-in-x part of the right-hand side.

2. Statement of the problem and the spectral problem

In the domain $\Omega = \{(x,y)\colon\ -a < x < a,\ 0 < y < 1\}$, $a > 0$, the following problem is considered. Throughout, m is a fixed number, $0 \le m < 1$, and ν is a fixed number, $\nu \in \{1, 2\}$.

Problem D_ν. Find a function $u(x,y) \in C^{3,2}_{x,y}(\Omega \setminus \{x=0\}) \cap C^{2,1}_{x,y}(\bar{\Omega})$ with finite one-sided limits $\partial^3 u/\partial x^3$ as $x \to \pm 0$, satisfying the conditions:

$$\begin{cases} u_{xxx} - \operatorname{sgn}(x) y^m u_{yy} = f(x,y), \quad x \neq 0; \\ u(-a,y) = u(a,y) = 0; \\ \frac{\partial^\nu u(-a,y)}{\partial x^\nu} = \gamma \frac{\partial^\nu u(a,y)}{\partial x^\nu}, \quad \gamma \in \mathbb{R}; \\ \frac{\partial^j u(-0,y)}{\partial x^j} = \frac{\partial^j u(+0,y)}{\partial x^j}, j = 0,1,2; \\ u(x,0) = u(x,1) = 0; \end{cases} \tag{2.1}$$

where $f(x,y)$ is a given sufficiently smooth function.

Given the conditions in y of problem D_ν, separation of variables leads to the spectral problem

$$Y''(y) + \lambda y^{-m} Y(y) = 0, \qquad Y(0) = Y(1) = 0. \tag{2.2}$$

Problem (2.2) is solved in Bessel functions: its solutions bounded at the origin and vanishing at $y = 0$ are $\sqrt{y}\, J_{\nu_0}\big(\frac{2\sqrt{\lambda}}{2-m} y^{\frac{2-m}{2}}\big)$, where $\nu_0 = \frac{1}{2-m} \in [\frac{1}{2}, 1)$, the second solution $\sqrt{y}\, J_{-\nu_0}(\,\cdot\,)$ being $O(1)$ as $y \to 0$ [29]; hence $Y(1) = 0$ means $J_{\nu_0}\big(\frac{2\sqrt{\lambda}}{2-m}\big) = 0$. Denoting by $0 < \mu_1 < \mu_2 < \dots$ the positive zeros of J_{ν_0}, all of which are simple [28, 29], we obtain the eigenvalues and the eigenfunctions orthonormal in

the space $L_{2,m}(0,1)$ of functions with finite norm $\|\varphi\| = \left(\int_0^1 y^{-m}\varphi^2(y)dy\right)^{1/2}$:

$$\lambda_k = \left(\frac{2-m}{2}\mu_k\right)^2, \qquad Y_k(y) = \frac{\sqrt{2-m}}{|J_{\nu_0+1}(\mu_k)|}\sqrt{y}\,J_{\nu_0}\left(\mu_k y^{\frac{2-m}{2}}\right), \qquad k \in N, \tag{2.3}$$

the normalising factor following from the substitution $z = y^{\frac{2-m}{2}}$. Since the kth positive zero $j_{\alpha,k}$ of J_α increases monotonically in α [29] and $\nu_0 \geq \frac{1}{2}$,

$$\mu_k = j_{\nu_0,k} \;\geq\; j_{\frac{1}{2},k} = \pi k, \qquad \lambda_k \;\geq\; \left(\frac{(2-m)\,\pi k}{2}\right)^2, \qquad k \in N, \tag{2.4}$$

with equality for $m = 0$, while McMahon's expansion [19, 29] gives

$$\mu_k = \pi\,(k + C_m) - \frac{4\nu_0^2 - 1}{8\pi\,(k + C_m)} + O\left(k^{-3}\right), \qquad C_m = \tfrac{m}{4(2-m)}, \qquad k \to \infty, \tag{2.5}$$

of which only the leading part

$$\mu_k = \pi\,(k + C_m) + O\left(k^{-1}\right), \qquad k \to +\infty, \tag{2.6}$$

is used below. Finally, the eigenfunctions satisfy the integral equation with the Green function $G(y,\xi)$ of the operator $-d^2/dy^2$ subject to the Dirichlet conditions:

$$Y_k(y) = \lambda_k \int_0^1 G(y,\xi)\,\xi^{-m} Y_k(\xi)\,d\xi, \qquad G(y,\xi) = \begin{cases} \xi\,(1-y), & 0 \leq \xi \leq y, \\ y(1-\xi), & y \leq \xi \leq 1. \end{cases} \tag{2.7}$$

Theorem 2.1. *The system $\{Y_k\}$ is complete in $L_{2,m}(0,1)$. Moreover, every function $\varphi(y) \in C^2[0,1]$ with $\varphi(0) = \varphi(1) = 0$ can be expanded into a uniformly and absolutely convergent Fourier series in the eigenfunctions $Y_k(y)$:*

$$\varphi(y) = \sum_{k=1}^{\infty} c_k Y_k(y), \qquad c_k = \int_0^1 y^{-m}\varphi(y)\,Y_k(y)\,dy.$$

Proof. The kernel $F(y,\xi) = y^{-\frac{m}{2}}G(y,\xi)\xi^{-\frac{m}{2}}$ is symmetric and continuous on $[0,1]^2$, since it does not exceed y^{1-m} and $1-m > 0$, and by (2.7) the functions $g_k = y^{-\frac{m}{2}}Y_k$ are its eigenfunctions. The corresponding operator is injective: applying $-d^2/dy^2$ to $\int_0^1 G(y,\xi)h(\xi)d\xi = 0$ gives $h = 0$. Hence $\{g_k\}$ is complete in $L_2(0,1)$ and $\{Y_k\}$ in $L_{2,m}(0,1)$. Further, $\varphi(0) = \varphi(1) = 0$ implies $y^{-\frac{m}{2}}\varphi = \int_0^1 F(y,\xi)\big(-\xi^{\frac{m}{2}}\varphi''(\xi)\big)d\xi$, so the Hilbert–Schmidt theorem yields the expansion of $y^{-m/2}\varphi$ in the system $\{g_k\}$; multiplying it by $y^{m/2}$ gives the assertion with $c_k = \int_0^1 y^{-m}\varphi Y_k\,dy$. □

Applying Bessel's inequality in $L_{2,m}(0,1)$ to the function $\xi \mapsto \partial_y^j G(y,\xi)$, whose Fourier coefficients equal $\lambda_k^{-1}d^jY_k(y)/dy^j$ by (2.7), and using $\left|\partial_y^j G\right| \leq 1$ and

$\int_0^1 \xi^{-m} d\xi = \frac{1}{1-m} < \infty$, we obtain, uniformly in $y \in [0,1]$,

$$\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \frac{d^j Y_k(y)}{dy^j} \right)^2 \leq \int_0^1 \xi^{-m} \left\{ \frac{\partial^j G(y,\xi)}{\partial y^j} \right\}^2 d\xi \leq M, \qquad j = 0, 1, \tag{2.8}$$

and in particular

$$\left| \frac{d^j Y_k(y)}{dy^j} \right| \leq M \lambda_k, \qquad j = 0, 1. \tag{2.9}$$

The bound (2.9) is rough for $j = 0$; the sharp one, together with an estimate of the Fourier coefficients, is given by the following lemma.

Lemma 2.1. *There exists a constant $M = M(m)$ such that*

$$|Y_k(y)| \leq M, \qquad y \in [0,1], \ k \in N. \tag{2.10}$$

If, in addition, φ satisfies the conditions of Theorem 2.1, then its Fourier coefficients satisfy

$$|c_k| \ \leq \ \frac{1}{\lambda_k} \left\| y^{\frac{m}{2}} \varphi'' \right\|_{L_2[0,1]}, \qquad \textit{in particular} \qquad \sum_{k=1}^{\infty} |c_k| < \infty. \tag{2.11}$$

Proof. By the Hankel expansion [19, 29] and $J_{\nu_0}(\mu_k) = 0$ one has $|J_{\nu_0+1}(\mu_k)| = \sqrt{2/(\pi\mu_k)}\,(1 + O(\mu_k^{-1}))$, so the normalising factor in (2.3) is of order $\sqrt{\mu_k}$. Put $t = \mu_k y^{(2-m)/2}$. For $t \geq 1$ the bound $|J_{\nu_0}(t)| \leq C t^{-1/2}$ gives $|Y_k(y)| \leq M\sqrt{\mu_k}\sqrt{y}\, t^{-1/2} = M y^{\frac{m}{4}} \leq M$; for $t \leq 1$ the bound $|J_{\nu_0}(t)| \leq C t^{\nu_0}$ together with $\nu_0 \frac{2-m}{2} = \frac{1}{2}$ gives $|Y_k(y)| \leq M \mu_k^{\frac{1}{2}+\nu_0} y \leq M \mu_k^{\frac{1}{2}-\nu_0} \leq M$, since $\nu_0 \geq \frac{1}{2}$. This proves (2.10). Next, $\varphi(0) = \varphi(1) = 0$ implies $\varphi = \int_0^1 G(\cdot,\xi)\big(-\varphi''(\xi)\big) d\xi$, whence, changing the order of integration and using (2.7), $c_k = -\lambda_k^{-1} \int_0^1 \varphi''(\xi) Y_k(\xi) d\xi$; applying the Cauchy–Bunyakovsky inequality to $\int_0^1 \big(\xi^{\frac{m}{2}} \varphi''\big)\big(\xi^{-\frac{m}{2}} Y_k\big) d\xi$ and taking into account $\|\xi^{-m/2} Y_k\|_{L_2} = 1$, we obtain the first inequality in (2.11); the second follows from (2.4). □

3. The main determinant and the uniqueness criterion

From now on we assume that the function $f(x,y)$ satisfies the conditions of Theorem 2.1 with respect to y (uniformly in x). We look for the solution of problem D_ν in the form

$$u(x,y) = \sum_{k=1}^{\infty} X_k(x)\, Y_k(y). \tag{3.1}$$

Substituting (3.1) into equation (2.1) and using the relations $y^m Y_k''(y) = -\lambda_k Y_k(y)$ and $\operatorname{sgn}(x) = \pm 1$, we get

$$\sum_k \big(X_k''' + \operatorname{sgn}(x) \lambda_k X_k \big) Y_k = f;$$

multiplying by $y^{-m}Y_n(y)$ and integrating over $y \in (0,1)$, we arrive at the problem for $X_k(x)$:

$$\begin{cases} X_k'''(x) + \lambda_k X_k(x) = f_k^+(x), \ 0 < x < a; \\ X_k'''(x) - \lambda_k X_k(x) = f_k^-(x), \ -a < x < 0; \\ X_k(a) = X_k(-a) = 0, \ X_k^{(\nu)}(-a) = \gamma X_k^{(\nu)}(a), \\ X_k(+0) = X_k(-0), \ X_k'(+0) = X_k'(-0), \ X_k''(+0) = X_k''(-0), \end{cases} \tag{3.2}$$

where

$$f_k^{\pm}(x) = \int_0^1 y^{-m} f(x,y) Y_k(y)\, dy, \tag{3.3}$$

for $x \in (0,a)$ and $x \in (-a,0)$ respectively.

