Gulf Journal of Mathematics
Online First, accepted manuscript

ISSN: 2309-4966
https://doi.org/10.56947/r1xnk541

ON ANNIHILATOR GRAPH OF REDUCED MODULES

NISRINA RAHADATUL 'AISY[1] and KHURUL WARDATI[2*]

Abstract. The focus of this study is to provide a new perspective on the annihilator graph of reduced modules, which has previously been investigated for modules over commutative unital rings and for reduced multiplication modules. In this graph, the non-zero torsion elements are the vertices, while the notions of residual and annihilator play an important role in determining the adjacency between vertices. Furthermore, the properties of the annihilator graph of a reduced module are investigated inductively and illustrated by several examples. Notably, every annihilator graph containing a cycle has a girth of either 3 or 4.

Keywords. Annihilator, annihilator graph, reduced module, residual, torsion element
2020 Mathematics Subject Classification. Primary 13C70; Secondary 05C25, 13C05

1. Introduction and preliminaries

1.1. **Introduction.** A graph is a mathematical structure consisting of a set of vertices and a set of edges, together with specific adjacency conditions between any two vertices [17]. Algebraic structures can be represented using graphs. For examples, graph representations have been investigated for finite groups [2, 5, 6], generalized quaternion groups [21], and cyclic groups [12, 23, 26]. In addition to group structures, graph representations of ring structures [15, 16, 20] as well as linear and multilinear algebras [5], have also been studied. In general, the graphs associated with these algebraic structures are undirected graphs. There are differences in the terminology used to describe the adjacency between two vertices.

In addition to being a generalization of vector spaces, the module structure can be viewed as a generalization of abelian groups and rings. Indeed, every abelian group is a module over the ring of integers, while any ring is a module over itself. The graph representations of modules are constructed based on the properties and structural characteristics of the underlying modules. In such representations, both the vertex set and the adjacency relation between vertices are

Date: Received: Oct 12, 2026; Revised: Nov 6, 2026; Accepted: Nov 22, 2026.
* Corresponding author
© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

determined by the module structure. Graph representations of module structures have previously been studied in the form of intersection graphs—specifically, intersection graph representations of left modules over a ring [27]. This establishes the possibility of conducting research related to the representation of modules as graphs.

The annihilator graph of a module over a commutative unital ring has been investigated in [8, 22]. Specifically, Nozari & Payrovi [22] required a commutative unital ring when representing the annihilator graph of a module. Meanwhile, Andari et al. [8] have studied the torsion graph and annihilator graph of $\mathbb{Z}_n$. These studies suggest that annihilator graph can also represent modules with other characteristics, such as the annihilator graphs of a reduced multiplication module [1].

Abdollah et al. [1] explored the torsion and annihilator graphs of a reduced multiplication module. In their work, the torsion graph of a reduced multiplication module generalizes the concept of zero-divisor graphs of rings to modules. Zero-divisor graphs of rings are further discussed by Anderson et al. [9, 10, 11]. Meanwhile, multiplication modules are further explained by Tavallaee & Mahtabi [24], Tolooei [25], and Alsuraiheed & Bavula [7].

Furthermore, Abdollah et al. [1] compared the torsion and annihilator graphs of reduced multiplication modules. However, they did not describe in detail how to extend the annihilator graph from a module over a commutative unital ring to a reduced multiplication module. Notably, Figure 1. in [1] provides an interesting example where the torsion and annihilator graph of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ have the same form. Both graphs consist of five vertices and two edges, as shown below:

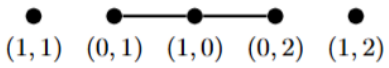

Fig. 1. [1] Annihilator graph of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ over $\mathbb{Z}$

In addition, their research established various structural properties of both graphs of reduced multiplication module, such as the diameter, cycle length, and the girth (shortest cycle length). Furthermore, the examples of the torsion graph and the annihilator graph are given in Figure 4. in [1] for a multiplication module that is not reduced.

Based on previous studies related to annihilator graphs and reduced modules, we conducted a study on the annihilator graph representation of a reduced right module. This is a generalization of the reduced multiplication module [1], because reduced modules are not always multiplication modules. In addition, this study is an extension of the annihilator graph of a module over a commutative unital ring, which is the expansion of the vertex set and the adjacency condition between vertices on the annihilator graph of a reduced module. There is an example of an annihilator graph of a reduced module over an integral domain, but it cannot be represented as an annihilator graph if viewed as a module over a commutative unital ring.

Inductively, some specific examples visually demonstrate the properties of the annihilator graphs of reduced modules. The main result of this study asserts that any annihilator graph of a reduced module containing a cycle has a girth of 3 or 4. The proof of this statement relies on several supporting lemmas detailing the graph's properties.

1.2. **Preliminaries.** Let R be a ring. A ring R is called reduced if for every $a \in R$, $a \neq 0$ then $a^2 \neq 0$. The definition of a reduced ring is restricted to rings that are not integral domains, as any integral domain naturally satisfies the reduced property [19]. For example, the ring $\mathbb{Z}_6$ is a reduced ring, and $\mathbb{Z}_4$ is not a reduced ring because $2^2 = 0$ in $\mathbb{Z}_4$.

A reduced module serves as generalization of a reduced ring, a concept previously discussed by Lee & Zhou [19] and Kyomuhangi & Ssevviiri [18]. Throughout this paper, a module M over a unital ring R is denoted by M_R. First, the definition of a reduced right module is given as follows.

Definition 1.1. [19] A module M_R is called a reduced module if for every $m \in M_R$ and $a \in R$ with $ma = 0$, then $mR \cap M_R a = \{0\}$.

A module M_R is said to be a multiplication module, if for each submodule $S \subset M_R, S = M_R I$, for some ideal $I \subset R$. To demonstrate that reduced modules are not necessarily multiplication modules and vice versa, examples covering all combinations of non-reduced module, reduced module, multiplication, and non-multiplication modules are provided below.

Example 1.2. The module $2\mathbb{Z}_8$ is not a reduced module over ring $\mathbb{Z}_8$, because there exist $4 \in 2\mathbb{Z}_8$ and $2 \in \mathbb{Z}_8$ such that $4 \cdot \mathbb{Z}_8 \cap 2\mathbb{Z}_8 \cdot 2 = \{0, 4\}$.

Example 1.2 demonstrates a non-reduced module, that is a multiplication module. This is because for every submodule N of $2\mathbb{Z}_8$, there exists an ideal I of $\mathbb{Z}_8$ such that $N = I \cdot 2\mathbb{Z}_8$. On the other hand, an example of non-multiplication module that satisfies the definition of a reduced module is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ over the ring $\mathbb{Z}$. This is because, there is a submodule $(\mathbb{Z}_2 \times \{0\}) \neq (\mathbb{Z}_2 \times \mathbb{Z}_2) \cdot J$ for every ideal J in $\mathbb{Z}$.

To strengthen the understanding of Definition 1.1, several examples of reduced modules are given below. The following examples illustrate some of the properties of reduced modules.

Example 1.3. The module $4\mathbb{Z}_8 = \{0, 4\}$ is a reduced module over the ring $\mathbb{Z}_8$. Indeed, for the element $4 \in 4\mathbb{Z}_8, 4 \cdot a = 0$ for $a = 0, 2, 4, 6 \in \mathbb{Z}_8$ and we obtain $4 \cdot \mathbb{Z}_8 \cap 4\mathbb{Z}_8 \cdot a = \{0\}$. It means that for every $m \in 4\mathbb{Z}_8$ and $a \in \mathbb{Z}_8$ with $ma = 0$, occur that $m\mathbb{Z}_8 \cap 4\mathbb{Z}_8 a = \{0\}$.

Example 1.4. Analogously, it is easy to show that the module $\mathbb{Z}_6$ is a reduced module over the ring $\mathbb{Z}_6$.

Example 1.2, 1.3, and 1.4, illustrate that neither the module $\mathbb{Z}_n$ nor its submodules $\mathbb{Z}_n$ are necessarily reduced modules over $\mathbb{Z}_n$. Inductively, based on Examples 1.2 and 1.3, the following property of reduced modules can be obtained. This property relates to the module $\mathbb{Z}_p$.

Proposition 1.5. [19] *Let p be a prime number and n be a non-negative integer, then the module $p^{n-1}\mathbb{Z}_{p^n}$ is a reduced module over $\mathbb{Z}_{p^n}$.*

Proof. Let p be a prime number and n be a non-negative integer. We will prove that $p^{n-1}\mathbb{Z}_{p^n}$ is a reduced module over $\mathbb{Z}_{p^n}$. Let $m \in p^{n-1}\mathbb{Z}_{p^n}$ and $a \in \mathbb{Z}_{p^n}$ with $ma = 0$. Note that, $mR = m\mathbb{Z}_{p^n}$. Since $m \in p^{n-1}\mathbb{Z}_{p^n}$, we obtain $mR = p^{n-1}\mathbb{Z}_{p^n}$. Next, $M_R a = p^{n-1}\mathbb{Z}_{p^n} \cdot a = \{ma \mid m \in p^{n-1}\mathbb{Z}_{p^n}\} = \{0\}$. We have $mR \cap M_R a = p^{n-1}\mathbb{Z}_{p^n} \cap \{0\} = \{0\}$. Thus, $p^{n-1}\mathbb{Z}_{p^n}$ is a reduced module. □

Proposition 1.5 in [19] was not proven in detail, so the proof is one of the findings of this research. The module $4\mathbb{Z}_8$ over the ring $\mathbb{Z}_8$ provides an illustration of the validity of Proposition 1.5, with $p = 2$ and $n = 3$.