Let us introduce the notation ($s_k := \sqrt[3]{\lambda_k}$):

$$T_1(x) = e^{-s_k x}, \qquad T_{2j}(x) = e^{\frac{1}{2}s_k x} \cos\left(\frac{\sqrt{3}}{2} s_k x + j\frac{\pi}{3}\right),$$

$$T_{3j}(x) = e^{\frac{1}{2}s_k x} \sin\left(\frac{\sqrt{3}}{2} s_k x + j\frac{\pi}{3}\right), \quad j = 0,1,2,3;$$

then the general solution of equations (3.2) has the form

$$X_k(x) = c_1 T_1(x) + c_2 T_{20}(x) + c_3 T_{30}(x) + \widetilde{X}_k^+(x), \qquad x > 0, \tag{3.4}$$

$$X_k(x) = d_1 T_1(-x) + d_2 T_{20}(-x) + d_3 T_{30}(-x) + \widetilde{X}_k^-(x), \qquad x < 0, \tag{3.5}$$

where c_j, d_j, $j = 1,2,3$, are arbitrary constants and $\widetilde{X}_k^+(x), \widetilde{X}_k^-(x)$ are particular solutions of equations (3.2). Everywhere below

$$\theta_k = \frac{\sqrt{3}}{2} a s_k, \qquad T_1(a) = e^{-a s_k},$$

$$T_{2j}(a) = e^{\frac{a s_k}{2}} \cos\left(\theta_k + j\frac{\pi}{3}\right), \qquad T_{3j}(a) = e^{\frac{a s_k}{2}} \sin\left(\theta_k + j\frac{\pi}{3}\right).$$

The following formulas for the derivatives hold:

$$X_k^{(j)}(x) = s_k^j \left((-1)^j c_1 T_1(x) + c_2 T_{2j}(x) + c_3 T_{3j}(x)\right) + \frac{d^j \widetilde{X}_k^+(x)}{dx^j}, \tag{3.6}$$

for $x > 0$, and

$$X_k^{(j)}(x) = (-1)^j s_k^j \big((-1)^j d_1 T_1(-x) + d_2 T_{2j}(-x) + d_3 T_{3j}(-x)\big) + \frac{d^j \widetilde{X}_k^-(x)}{dx^j}, \tag{3.7}$$

for $x < 0$. Formulas (3.6), (3.7) are proved by induction: one checks directly that $T_1' = -s_k T_1$, $T_{2j}' = s_k T_{2,j+1}$, $T_{3j}' = s_k T_{3,j+1}$; in particular $T_{2,3} = -T_{20}$ and $T_{3,3} = -T_{30}$, which agrees with the equation $X''' = -\lambda_k X$. In what follows it is convenient to treat the index j in T_{2j}, T_{3j} as an arbitrary integer, the functions being defined by the same formulas.

Convention. All constants independent of k are denoted by M. Where the dependence on the parameters a, ν, γ, m is essential, it is indicated explicitly: $M = M(\nu, a)$ and so on.

Substituting (3.4), (3.5) into the conditions (3.2) gives a system of six equations for $c_1, c_2, c_3, d_1, d_2, d_3$ (the particular solutions $\widetilde{X}_k^{\pm}$ are arbitrary here; a specific choice will be needed only in Section 4):

$$\begin{cases} c_1T_1(a) + c_2T_{20}(a) + c_3T_{30}(a) = -\widetilde{X}_k^+(a), \\ d_1T_1(a) + d_2T_{20}(a) + d_3T_{30}(a) = -\widetilde{X}_k^-(-a), \\ d_1T_1(a) + (-1)^{\nu}\left(d_2T_{2\nu}(a) + d_3T_{3\nu}(a)\right) \\ \qquad -\gamma\left((-1)^{\nu}c_1T_1(a) + c_2T_{2\nu}(a) + c_3T_{3\nu}(a)\right) = R_k, \\ c_1 + c_2 - d_1 - d_2 = P_0, \\ -c_1 + \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_3 - d_1 + \frac{1}{2}d_2 + \frac{\sqrt{3}}{2}d_3 = P_1, \\ c_1 - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_3 - d_1 + \frac{1}{2}d_2 - \frac{\sqrt{3}}{2}d_3 = P_2, \end{cases} \tag{3.8}$$

where

$$R_k = s_k^{-\nu}\left[\gamma\frac{d^{\nu}\widetilde{X}_k^+}{dx^{\nu}}(a) - \frac{d^{\nu}\widetilde{X}_k^-}{dx^{\nu}}(-a)\right],$$

$$P_j = s_k^{-j}\left[\frac{d^j\widetilde{X}_k^-}{dx^j}(0) - \frac{d^j\widetilde{X}_k^+}{dx^j}(0)\right], \qquad j = 0, 1, 2.$$

The main determinant of system (3.8) has the following form (the argument a of the functions T_1, T_{2j}, T_{3j} is omitted below, that is $T_1 \equiv T_1(a)$ and so on; moreover, for brevity we put $\varsigma = (-1)^{\nu}$):

$$\Delta(a,k,\nu) = \begin{vmatrix} T_1 & T_{20} & T_{30} & 0 & 0 & 0 \\ 0 & 0 & 0 & T_1 & T_{20} & T_{30} \\ -\gamma\varsigma T_1 & -\gamma T_{2\nu} & -\gamma T_{3\nu} & T_1 & \varsigma T_{2\nu} & \varsigma T_{3\nu} \\ 1 & 1 & 0 & -1 & -1 & 0 \\ -1 & \frac{1}{2} & \frac{\sqrt{3}}{2} & -1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -1 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{vmatrix}.$$

Adding the columns $1, 2, 3$ to the columns $4, 5, 6$ respectively, subtracting the second row from the first and interchanging the first and the fifth rows, we bring the matrix to a block form whose lower right 3×3 block vanishes. For such a matrix $\left|\begin{smallmatrix} P & Q \\ S & 0 \end{smallmatrix}\right| = -|Q||S|$, because the permutation of the first three columns with the last three is odd. Since after these operations the third row of the block Q equals $(\varsigma - \gamma)\big(\varsigma T_1,\, T_{2\nu},\, T_{3\nu}\big)$, we arrive at

$$\Delta(a,k,\nu) = \left((-1)^{\nu} - \gamma\right)\Delta_1^{\nu}\Delta_2, \tag{3.9}$$

where

$$\Delta_1^{\nu} = \begin{vmatrix} -2 & 1 & \sqrt{3} \\ T_1 & T_{20} & T_{30} \\ (-1)^{\nu}T_1 & T_{2\nu} & T_{3\nu} \end{vmatrix} = 2\begin{vmatrix} -1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ T_1 & T_{20} & T_{30} \\ (-1)^{\nu}T_1 & T_{2\nu} & T_{3\nu} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ T_1 & T_{20} & T_{30} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix}.$$

Each of these determinants is computed directly:

$$\Delta_1^1 = -\sqrt{3}\left[e^{as_k} - 2e^{-\frac{as_k}{2}}\cos\left(\theta_k + \frac{\pi}{3}\right)\right], \quad \Delta_1^2 = -\sqrt{3}\left[e^{as_k} + 2e^{-\frac{as_k}{2}}\cos\theta_k\right], \tag{3.10}$$

$$\Delta_2 = -\frac{\sqrt{3}}{2}\left[T_1 - T_{20} - \sqrt{3}T_{30}\right] = -\frac{\sqrt{3}}{2}\left[e^{-as_k} - 2e^{\frac{as_k}{2}}\sin\left(\theta_k + \frac{\pi}{6}\right)\right], \tag{3.11}$$

where the identity $\cos\left(\theta_k - \frac{\pi}{3}\right) = \sin\left(\theta_k + \frac{\pi}{6}\right)$ has been used; from now on the sine form is employed everywhere. Thus, by (3.9)–(3.11),

$$\Delta(a,k,\nu) = \frac{3}{2}\left((-1)^\nu - \gamma\right)\Delta_3(a,k,\nu)\, e^{\frac{3as_k}{2}}\left[e^{-\frac{3as_k}{2}} - 2\sin\left(\theta_k + \frac{\pi}{6}\right)\right]$$

where

$$\Delta_3(a,k,\nu) = \begin{cases} 1 - 2e^{-\frac{3as_k}{2}}\cos\left(\theta_k + \frac{\pi}{3}\right), & \nu = 1; \\ 1 + 2e^{-\frac{3as_k}{2}}\cos\theta_k, & \nu = 2. \end{cases}$$

Consequently,

$$1 - 2e^{-\frac{3as_k}{2}} \le |\Delta_3(a,k,\nu)| \le 1 + 2e^{-\frac{3as_1}{2}},$$

and hence $\Delta_3 \neq 0$ for all k with $e^{\frac{3}{2}as_k} > 2$; by (2.4) this holds at once for all $k \in \mathbb{N}$ provided that

$$a\left(\frac{(2-m)\pi}{2}\right)^{2/3} > \frac{2}{3}\ln 2 \approx 0.4621,$$

and in any case for all $k \ge k_0(a,m)$.

Theorem 3.1. *The homogeneous problem D_ν has only the zero solution if and only if $\Delta(a,k,\nu) \neq 0$ for all $k \in \mathbb{N}$, that is, if simultaneously*

$$\gamma \neq (-1)^\nu, \qquad \Delta_1^\nu \neq 0 \quad \textit{and} \quad \Delta_2 \neq 0 \qquad \textit{for all } k \in \mathbb{N}.$$

Proof. Sufficiency. Let u be a solution of the homogeneous problem D_ν ($f \equiv 0$). Put $X_k(x) = \int_0^1 y^{-m}u(x,y)Y_k(y)dy$. Multiplying the homogeneous equation by $y^{-m}Y_k(y)$, integrating over $y \in (\epsilon, 1-\epsilon)$ twice by parts and passing to the limit as $\epsilon \to 0$ (the passage is legitimate because $u \in C^{2,1}_{x,y}(\bar{\Omega})$, $u(x,0) = u(x,1) = 0$, $Y_k(0) = Y_k(1) = 0$ and y^{-m} is integrable for $m < 1$), we obtain that X_k satisfies (3.2) with $f_k^\pm \equiv 0$. Then one may take $\widetilde{X}_k^\pm \equiv 0$, system (3.8) is homogeneous and, by assumption, its determinant is nonzero; hence $c_i = d_i = 0$ and $X_k \equiv 0$ for all k. By Theorem 2.1 the system $\{Y_k\}$ is complete in $L_2\left(y^{-m}dy\right)$, and therefore $u \equiv 0$.

Necessity. If $\Delta(a,k_0,\nu) = 0$ for some k_0, then the homogeneous system (3.8) has a nonzero solution (c_i, d_i); the function $u = X_{k_0}(x)Y_{k_0}(y)$ built from it is a nonzero solution of the homogeneous problem D_ν. It remains to note that, by (3.9), the vanishing of Δ is equivalent to the vanishing of one of the three factors listed above. □

Corollary 3.1. *Under the conditions of Theorem 3.1 problem D_ν has at most one solution; the existence is established in Section 4.*

4. Two cases of the right-hand side and the estimate of the solution

The right-hand side is uniquely representable as the sum of an odd and an even term in x, and problem D_ν is linear; hence it suffices to solve it in two cases: f odd in x and f even in x, the general case being obtained by addition (Subsection 4.5). Everywhere below

$$F_k = \max_{0\le x\le a} \left| f_k^+(x) \right|,$$

and $k_0 = k_0(a,m)$ is defined by the condition $e^{\frac{3}{2}as_k} > 2$.

We first construct a particular solution of the equation $W''' + \lambda_k W = h$ on $(0,a)$ which, together with its derivatives, is bounded uniformly in k.

Lemma 4.1. *Let $s_k \ge 1$, $h \in C[0,a]$, $\|h\| = \max_{[0,a]} |h|$. Then the function*

$$\widehat{Z}[h](x) = \frac{1}{3s_k^2}\int_0^x e^{-s_k(x-\xi)}h(\xi)d\xi - \frac{2}{3s_k^2}\int_x^a e^{\frac{s_k(x-\xi)}{2}} \sin\left(\frac{\sqrt{3}s_k(x-\xi)}{2} - \frac{\pi}{6}\right) h(\xi)d\xi \tag{4.1}$$

is a particular solution of the equation $W''' + \lambda_k W = h$ on $[0,a]$, and

$$\max_{[0,a]} \left| \frac{d^j \widehat{Z}[h]}{dx^j} \right| \le \frac{8}{3} s_k^{j-3} \|h\|, \qquad j = 0,1,2,3. \tag{4.2}$$

Proof. Formula (4.1) is the particular solution given by the variation of constants, the directions of integration being chosen according to the sign of the real part of the roots of $r^3 = -\lambda_k$: the decaying mode $r_1 = -s_k$ is integrated forward from zero, and the pair $r_{2,3} = s_k e^{\pm i\pi/3}$ backward from the point a. Differentiating under the integral sign and using

$$\left| \frac{d^j}{dx^j} e^{-s_k(x-\xi)} \right| = s_k^j e^{-s_k(x-\xi)},$$

$$\left| \frac{d^j}{dx^j} \left[e^{\frac{s_k(x-\xi)}{2}} \sin\left(\frac{\sqrt{3}s_k(x-\xi)}{2} - \frac{\pi}{6}\right)\right] \right| \le s_k^j e^{\frac{s_k(x-\xi)}{2}},$$

as well as $\int_0^x e^{-s_k(x-\xi)}d\xi \le s_k^{-1}$ and $\int_x^a e^{s_k(x-\xi)/2}d\xi \le 2s_k^{-1}$, we obtain for $j = 0,1,2$

$$\left| \frac{d^j \widehat{Z}[h]}{dx^j} \right| \le \frac{s_k^j}{3s_k^2}\left(\frac{1}{s_k} + \frac{4}{s_k}\right)\|h\| = \frac{5}{3}s_k^{j-3}\|h\|,$$

while for $j = 3$ the equation itself, $\widehat{Z}''' = h - \lambda_k \widehat{Z}$, gives $\|\widehat{Z}'''\| \le \left(1 + \frac{5}{3}\right)\|h\|$. □

Denote by P_k and Q_k the even and the odd parts of the solution, considered on $(0,a)$:

$$P_k(x) = \frac{X_k(x) + X_k(-x)}{2}, \qquad Q_k(x) = \frac{X_k(x) - X_k(-x)}{2},$$

so that

$$X_k(x) = P_k(|x|) + \operatorname{sgn}(x) Q_k(|x|), \qquad -a < x < a. \tag{4.3}$$

Let us derive the equations satisfied by P_k and Q_k. By (3.2), for $0 < x < a$ the function $X_k(-x)$ satisfies

$$-X_k'''(-x) = -\lambda_k X_k(-x) - f_k^-(-x),$$

hence, adding and subtracting,

$$P_k''' + \lambda_k P_k = \frac{f_k^+(x) - f_k^-(-x)}{2}, \qquad Q_k''' + \lambda_k Q_k = \frac{f_k^+(x) + f_k^-(-x)}{2}. \tag{4.4}$$

Observe how the right-hand side is distributed between the two parts: if f is odd in x, then $f_k^-(-x) = -f_k^+(x)$, so that the equation for P_k is inhomogeneous while that for Q_k is homogeneous; if f is even in x, the roles are reversed.

We next recompute the boundary conditions. From $X_k^{(j)}(+0) = X_k^{(j)}(-0)$ we get $P_k^{(j)}(0) = \frac{1+(-1)^j}{2} X_k^{(j)}(0)$ and $Q_k^{(j)}(0) = \frac{1-(-1)^j}{2} X_k^{(j)}(0)$, that is, the conjugation conditions are distributed unequally: one of them falls to the even part and two to the odd part. Together with $X_k(\pm a) = 0$ this gives

$$P_k'(0) = 0, \quad P_k(a) = 0; \qquad Q_k(0) = 0, \quad Q_k''(0) = 0, \quad Q_k(a) = 0. \tag{4.5}$$

Finally, $X_k^{(\nu)}(a) = P_k^{(\nu)}(a) + Q_k^{(\nu)}(a)$ and $X_k^{(\nu)}(-a) = (-1)^\nu \big[P_k^{(\nu)}(a) - Q_k^{(\nu)}(a)\big]$, so that the nonlocal condition takes the form

$$\left((-1)^\nu - \gamma\right) P_k^{(\nu)}(a) = \left((-1)^\nu + \gamma\right) Q_k^{(\nu)}(a). \tag{4.6}$$

Thus Q_k has three conditions of its own and P_k has two, while one further condition, (4.6), couples the two parts; this is precisely what gives rise to the two cases considered below. We emphasize that the passage from (3.2) to (4.4)–(4.6) is an equivalence, the converse passage being verified directly.

4.1. **Right-hand side odd in x.** Let $f(-x, y) = -f(x, y)$. Then, by (4.4), the function Q_k satisfies the homogeneous equation and the three conditions (4.5), whose determinant equals

$$\begin{vmatrix} 1 & 1 & 0 \\ T_1(a) & T_{20}(a) & T_{30}(a) \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = \Delta_2 \neq 0,$$

that is, exactly the determinant from (3.9). Consequently $Q_k \equiv 0$ and, by Corollary 3.1, the solution is even in x. Condition (4.6) then becomes $\big((-1)^\nu - \gamma\big) P_k^{(\nu)}(a) = 0$, that is $P_k^{(\nu)}(a) = 0$, and P_k solves a three-point problem on half of the interval,

$$P_k''' + \lambda_k P_k = f_k^+(x), \qquad P_k'(0) = 0, \quad P_k(a) = 0, \quad P_k^{(\nu)}(a) = 0. \tag{4.7}$$

We now solve it. Put $P_k = \widehat{Z}[f_k^+] + \sigma_1 T_1 + \sigma_2 T_{20} + \sigma_3 T_{30}$; then by (3.6) the vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ solves the system $A^\nu \sigma = b$ with the matrix and the right-hand

side

$$A^\nu = \begin{pmatrix} -1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ T_1 & T_{20} & T_{30} \\ (-1)^\nu T_1 & T_{2\nu} & T_{3\nu} \end{pmatrix}, \qquad b = \begin{pmatrix} -s_k^{-1}\widehat{Z}'[f_k^+](0) \\ -\widehat{Z}[f_k^+](a) \\ -s_k^{-\nu}\widehat{Z}^{(\nu)}[f_k^+](a) \end{pmatrix}, \tag{4.8}$$

where the argument a of the functions T is omitted and $\det A^\nu = \frac{1}{2}\Delta_1^\nu$. By Cramer's rule

$$\sigma_i = \frac{2}{\Delta_1^\nu}\det A_i^\nu, \qquad i = 1, 2, 3, \tag{4.9}$$

where A_i^ν is obtained from A^ν by replacing its ith column by b. The structure of these determinants is what matters below. In $\det A_1^\nu$ both remaining columns carry the growing functions T_{2j}, T_{3j}, and the product $T_{20}T_{3\nu} - T_{30}T_{2\nu} = \frac{\sqrt{3}}{2}e^{as_k}$ occurs; in $\det A_2^\nu$ and $\det A_3^\nu$ only one column is growing, the other two containing either b or $T_1 = e^{-as_k}$. This is what produces the gain of the exponent in Subsection 4.3.