In Example 1.3, it is known that $4 \cdot 2 = 0 = 4 \cdot 4 = 4 \cdot 6$ so $4\mathbb{Z}_8 \cdot 2 = \{0\} = 4\mathbb{Z}_8 \cdot 4 = 4\mathbb{Z}_8 \cdot 6$. The same applies to Example 1.4. This illustrates another property of reduced modules stated by Agayev al. [3] in the following lemma.

Lemma 1.6. [3] *Let M_R be a reduced module, then for every $m \in M_R, a \in R$,*

(1) *If $ma = 0$ then $mRa = \{0\}$.*
(2) *If $ma^2 = 0$ then $ma = 0$.*

Proof. [3] Given a reduced module M_R, it means that for every $m \in M_R$ and $a \in R$, the implication $ma = 0 \Rightarrow mR \cap M_R a = \{0\}$ holds.

(1) We will prove that if $ma = 0$ then $mRa = \{0\}$. Since $mR = \{mr \mid r \in R\} \subset M_R$ then $mRa = \{(mr)a \mid r \in R\} = \{m(ra) \mid ra \in R\} \subset mR$ implying that $mRa \subset M_R$. Let $x \in mRa$, then $x = mra$ for some $r \in R$. Note that $x = mra = (mr) \cdot a \in M_R a$, because $mr \in mR \subset M_R$. On the other hand, $x = m(ra)$ with $ra \in R$, so $x \in mR$. Thus, $x \in mR \cap M_R a = \{0\}$, which proves $mRa = \{0\}$.
(2) We will show that if $ma^2 = 0$ then $ma = 0$. We know that $ma \in M_R$ and $ma^2 = 0$. Note that $ma^2 = ma \cdot a = 0$. Since M_R is a reduced module, $ma \in M_R$ and $(ma) \cdot a = 0$, then $(ma)R \cap M_R a = \{0\}$. In addition, $ma = (ma)1_R \in (ma)R$ and $ma \in M_R a$, hence $ma = 0$.

□

Let $\mu : R \to S$ be a ring homomorphism, M_R be a right module over the ring R, and M_S be a right module over ring S. We define the mapping

$$f : M_R \times R \mapsto M_S$$

$$(m, r) \mapsto f(mr) = m\mu(r)$$

This mapping provides a guide to the relationship between ring homomorphisms and reduced right modules, which is expressed in the following properties.

Proposition 1.7. [13] *Let $\mu : R \to S$ be a ring homomorphism, M_R and M_S be right modules, and f be the mapping defined above.*

(1) *If M_S is a reduced module, then M_R is a reduced module.*
(2) *If μ is an epimorphism, then it holds that if the module M_R is reduced, then M_S is reduced.*

According to Example 1.4, Z_6 is a reduced module over the ring Z_6. Since we can construct the ring homomorphism $\mu : Z \to \mathbb{Z}_6$, by Proposition 1.7, $\mathbb{Z}_6$ is a reduced module over the ring $\mathbb{Z}$.

The concepts of torsion and annihilator elements in a module play an important role in the discussion of annihilator graphs. The torsion and annihilator elements in a module M_R are defined by Bordbar [14] as follows.

Definition 1.8. [14] Let M_R be a module and $m \in M_R$. An element $m \in M_R$ is called a torsion element if there exists an element $r \in R \setminus \{0\}$ such that $mr = 0$. An element $r \in R$ is called an annihilator of $m \in M_R$ if $mr = 0$, so the set of all annihilators of $m \in M_R$ is denoted by $Ann_R(m) = \{r \in R \mid mr = 0\}$.

The set of torsion elements of a module M_R is written by $T(M_R)$. Based on the definition of torsion elements, we obtain the theorem concerning the relationship between torsion elements in a reduced module M_R over a given ring and its reduced submodule.