4.2. **Right-hand side even in x.** Let now $f(-x, y) = f(x, y)$. By (4.4) the roles are interchanged: the equation for Q_k becomes inhomogeneous, while P_k satisfies the homogeneous one. Therefore we first find the odd part, and then, by (4.6), the even one.

The function Q_k is the solution of the three-point problem

$$Q_k''' + \lambda_k Q_k = f_k^+(x), \qquad Q_k(0) = 0, \quad Q_k''(0) = 0, \quad Q_k(a) = 0. \tag{4.10}$$

Putting $Q_k = \widehat{Z}[f_k^+] + \omega_1 T_1 + \omega_2 T_{20} + \omega_3 T_{30}$, we are led to the system $B\omega = c$ with

$$B = \begin{pmatrix} 1 & 1 & 0 \\ T_1 & T_{20} & T_{30} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, \qquad c = \begin{pmatrix} -\widehat{Z}[f_k^+](0) \\ -\widehat{Z}[f_k^+](a) \\ -s_k^{-2}\widehat{Z}''[f_k^+](0) \end{pmatrix}, \qquad \det B = \Delta_2. \tag{4.11}$$

By Cramer's rule, expanding the determinants and using the identity $\frac{\sqrt{3}}{2}T_{20} + \frac{1}{2}T_{30} = T_{31}$, we obtain

$$\omega_i = \frac{1}{\Delta_2}\sum_{j=1}^{3}\beta_{ij}c_j, \qquad i = 1, 2, 3, \tag{4.12}$$

where every coefficient β_{ij} is one of the quantities T_{31}, T_{30}, T_{20}, T_1 or a linear combination of them with numerical coefficients, so that $|\beta_{ij}| \le Me^{\frac{as_k}{2}}$. In contrast to (4.9), here each of the three numerators contains a growing function in the first power, while the denominator Δ_2 is only of order $e^{as_k/2}\left|\sin(\theta_k + \frac{\pi}{6})\right|$; no gain of the exponent occurs.

Let us turn to the even part. The function P_k satisfies the homogeneous equation and, by (4.5) and (4.6), the three conditions

$$P_k'(0) = 0, \quad P_k(a) = 0, \quad P_k^{(\nu)}(a) = \kappa_\nu Q_k^{(\nu)}(a), \quad \kappa_\nu = \frac{(-1)^\nu + \gamma}{(-1)^\nu - \gamma}, \tag{4.13}$$

where κ_ν is finite because $\gamma \neq (-1)^\nu$. This is the same system as in (4.8), but with the right-hand side $(0,0,\rho)^{\mathsf{T}}$, where $\rho = s_k^{-\nu}\kappa_\nu Q_k^{(\nu)}(a)$. Putting $P_k = \pi_1 T_1 + \pi_2 T_{20} + \pi_3 T_{30}$ and applying Cramer's rule (the expansion being along the column whose only nonzero entry is ρ in the third row), we obtain

$$\pi_i = \frac{2\rho}{\Delta_1^\nu}\,\tau_i, \qquad \tau_1 = \tfrac{1}{2}T_{30} - \tfrac{\sqrt{3}}{2}T_{20}, \quad \tau_2 = T_{30} + \tfrac{\sqrt{3}}{2}T_1, \quad \tau_3 = -\left(T_{20} + \tfrac{1}{2}T_1\right), \tag{4.14}$$

so that $|\tau_i| \le 2e^{\frac{as_k}{2}}$. The solution of problem (3.2) is given by (4.3) with these P_k and Q_k.

4.3. **The estimate in the odd case.**

Theorem 4.1. *Let $\gamma \neq (-1)^\nu$, $\Delta(a,k,\nu) \neq 0$ and $f(-x,y) = -f(x,y)$. Then for $k \ge k_0$*

$$\max_{-a\le x\le a}\left|X_k^{(j)}(x)\right| \le M\lambda_k^{\frac{j-3}{3}}F_k, \qquad j = 0,1,2,3, \tag{4.15}$$

and no small denominators arise.

Proof. By Lemma 4.1 all three components of the right-hand side b in (4.8) do not exceed $\frac{8}{3}\lambda_k^{-1}F_k$ in absolute value, while by (3.10) $\left|\Delta_1^\nu\right| \ge \sqrt{3}\left(e^{as_k} - 2e^{-\frac{as_k}{2}}\right) \ge Me^{as_k}$ for $k \ge k_0$. All the values T_1, T_{2j}, T_{3j} at the point a do not exceed $e^{as_k/2}$ in absolute value.

By the structure of the determinants in (4.9) described above, $|\det A_1^\nu| \le M\lambda_k^{-1}F_k e^{as_k}$, whereas $|\det A_2^\nu|, |\det A_3^\nu| \le M\lambda_k^{-1}F_k e^{\frac{as_k}{2}}$. Consequently,

$$|\sigma_1| \le M\lambda_k^{-1}F_k, \qquad |\sigma_2|, |\sigma_3| \le M\lambda_k^{-1}e^{-\frac{as_k}{2}}F_k.$$

Since $|T_1(x)| \le 1$ and $|T_{2j}(x)|, |T_{3j}(x)| \le e^{\frac{as_k}{2}}$ on $[0,a]$, formulas (3.6) and (4.2) give

$$\left|P_k^{(j)}(x)\right| \le s_k^j\left[|\sigma_1| + (|\sigma_2| + |\sigma_3|)\,e^{\frac{as_k}{2}}\right] + \left|\frac{d^j\widehat{Z}[f_k^+]}{dx^j}\right| \le M\lambda_k^{\frac{j-3}{3}}F_k,$$

which together with $Q_k \equiv 0$ and (4.3) yields (4.15). □

4.4. **The estimate in the even case.** Put

$$\Theta_k = \frac{2}{\sqrt{3}}\cdot\frac{e^{\frac{as_k}{2}}}{\left|e^{-\frac{3as_k}{2}} - 2\sin\left(\theta_k + \frac{\pi}{6}\right)\right|}, \qquad \text{so that} \qquad |\Delta_2| = \frac{e^{as_k}}{\Theta_k}; \tag{4.16}$$

the quantity Θ_k is finite precisely when $\Delta_2 \neq 0$.

Since $\left|e^{-\frac{3as_k}{2}} - 2\sin(\theta_k + \frac{\pi}{6})\right| \le 3$, one always has $\Theta_k \ge \frac{2}{3\sqrt{3}}e^{\frac{as_k}{2}}$. If, moreover,

$$\left|\sin\left(\theta_k + \frac{\pi}{6}\right)\right| \ge e^{-\frac{3as_k}{2}}, \tag{4.17}$$

then $\left|e^{-\frac{3as_k}{2}} - 2\sin\right| \ge 2|\sin| - e^{-\frac{3as_k}{2}} \ge |\sin|$, and therefore

$$\Theta_k \le \frac{2}{\sqrt{3}}\cdot\frac{e^{\frac{as_k}{2}}}{\left|\sin\left(\theta_k + \frac{\pi}{6}\right)\right|}. \tag{4.18}$$

Condition (4.17) certainly holds for $k \geq k_0$ whenever some power lower bound $\left|\sin(\theta_k + \frac{\pi}{6})\right| \geq Mk^{-p}$ is available; such bounds are supplied by Lemmas 5.1 and 5.2 of Section 5. The condition $\Delta(a,k,\nu) \neq 0$ alone does not imply (4.17): it only means that $\sin(\theta_k + \frac{\pi}{6}) \neq \frac{1}{2}e^{-\frac{3as_k}{2}}$.

Theorem 4.2. *Let $\gamma \neq (-1)^\nu$, $\Delta(a,k,\nu) \neq 0$ and $f(-x,y) = f(x,y)$. Then for $k \geq k_0$*

$$\max_{-a\leq x\leq a}\left|X_k^{(j)}(x)\right| \leq M\lambda_k^{\frac{j-3}{3}}\Theta_k F_k, \qquad j = 0,1,2,3. \tag{4.19}$$

Proof. Estimate of Q_k. By Lemma 4.1 the components of the right-hand side c of problem (4.10) in (4.11) do not exceed $\frac{8}{3}\lambda_k^{-1}F_k$, and $|\det B| = |\Delta_2| = e^{as_k}\Theta_k^{-1}$ by (4.16). Since $|\beta_{ij}| \leq Me^{\frac{as_k}{2}}$ in (4.12), every numerator there does not exceed $M\lambda_k^{-1}F_k e^{\frac{as_k}{2}}$, whence

$$|\omega_i| \leq M\lambda_k^{-1}F_k\Theta_k e^{-\frac{as_k}{2}}, \qquad i = 1,2,3,$$

and, as in Theorem 4.1,

$$\max_{[0,a]}\left|Q_k^{(j)}(x)\right| \leq M\lambda_k^{\frac{j-3}{3}}\Theta_k F_k, \qquad j = 0,1,2,3. \tag{4.20}$$

Estimate of P_k. By (4.20) with $j = \nu$,

$$|\rho| = s_k^{-\nu}\,|\kappa_\nu|\left|Q_k^{(\nu)}(a)\right| \leq M\lambda_k^{-1}\Theta_k F_k,$$

while in (4.14), which solves (4.13), one has $|\tau_i| \leq 2e^{\frac{as_k}{2}}$ and $\left|\Delta_1^\nu\right| \geq Me^{as_k}$; consequently,

$$|\pi_i| \leq M\,|\rho|\,e^{-\frac{as_k}{2}} \leq M\lambda_k^{-1}\Theta_k F_k e^{-\frac{as_k}{2}}, \qquad i = 1,2,3,$$

whence $\left|P_k^{(j)}(x)\right| \leq s_k^j\big(|\pi_1| + (|\pi_2| + |\pi_3|)e^{\frac{as_k}{2}}\big) \leq M\lambda_k^{(j-3)/3}\Theta_k F_k$. Adding this to (4.20) and taking (4.3) into account, we obtain (4.19). □

4.5. **The general case.** Put

$$F_k^{\mathrm{odd}} = \max_{[0,a]}\left|\frac{f_k^+(x) - f_k^-(-x)}{2}\right|, \qquad F_k^{\mathrm{ev}} = \max_{[0,a]}\left|\frac{f_k^+(x) + f_k^-(-x)}{2}\right|;$$

by (4.4) these are precisely the quantities F_k for the odd and the even terms of the right-hand side. Adding the corresponding solutions, we obtain from Theorems 4.1 and 4.2 the following statement.

Corollary 4.1. *Let $\gamma \neq (-1)^\nu$ and $\Delta(a,k,\nu) \neq 0$. Then for $k \geq k_0$*

$$\max_{-a\leq x\leq a}\left|X_k^{(j)}(x)\right| \leq M\lambda_k^{\frac{j-3}{3}}\left(F_k^{\mathrm{odd}} + \Theta_k F_k^{\mathrm{ev}}\right), \qquad j = 0,1,2,3. \tag{4.21}$$

The point is that the factor Θ_k, which contains both the exponential growth $e^{as_k/2}$ and the small denominator, enters (4.21) only together with F_k^{ev}.

Theorem 4.3. *Let, in addition to the assumptions of Corollary 4.1, the function f be odd in x, continuous on $\overline{\Omega^+}$, $\Omega^+ = \Omega \cap \{x > 0\}$, and let $f(x,\cdot)$ satisfy the conditions of Theorem 2.1 uniformly in x for every x. Then problem D_ν has a unique solution (3.1); it is even in x, and no restrictions on a are required.*

Proof. By (2.11) $F_k \le M\lambda_k^{-1}$, and $\lambda_k \ge Mk^2$ by (2.4); consequently $\sum_k F_k < \infty$; hence, by (2.8), (2.10) and (4.15),

$$\sum_k \left|X_k^{(j)}\right| |Y_k| \le M \sum_k \lambda_k^{\frac{j-3}{3}} F_k \le M \sum_k \lambda_k^{-1} < \infty, \quad j = 0,1,2,3,$$

$$\sum_k |X_k|\,|Y_k'| \le M \sum_k \lambda_k^{-1} F_k \lambda_k < \infty, \qquad \sum_k |X_k|\,|y^m Y_k''| \le M \sum_k F_k < \infty,$$

and the series (3.1) therefore admits termwise differentiation three times in x and twice in y, and all the conditions (2.1) are satisfied termwise. □

For a right-hand side even in x it follows from (4.19) that

$$|u_{xxx}| \le M \sum_{k=1}^{\infty} \lambda_k \Theta_k F_k^{\mathrm{ev}} \left|\frac{Y_k(y)}{\lambda_k}\right| \le M \sqrt{\sum_{k=1}^{\infty} \left|\frac{Y_k(y)}{\lambda_k}\right|^2} \sqrt{\sum_{k=1}^{\infty} (\lambda_k \Theta_k F_k^{\mathrm{ev}})^2}. \quad (4.22)$$

The convergence of the first factor follows from (2.8). The second one requires the convergence of the series $\sum_k \left(\lambda_k \Theta_k F_k^{\mathrm{ev}}\right)^2$, and for this one needs a lower bound for $\left|\sin(\theta_k + \frac{\pi}{6})\right|$ and a sufficiently rapid decay of F_k^{ev}. The first question is the subject of Section 5.

5. Small denominators

Denote

$$A_k = \frac{\theta_k}{\pi} = \frac{\sqrt{3}\,a\,s_k}{2\pi}, \qquad B_n = n - \frac{1}{6} \ (n \in \mathbb{Z}), \qquad \delta(k,n) = |A_k - B_n|,$$

so that $\left|\sin(\theta_k + \frac{\pi}{6})\right| = |\sin \pi A_k| \ge 2 \min_n \delta(k,n)$. The problem is to bound $\delta(k,n)$ from below.

Put

$$C_m = \frac{1}{2(2-m)} - \frac{1}{4} = \frac{m}{4(2-m)}, \qquad E = \frac{\sqrt{3}}{2\pi}\left(\frac{2-m}{2}\pi\right)^{2/3} a, \qquad \nu_0 = \frac{1}{2-m}.$$

It follows from (2.3), (2.6) and (2.5) that

$$A_k = Ek^{2/3} + O\left(k^{-1/3}\right). \quad (5.1)$$

One cannot replace A_k here by the leading term $Ek^{2/3}$ and estimate $\left|\sin \pi\left(Ek^{2/3} + \frac{1}{6}\right)\right|$ from below: the discarded correction is of order $k^{-1/3}$, whereas the estimates proved below are of order $k^{-4/3}$ and smaller; the ratio of the discarded term to the one being proved grows like k. The way out is to carry out the Diophantine analysis not for the phase A_k itself but for its cube: upon cubing, the correction turns from $O(k^{-1/3})$ into a *constant*, while the remainder drops to $O(k^{-2})$ and is compared no longer with $k^{-4/3}$ but with a constant (Lemma 5.1). Indeed,

$$A_k^3 = \frac{3\sqrt{3}a^3}{8\pi^3}\lambda_k = \frac{E^3}{\pi^2}\mu_k^2, \qquad \mu_k^2 = \pi^2(k + C_m)^2 - \frac{4\nu_0^2 - 1}{4} + O\left(k^{-2}\right),$$

whence

$$A_k^3 = E^3(k + C_m)^2 - c + O\left(k^{-2}\right), \qquad c = \frac{E^3\left(4\nu_0^2 - 1\right)}{4\pi^2}. \tag{5.2}$$

We use the difference-of-cubes identity: for $A_k > 0$ and B_n,

$$A_k - B_n = \frac{A_k^3 - B_n^3}{A_k^2 + A_k B_n + B_n^2}, \tag{5.3}$$

and in the region where $B_n \to A_k$ the denominator is bounded above,

$$A_k^2 + A_k B_n + B_n^2 \le 4E^2 k^{4/3}, \tag{5.4}$$

for all sufficiently large k. It remains to bound the numerator from below.

5.1. **The case of a rational cube of the parameter E.** Let the degeneracy parameter m be rational and let the geometric sizes of the domain be chosen so that $E^3 = \frac{p}{q} \in \mathbb{Q}$. Write $C_m = \frac{u}{v}$ in lowest terms and put $L = 216\, q\, v^2$. Then

$$E^3(k + C_m)^2 - B_n^3 = \frac{216p(vk + u)^2 - qv^2(6n - 1)^3}{216qv^2} = \frac{N_{k,n}}{L}, \quad N_{k,n} \in \mathbb{Z}$$

that is, the rational part of expression (5.2) runs over a lattice of step $1/L$. Hence, by (5.2),

$$A_k^3 - B_n^3 = \frac{N_{k,n}}{L} - c + O\left(k^{-2}\right).$$

Lemma 5.1. *Let $m \in [0, 1)$ be rational and $E^3 = \frac{p}{q} \in \mathbb{Q}$.*

1) If $m \neq 0$, then $c \neq 0$ is transcendental, hence $d_0 = \operatorname{dist}\left(c, \frac{1}{L}\mathbb{Z}\right) > 0$, and there exists k_0 such that for all $k \ge k_0$ and all $n \in \mathbb{Z}$

$$\delta(k, n) \ge \frac{d_0}{8E^2} \cdot \frac{1}{k^{4/3}}.$$

2) If $m = 0$, then $c = 0$, and under the nonresonance condition

$$N_{k,n} = 216pk^2 - q(6n - 1)^3 \neq 0 \qquad \forall k \in \mathbb{N},\ n \in \mathbb{Z}, \tag{5.5}$$

for all $k \in \mathbb{N}$ one has

$$\delta(k, n) \ge \frac{1}{864\, qE^2} \cdot \frac{1}{k^{4/3}}.$$

Proof. 1) For $m \neq 0$ we have $4\nu_0^2 \neq 1$, and $c = \frac{p\left(4\nu_0^2 - 1\right)}{4q\pi^2}$ is a nonzero rational number divided by π^2. Since π is transcendental, c is transcendental as well; in particular $c \notin \frac{1}{L}\mathbb{Z}$, and therefore $d_0 > 0$. Choose k_0 so that the remainder in (5.2) does not exceed $d_0/2$ for $k \ge k_0$; this is possible because the remainder is $O(k^{-2})$ while d_0 does not depend on k. Then

$$\left|A_k^3 - B_n^3\right| \ge \operatorname{dist}\left(c, \tfrac{1}{L}\mathbb{Z}\right) - \frac{d_0}{2} = \frac{d_0}{2},$$

and substitution into (5.3) with (5.4) taken into account gives the required bound.

2) For $m = 0$ we have $\nu_0 = \frac{1}{2}$, $4\nu_0^2 - 1 = 0$, hence $c = 0$; moreover $\mu_k = \pi k$ exactly (the zeros of $J_{1/2}$), so that the remainder in (5.2) is absent and $A_k^3 = E^3 k^2$.

The number $N_{k,n}$ is an integer and, by condition (5.5), nonzero; hence $|N_{k,n}| \geq 1$ and $\left|A_k^3 - B_n^3\right| \geq \frac{1}{L} = \frac{1}{216q}$. It remains to use (5.3) and (5.4). □

We now check that for $m = 0$ the set of parameters satisfying condition (5.5) is nonempty.

Proposition 5.1. *Let $m = 0$ and let q be odd. Then the equation $N_{k,n} = 0$ has no solutions for any $k \in \mathbb{N}$, $n \in \mathbb{Z}$.*

Proof. Suppose $216pk^2 = q(6n-1)^3$. The left-hand side is divisible by $216 = 2^3 \cdot 3^3$ and hence is even. On the right-hand side $6n-1$ is odd for every integer n, so its cube is odd as well; since q is odd, the right-hand side is odd. A contradiction. □

By Proposition 5.1, for example, the pair $p = 1$, $q = 5$ satisfies condition (5.5).

5.2. **The case of an algebraic irrational cube of the parameter E.**

Lemma 5.2. *Let $m = 0$ and let E^3 be an algebraic irrational number. Then for every $\epsilon > 0$ there exists $C_\epsilon = C(E, \epsilon) > 0$ such that for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$*

$$\delta(k, n) \geq \frac{C_\epsilon}{k^{10/3+2\epsilon}}.$$

Proof. For $m = 0$, by Lemma 5.1 2), $A_k^3 = E^3k^2$ exactly. Taking the factor $216 = 6^3$ inside the absolute value,

$$\left|A_k^3 - B_n^3\right| = \frac{k^2}{216}\left|\vartheta - \frac{(6n-1)^3}{k^2}\right|, \qquad \vartheta = 216E^3. \tag{5.6}$$

By assumption ϑ is an algebraic irrational number. The expression under the absolute value is the problem of rational approximation of ϑ by the fraction N/M with $N = (6n-1)^3$ and $M = k^2$; if the fraction is reducible, its irreducible denominator M' divides k^2, whence $M' \leq k^2$. By the Thue–Siegel–Roth theorem [22] the irrationality measure of an algebraic irrational number equals 2: for every $\epsilon > 0$ there exists $c_0 = c_0(\vartheta, \epsilon) > 0$ such that

$$\left|\vartheta - \frac{N}{M'}\right| \geq \frac{c_0}{(M')^{2+\epsilon}} \geq \frac{c_0}{k^{4+2\epsilon}} \tag{5.7}$$

for all rationals N/M'. There are finitely many exceptions here, and for each of them $|\vartheta - N/M'| > 0$ by the irrationality of ϑ, and all of them are absorbed by decreasing c_0; thus (5.7) holds for all $k \in \mathbb{N}$. Substituting (5.7) into (5.6) gives

$$\left|A_k^3 - B_n^3\right| \geq \frac{c_0}{216} \cdot \frac{1}{k^{2+2\epsilon}}$$

and together with (5.3), (5.4) this gives $\delta(k, n) \geq \frac{c_0}{864E^2} k^{-10/3-2\epsilon}$. □

Remark 5.1. The restriction $m = 0$ in Lemma 5.2 is essential: for $m \neq 0$ the shift c in (5.2) is transcendental, and the number being approximated is no longer algebraic and Roth's theorem does not apply, while the lattice argument of Lemma 5.1 1) fails as well, since for irrational E^3 the quantity $E^3(k + C_m)^2 - B_n^3$ does not form a lattice. Note also that the constant in Roth's theorem is ineffective, so that C_ϵ cannot be computed, unlike the explicit constants of Lemma 5.1

(see [5]). Finally, $\vartheta = 216E^3$ is algebraic precisely when a^3/π is, and the set of such a is countable.

It follows from Lemmas 5.1 and 5.2 that, under the assumptions made, $\left|\sin(\theta_k + \frac{\pi}{6})\right| \geq Mk^{-p}$ with $p = 4/3$ or $p = 10/3 + 2\epsilon$ respectively. Since k^{-p} decays slower than any exponential, condition (4.17) holds for all $k \geq k_0$, and by (4.18) $\Theta_k \leq Mk^p e^{\frac{as_k}{2}}$. Together with (4.22) this gives the following statement.

Theorem 5.1. *Let $\gamma \neq (-1)^\nu$, let $\Delta(a,k,\nu) \neq 0$ for all $k \in \mathbb{N}$, let the assumptions of Lemma 5.1 or Lemma 5.2 be fulfilled, and let the even in x part of the right-hand side be such that*

$$F_k^{\mathrm{ev}} \leq C \exp\left(-\varrho k^{2/3}\right), \qquad \varrho > \frac{a}{2}\left(\frac{(2-m)\pi}{2}\right)^{2/3}. \tag{5.8}$$

Then problem D_ν has a unique solution (3.1).

Classes of data ensuring (5.8) are described in Corollary 6.1 below.

Proof. By Theorem 4.3 it suffices to consider a right-hand side even in x. By (5.1), $\frac{as_k}{2} = \frac{\pi A_k}{\sqrt{3}} = \frac{\pi E}{\sqrt{3}}k^{2/3} + O(k^{-1/3})$, where $\frac{\pi E}{\sqrt{3}} = \frac{a}{2}\left(\frac{(2-m)\pi}{2}\right)^{2/3}$; therefore

$$\lambda_k \Theta_k F_k^{\mathrm{ev}} \leq Mk^{2+p} \exp\left(-\left(\varrho - \frac{\pi E}{\sqrt{3}}\right)k^{2/3}\right),$$

and the series in (4.22) converges for any finite p. The remaining series are estimated exactly as in Theorem 4.3. □

The exponent p from Lemmas 5.1 and 5.2 does not affect the result: the exponential suppresses any power factor, and the role of these lemmas reduces to guaranteeing the finiteness of Θ_k and the polynomial character of its growth.

6. Convergence of the series representing the solution

We collect the results of Sections 4 and 5 as statements about convergence. Denote the majorant from estimate (4.21) by

$$\Lambda_k = F_k^{\mathrm{odd}} + \Theta_k F_k^{\mathrm{ev}}, \qquad \max_{-a\leq x\leq a}\left|X_k^{(j)}(x)\right| \leq M\lambda_k^{\frac{j-3}{3}}\Lambda_k, \quad j = 0,1,2,3. \tag{6.1}$$

Theorem 6.1. *Let the conditions of Theorem 3.1 be fulfilled and let*

$$\sum_{k=1}^{\infty} \Lambda_k < \infty. \tag{6.2}$$

Then the series (3.1) represents a solution of problem D_ν, namely:

1) the series $\sum_k X_k^{(j)}(x)Y_k(y)$, $j = 0,1,2,3$, converge absolutely and uniformly on $\overline{\Omega^+}$ and on $\overline{\Omega^-}$;

2) the series $\sum_k X_k(x)Y_k'(y)$ converges absolutely and uniformly on $\bar{\Omega}$;

3) the series $\sum_k X_k(x)\, y^m Y_k''(y)$ converges absolutely and uniformly on $\bar{\Omega}$, and the series $\sum_k X_k(x)Y_k''(y)$ converges absolutely and uniformly on $\bar{\Omega}_\epsilon = \bar{\Omega}\cap\{y \geq \epsilon\}$ for every $\epsilon > 0$;

4) the sum $u(x,y)$ of the series (3.1) belongs to the class $C^{3,2}_{x,y}(\Omega\setminus\{x=0\})\cap C^{2,1}_{x,y}(\bar\Omega)$ and satisfies all the conditions (2.1); moreover, the stability estimate

$$\|u\|_{C(\bar\Omega)}\le M\sum_{k=1}^{\infty}\lambda_k^{-1}\Lambda_k$$

holds.

Proof. By Lemma 2.1, $|Y_k(y)|\le M$; by (2.9) $|Y_k'(y)|\le M\lambda_k$, and $y^mY_k''(y)=-\lambda_kY_k(y)$. Hence, by (6.1),

$$\sum_k\left|X_k^{(j)}\right||Y_k|\le M\sum_k\lambda_k^{\frac{j-3}{3}}\Lambda_k\le M\sum_k\Lambda_k<\infty,$$

for $j=0,1,2,3$, and

$$\sum_k|X_k|\,|Y_k'|\le M\sum_k\lambda_k^{-1}\Lambda_k\cdot\lambda_k=M\sum_k\Lambda_k<\infty,$$

$$\sum_k|X_k|\,|y^mY_k''|\le M\sum_k\lambda_k^{-1}\Lambda_k\lambda_k=M\sum_k\Lambda_k<\infty,$$

all the majorants being independent of (x,y), which gives 1)–3); for $\sum_k X_kY_k''$ the majorant contains the additional factor y^{-m}, bounded on $\bar\Omega_\epsilon$. The uniform convergence of the series of derivatives implies that the series (3.1) may be differentiated termwise three times in x and twice in y, while each term X_kY_k satisfies the equation and all the conditions (2.1) by construction. The stability estimate follows from the first chain with $j=0$. □

It remains to find out under which conditions on the right-hand side assumption (6.2) of Theorem 6.1 holds.

Theorem 6.2. *Let f be continuous on $\overline{\Omega^+}$ and $\overline{\Omega^-}$, and let $f(x,\cdot)$ satisfy the conditions of Theorem 2.1 uniformly in x for every x. Then*

$$F_k^{\mathrm{odd}}\le M\lambda_k^{-1},\qquad F_k^{\mathrm{ev}}\le M\lambda_k^{-1},$$

and:

1) if f is odd in x, then $\Lambda_k=F_k^{\mathrm{odd}}\le M\lambda_k^{-1}$ and condition (6.2) holds automatically, since $\lambda_k\ge Mk^2$; no additional requirements on f, on a or on m are needed;

2) in the general case condition (6.2) holds provided that one of the estimates of Lemmas 5.1, 5.2 is valid and the even in x part of the right-hand side satisfies (5.8).

Proof. The first inequality is (2.11) applied to $f(x,\cdot)$; the second one is the same for $f(-x,\cdot)$, since $F_k^{\mathrm{odd}},F_k^{\mathrm{ev}}\le\max_{[0,a]}|f_k^+|+\max_{[-a,0]}|f_k^-|$. In case 1) we have $F_k^{\mathrm{ev}}\equiv0$, hence $\Lambda_k=F_k^{\mathrm{odd}}\le M\lambda_k^{-1}\le Mk^{-2}$, and the series (6.2) converges. In case 2), by Lemmas 5.1, 5.2, $\Theta_k\le Mk^pe^{\frac{as_k}{2}}$ with $p=4/3$ or $p=10/3+2\epsilon$, while by (5.8) and (5.1)

$$\Theta_kF_k^{\mathrm{ev}}\le Mk^p\exp\left(-\left(\varrho-\frac{\pi E}{\sqrt3}\right)k^{2/3}\right),\qquad\frac{\pi E}{\sqrt3}=\frac a2\left(\frac{(2-m)\pi}{2}\right)^{2/3},$$

and the series converges for any finite p. □

Corollary 6.1. *Denote by* $f_{\mathrm{ev}}(x,y) = \frac{f(x,y)+f(-x,y)}{2}$ *the even in* x *part of the right-hand side, put* $\mathcal{L} = -y^m\partial_y^2$, $\varphi_0 = f_{\mathrm{ev}}$, $\varphi_{j+1} = \mathcal{L}\varphi_j$, *and let* $\varphi_j(x,\cdot)$ *satisfy the conditions of Theorem 2.1 uniformly in* x *for every* j. *Condition (5.8) holds in each of the following cases:*

(a) *the iterates of the operator* $\mathcal{L}$ *grow no faster than*

$$M_N := \sup_x \left\|y^{-m/2}\varphi_N(x,\cdot)\right\|_{L_2[0,1]} \le AB^N(N!)^3, \qquad N = 0,1,2,\dots, \tag{6.3}$$

with a sufficiently small B *(it is exactly* B *that determines* ϱ *in (5.8));*

(b) $m = 0$ *and* $f_{\mathrm{ev}}(x,\cdot)$ *belongs to the Gevrey class* $G^{3/2}$ [9, 21], *that is,*

$$\left\|\partial_y^n f_{\mathrm{ev}}(x,\cdot)\right\|_{L_2} \le CR^n(n!)^{3/2}, \qquad n = 0,1,2,\dots, \tag{6.4}$$

uniformly in x, *with a sufficiently small* R, *and moreover* $\partial_y^{2j} f_{\mathrm{ev}}(x,0) = 0$ *and* $\partial_y^{2j} f_{\mathrm{ev}}(x,1) = 0$ *for all* $j \ge 0$; *in particular, any function analytic in* y *and satisfying these conditions at the endpoints will do;*

(c) f_{ev} *is a finite combination of eigenfunctions,*

$$f_{\mathrm{ev}}(x,y) = \sum_{k=1}^{K} g_k(x)Y_k(y),$$

and then $F_k^{\mathrm{ev}} = 0$ *for* $k > K$ *and (5.8) holds trivially.*

Proof. Iterating (2.11), we obtain $f_k^{\pm} = \lambda_k^{-N}(\varphi_N)_k$; since the system $\{Y_k\}$ is orthonormal in $L_2\big(y^{-m}dy\big)$, we have $\big|(\varphi_N)_k\big| \le \left\|y^{-m/2}\varphi_N\right\|_{L_2}$ and therefore $F_k^{\mathrm{ev}} \le M_N\lambda_k^{-N}$ for every N. In case (a) we minimize $AB^N(N!)^3\lambda_k^{-N}$ in N: for $\lambda_k \asymp k^2$ the optimal choice is $N \asymp (k^2/B)^{1/3}$, and the value is of order $\exp\big(-3B^{-1/3}k^{2/3}(1+o(1))\big)$; thus $\varrho \asymp 3B^{-1/3}$ increases as B decreases, and condition (5.8) holds for a sufficiently small B.

In case (b) $m = 0$, the operator $\mathcal{L}$ becomes $-\partial_y^2$, so that $\varphi_N = (-1)^N\partial_y^{2N}f_{\mathrm{ev}}$, the conditions on the traces reduce to the vanishing of the derivatives of even order, and (6.4) gives

$$M_N \le CR^{2N}((2N)!)^{3/2} \le C\big(8R^2\big)^N(N!)^3$$

by $(2N)! \le 4^N(N!)^2$. This is (6.3) with $B = 8R^2$, so (b) follows from (a). Case (c) is obvious. □

Condition (6.3) is stated in terms of the iterates of the operator $\mathcal{L} = -y^m\partial_y^2$ rather than in terms of the derivatives $\partial_y^n f_{\mathrm{ev}}$, and for $0 < m < 1$ it cannot be otherwise. Already the second application of $\mathcal{L}$ gives

$$\varphi_2 = m(m-1)y^{2m-2}\partial_y^2 f_{\mathrm{ev}} + 2my^{2m-1}\partial_y^3 f_{\mathrm{ev}} + y^{2m}\partial_y^4 f_{\mathrm{ev}},$$

where the exponent $2m-2$ is negative; hence φ_2 is unbounded at the origin unless $\partial_y^2 f_{\mathrm{ev}}(x,0) = 0$; further iterations produce powers y^{Nm-r} with arbitrarily large r. Therefore neither the smoothness nor the analyticity of f_{ev} in y by itself guarantees estimate (6.3): matching conditions at the point $y = 0$ are also needed, and it is more convenient to impose them at once in the form (6.3), that

is, to require $f_{\mathrm{ev}}(x,\cdot)$ to belong to the Gevrey class associated with the operator $\mathcal{L}$ itself. For $m=0$ there is no degeneracy, the operator $\mathcal{L}$ becomes the classical one, and the condition takes the usual form (6.4).

7. The inverse problem

The direct problem studied above exhibits a sharp dichotomy: by Theorem 4.3 problem D_ν is well posed in the classical sense, with finitely smooth data and without any restriction on the size of the domain, exactly when the right-hand side is odd in x, whereas for an even right-hand side the small denominators force a Gevrey-type condition. This dichotomy makes the odd class the natural setting for an inverse problem.

Inverse problems of determining a factor of the right-hand side from an additional measurement of the solution on an interior line are classical for equations of even order; for odd-order equations with multiple characteristics they have been considered in [3]. In the present situation such a problem has a further feature worth emphasising. Since the solution corresponding to an odd right-hand side is even in x, the data may be prescribed on a single line $x=x_0$ without loss of generality, and the problem separates with respect to the spectral index k: the whole question of solvability is reduced to the behaviour of the single sequence of numbers $\Pi_k(x_0)$, for which the asymptotics $\lambda_k\Pi_k(x_0)\to p(x_0)$ is established in Lemma 7.1 below. In particular, no small denominators appear in the inverse problem, in contrast with the even case of the direct one.

Problem I_ν. Suppose we are given a point $x_0\in(0,a)$, an odd function $p\in C[-a,a]$ and a function $\psi(y)$. Find a pair of functions $\{u(x,y),\ g(y)\}$, $g\in C[0,1]$, such that u is a solution of problem D_ν with the right-hand side $f(x,y)=p(x)g(y)$ and, in addition,

$$u(x_0,y)=\psi(y),\qquad 0\le y\le 1. \tag{7.1}$$

Since p is odd, the right-hand side is odd in x; by Subsection 4.1 the solution is even in x, and therefore prescribing condition (7.1) on one of the two lines $x=\pm x_0$ involves no loss of generality.

Substituting $f=p(x)g(y)$ into (3.3) gives $f_k^{\pm}(x)=p(x)g_k$, where

$$g_k=\int_0^1 y^{-m}g(y)Y_k(y)\,dy.$$

Denote by Π_k the solution of problem (4.7) with the right-hand side $p(x)$ instead of $f_k^+(x)$:

$$\Pi_k'''+\lambda_k\Pi_k=p(x),\quad \Pi_k'(0)=0,\quad \Pi_k(a)=0,\quad \Pi_k^{(\nu)}(a)=0. \tag{7.2}$$

By linearity $X_k=g_k\Pi_k$, and condition (7.1) gives

$$\psi_k=g_k\Pi_k(x_0),\qquad \psi_k=\int_0^1 y^{-m}\psi(y)Y_k(y)dy. \tag{7.3}$$

Thus problem I_ν splits with respect to the index k, and all the information about its solvability is contained in the numbers $\Pi_k(x_0)$.

Lemma 7.1. *Let $\gamma \neq (-1)^\nu$, $\Delta(a,k,\nu) \neq 0$, let the function $p \in C[-a,a]$ be odd and $x_0 \in (0,a)$. Then*

$$\lim_{k\to\infty} \lambda_k \Pi_k(x_0) = p(x_0). \tag{7.4}$$

Proof. By the proof of Theorem 4.1 applied to (7.2), $\Pi_k = \widehat{Z}[p] + \sigma_1 T_1 + \sigma_2 T_{20} + \sigma_3 T_{30}$, where $|\sigma_1| \le M\lambda_k^{-1} P$ and $|\sigma_2|, |\sigma_3| \le M\lambda_k^{-1} e^{-\frac{as_k}{2}} P$ with $P = \max_{[-a,a]} |p|$. Since $T_1(x_0) = e^{-s_k x_0}$ and $|T_{20}(x_0)|, |T_{30}(x_0)| \le e^{\frac{s_k x_0}{2}}$, the homogeneous part is $O\big(\lambda_k^{-1} e^{-\varrho_0 s_k}\big)$ with $\varrho_0 = \min\big\{x_0, \frac{a-x_0}{2}\big\} > 0$; hence $\lambda_k \Pi_k(x_0) = \lambda_k \widehat{Z}[p](x_0) + o(1)$.

Let us compute $\lambda_k \widehat{Z}[p](x_0)$. By (4.1), $\widehat{Z}[p](x_0) = \sum_{i=1}^3 \frac{J_i(x_0)}{3r_i^2}$, where $r_1 = -s_k$, $r_{2,3} = s_k e^{\pm i\pi/3}$ and, after the substitutions $t = x_0 - \xi$ and $t = \xi - x_0$,

$$J_1(x_0) = \int_0^{x_0} e^{r_1 t} p(x_0 - t)\,dt, \qquad J_i(x_0) = -\int_0^{a-x_0} e^{-r_i t} p(x_0 + t)\,dt, \quad i = 2,3.$$

Since $\operatorname{Re} r_1 = -s_k$, $\operatorname{Re} r_i = \frac{s_k}{2}$ $(i = 2,3)$ and p is continuous at the point x_0, in all three cases $J_i(x_0) = -\frac{p(x_0)}{r_i} + o\big(s_k^{-1}\big)$, whence

$$\widehat{Z}[p](x_0) = -\frac{p(x_0)}{3} \sum_{i=1}^3 \frac{1}{r_i^3} + o\left(\lambda_k^{-1}\right) = \frac{p(x_0)}{\lambda_k} + o\left(\lambda_k^{-1}\right),$$

because $r_i^3 = -\lambda_k$ for all i. □

Theorem 7.1. *Let*

1) $\gamma \neq (-1)^\nu$ and $\Delta(a,k,\nu) \neq 0$ for all $k \in \mathbb{N}$;

2) $p \in C[-a,a]$ be odd, $x_0 \in (0,a)$ and $p(x_0) \neq 0$;

3) $\Pi_k(x_0) \neq 0$ for all $k \in \mathbb{N}$;

4) the functions ψ and $-y^m\psi''$ satisfy the conditions of Theorem 2.1.

Then problem I_ν has a unique solution, and

$$g(y) = \sum_{k=1}^\infty \frac{\psi_k}{\Pi_k(x_0)} Y_k(y), \qquad u(x,y) = \sum_{k=1}^\infty \frac{\psi_k}{\Pi_k(x_0)} \Pi_k(|x|) Y_k(y). \tag{7.5}$$

Condition 1) is the hypothesis of Theorem 3.1: it guarantees the well-posedness of the direct problem for the reconstructed g. For the auxiliary problem (7.2) defining Π_k only part of it is used, namely $\gamma \neq (-1)^\nu$ and $\Delta_1^\nu \neq 0$; the factor $\Delta_2 \neq 0$ enters through the uniqueness of u for a given g. By Lemma 7.1, under condition 2) the quantity $\lambda_k \Pi_k(x_0)$ tends to $p(x_0) \neq 0$, and the zeros of $\Pi_k(x_0)$ can therefore occur only among finitely many initial indices; condition 3) has to be verified only for those.

Proof. Uniqueness. If $\{u, g\}$ and $\{\tilde{u}, \tilde{g}\}$ are two solutions, then their difference solves problem I_ν with $\psi \equiv 0$, and by (7.3) $(g_k - \tilde{g}_k)\Pi_k(x_0) = 0$, whence $g_k = \tilde{g}_k$ for all k by condition 3); hence $g = \tilde{g}$ by the completeness of the system $\{Y_k\}$, and then $u = \tilde{u}$ by Theorem 3.1. Note that the recovery of g itself uses only $\gamma \neq (-1)^\nu$ and $\Delta_1^\nu \neq 0$, which guarantee the solvability of (7.2); the remaining factor $\Delta_2 \neq 0$ in condition 1) is needed for the uniqueness of u for a given g.

Existence. First we show that $\psi_k = O(\lambda_k^{-2})$. Put $\chi = \mathcal{L}\psi = -y^m\psi''$; by condition 4) both ψ and χ satisfy the conditions of Theorem 2.1. Integrating twice by parts and using $\psi(0) = \psi(1) = 0$, $Y_k(0) = Y_k(1) = 0$ and $Y_k'' = -\lambda_k y^{-m} Y_k$, we get

$$\chi_k = \int_0^1 y^{-m}\chi(y)Y_k(y)dy = -\int_0^1 \psi''(y)Y_k(y)dy = \lambda_k\psi_k,$$

and in the same way $(\mathcal{L}\chi)_k = \lambda_k\chi_k$. Hence $\lambda_k^2\psi_k = \lambda_k\chi_k = \bigl(-y^m\chi''\bigr)_k =: \Psi_k^*$, and Bessel's inequality in $L_2\bigl(y^{-m}dy\bigr)$ gives

$$\sum_{k=1}^{\infty} |\Psi_k^*|^2 \le \int_0^1 y^m(\chi'')^2 dy =: M^2 < \infty,$$

the integral being finite because $y^{m/2}\chi'' \in L_2[0,1]$ by the conditions of Theorem 2.1 applied to χ. Thus $\psi_k = \lambda_k^{-2}\Psi_k^*$ with $\sum_k |\Psi_k^*|^2 \le M^2$. By Lemma 7.1 and condition 2) there exists k_2 such that $|\Pi_k(x_0)| \ge \frac{|p(x_0)|}{2\lambda_k}$ for $k \ge k_2$; for the finitely many indices $k < k_2$ the quantities $\Pi_k(x_0)$ are nonzero by condition 3), whence $\min_{k<k_2} \lambda_k|\Pi_k(x_0)| =: \varkappa > 0$. Putting $C = \max\bigl\{2|p(x_0)|^{-1}, \varkappa^{-1}\bigr\}$, we obtain an estimate valid for all k:

$$|g_k| = \frac{|\psi_k|}{|\Pi_k(x_0)|} \le C\frac{|\Psi_k^*|}{\lambda_k}, \quad \sum_{k=1}^{\infty} |g_k| \le C\Bigl(\sum_k \lambda_k^{-2}\Bigr)^{1/2}\Bigl(\sum_k |\Psi_k^*|^2\Bigr)^{1/2} < \infty \quad (7.6)$$

by $\lambda_k \ge Mk^2$. Therefore the series for g in (7.5) converges absolutely and uniformly on $[0,1]$, so that $g \in C[0,1]$ and $g(0) = g(1) = 0$. Note in passing that (7.6) implies a more precise statement on the regularity of the recovered function:

$$\sum_{k=1}^{\infty} \lambda_k^2|g_k|^2 \le C^2M^2 < \infty$$

that is, $g \in D(\mathcal{L})$ and $\mathcal{L}g \in L_2\bigl(0,1; y^{-m}dy\bigr)$. This is a statement in the spectral scale of $\mathcal{L}$: the inverse transition $\psi \mapsto g$ is of order two in that scale. The inclusion $g \in C^2[0,1]$ is not claimed and is not needed below.

Further, $f(x,y) = p(x)g(y)$ is continuous and odd in x, and by the definition (3.3)

$$f_k^{\pm}(x) = \int_0^1 y^{-m}p(x)g(y)Y_k(y)dy = p(x)g_k, \qquad \text{whence} \qquad F_k = P\,|g_k|\,;$$

this is an *identity*, not an estimate, and for it the mere existence of the coefficients g_k suffices. We emphasize that Theorem 4.3 is not used here: its smoothness assumptions on f serve only to derive $F_k = O(\lambda_k^{-1})$ via (2.11), whereas our estimate $F_k \le CP\lambda_k^{-1}|\Psi_k^*|$ follows directly from the construction and is stronger. Theorem 4.1, on the other hand, is an a priori estimate for each index k separately and contains no smoothness assumptions on the right-hand side.

By Theorem 4.1 and (7.6), taking into account $\lambda_k^{(j-3)/3} \le 1$ for $j \le 3$, the estimates (2.8), (2.10) and the equality $y^m Y_k'' = -\lambda_k Y_k$,

$$\sum_k \left|X_k^{(j)}\right| |Y_k| \le MP \sum_k |g_k| < \infty, \qquad j = 0,1,2,3,$$

$$\sum_k |X_k|\,|Y_k'| \le MP \sum_k \lambda_k^{-1} |g_k|\,\lambda_k < \infty, \qquad \sum_k |X_k|\,|y^m Y_k''| \le MP \sum_k |g_k| < \infty,$$

the last series without the factor y^m converging uniformly on $\bar{\Omega}_\epsilon$ for every $\epsilon > 0$. Hence the series (7.5) admits termwise differentiation three times in x and twice in y, while the termwise substitution into the equation gives $\sum_k \big(X_k''' + \lambda_k X_k\big) Y_k = p(x) \sum_k g_k Y_k = p(x) g(y)$. Thus u from (7.5) is a classical solution of problem D_ν with the right-hand side $p(x)g(y)$, and by (7.3)

$$u(x_0, y) = \sum_{k=1}^{\infty} g_k \Pi_k(x_0) Y_k(y) = \sum_{k=1}^{\infty} \psi_k Y_k(y) = \psi(y),$$

the last equality being the expansion of Theorem 2.1 for the function ψ. □

8. Conclusion

A uniqueness criterion for problem D_ν is established (Theorem 3.1), and the a priori estimate (4.21) is proved, in which the factor Θ_k, carrying both the exponential growth and the small denominator, accompanies only the even in x part of the right-hand side. Hence problem D_ν is uniquely solvable in the classical sense for a right-hand side odd in x with finitely smooth data and without restrictions on the size of the domain (Theorem 4.3), while in the general case a Gevrey-type condition on the even part suffices (Theorem 5.1). For the small denominators two unconditional lower bounds are obtained by passing to the cube of the phase (Lemmas 5.1 and 5.2). For the inverse problem the relation (7.4) is proved (Lemma 7.1), whence the absence of small denominators and the unique solvability for $p(x_0) \ne 0$ (Theorem 7.1), the inverse transition $\psi \mapsto g$ being of order two in the spectral scale of $\mathcal{L} = -y^m \partial_y^2$.

Three questions remain open: the case of an algebraic irrational E^3 for $m \ne 0$ (Remark 5.1); the necessity of the exponential condition (5.8), whose proof requires a counterexample sequence; and the transfer of the scheme to equations of order $2n+1$ with the derivative $\partial_y^{2m_0}$.

Declaration of generative AI use

During the preparation of this work the authors used Claude (Anthropic) for translation into English, language editing and symbolic–numerical verification of intermediate computations. The authors reviewed and edited the output and take full responsibility for the content of the publication.

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