Theorem 1.9. [3] *Let M_R be a reduced module and R be an integral domain, then $T(M_R)$ is a reduced submodule of M_R.*

Proof. [3] Given R be an integral domain. We will prove that $T(M_R)$ is a reduced submodule of M_R.
First, we will show that $T(M_R)$ is a submodule of M_R. Let $m_1, m_2 \in T(M_R)$ and $r \in R$. Then there exist $0 \neq t_1, t_2 \in R$ such that $m_1 \cdot t_1 = 0$ and $m_2 \cdot t_2 = 0$. Next, by Definition 1.1, we obtain $m_1 \cdot R \cdot t_1 = \{0\}$. This means that for every $x \in R$ it holds that $(m_1 \cdot x) \cdot t_1 = 0$. By choosing $x = t_2$, we get $(m_1 \cdot t_2) \cdot t_1 = 0 \Leftrightarrow m_1 \cdot (t_2 t_1) = 0$. Since $m_2 \cdot t_2 = 0$ also holds, we get $(m_2 \cdot t_2) \cdot t_1 = 0 \cdot t_1 \Leftrightarrow m_2 \cdot (t_2 t_1) = 0$. Note that, $(m_1 - m_2) \cdot (t_2 t_1) = (m_1 \cdot t_2 t_1) - (m_2 \cdot t_2 t_1) = 0$. Since R has no non-trivial zero-divisors and t_1, t_2 are both non-zero, then $t_2 t_1 \neq 0$. Thus, we obtain a non-zero element $s \in R$, with $s = t_2 t_1$ such that $(m_1 - m_2) \cdot s = 0$. Hence, it is proven that $m_1 - m_2 \in T(M_R)$. Next, using Definition 1.1, we have $m_1 R t_1 = \{0\}$, then $(m_1 \cdot r) \cdot t_1 = 0$. Consequently, $m_1 \cdot r \in T(M_R)$, which establishes that $T(M_R)$ is a submodule of M_R.
Second, we will show that $T(M_R)$ is a reduced module. Let $n \in T(M_R)$ and $s \in R$, with $ns = 0$. Since $T(M_R) \subseteq M_R$, then $n \in M_R$. It is known that M_R is reduced and $ns = 0$, then by Definition 1.1 we have $nR \cap M_R s = \{0\}$. On the other hand, $nR \cap T(M_R)s \subseteq nR \cap M_R s = \{0\}$. In other words, $T(M_R)$ is a reduced module. □

Note again in Example 1.4, that $T(Z_6) = \{0, 2, 3, 4\}$ is not a submodule of $\mathbb{Z}_6$ over itself. This is because the ring $\mathbb{Z}_6$ contains non-trivial zero-divisors.

Submodules of a module can form the residuals of the ring, which are essential construction in defining the annihilator graph. Let N be a submodule of M_R, then $[N : M_R] = \{r \in R \mid M_R r \subseteq N\}$ is an ideal of the ring R, and $[N : M_R]$ is called the residual of N by M_R. If $x \in M_R$, then xR is the submodule of M_R generated by x. These two definitions are used by Abdollah et al. [1] to define the residual as follows.

Definition 1.10. [1] The residual of xR by M_R is defined and denoted by

$$[x : M_R] = [xR : M_R] = \{r \in R \mid M_R r \subseteq xR\}.$$

The non-zero torsion elements in Definition 1.8, constitutes the vertices of the annihilator graph of a reduced module. Meanwhile, the definition of residual of xR in Definition 1.10 when collaborated with the annihilator element in Definition 1.8 serves as a bridge between the structure of reduced modules and graph theory in this research.

2. Main results

It is worth restating that M_R is a right module over a unital ring R (unless a commutative ring is explicitly specified)—simply referred to as a module. If $m \in M_R$ and $a \in R$ then it is easy to show that, $mR = \{mr \mid r \in R\}$ and $M_R a = \{ma \mid m \in M_R\}$ are both submodules of M_R. It should be emphasized again that the right module M_R is reduced, as in Definition 1.1, if for every $m \in M_R$ and $a \in R$ with $ma = 0$, then $mR \cap M_R a = \{0\}$.

The focus of this research is the structure of a reduced module represented by an annihilator graph. An annihilator graph of a reduced module M_R denoted by $AG(M_R)$, is an extension of the annihilator graph $AG(M)$ over a commutative unital ring R. The adjacency between any two vertices a, b in the graph is denoted by $a \sim b$, with $a, b \in T(M_R) \setminus \{0\}$.

The annihilator graph of an R-module M over a commutative unital ring in [22] has a vertex set consisting of non-trivial zero-divisors, namely $Z_R(M) \setminus Ann_R(M) \subseteq R$, where $Z_R(M) = \{r \in R \mid rm = 0 \text{ for some } m \in M\}$, and $Ann_R(M) = \{r \in R \mid rM = \{0\}\}$. We know that a non-trivial zero-divisor in a ring can be generalized to a non-zero torsion element in a module. Analogously, the annihilator graph of a reduced module M_R is denoted as $AG(M_R)$. The non-zero torsion elements are the vertices of the annihilator graph $AG(M_R)$. That is, $T(M_R) \setminus \{0\}$ is the set of vertices in $AG(M_R)$.

According to [22], any two vertices a, b in $AG(M)$ are adjacent if and only if

$$Ann_M(ab) \neq Ann_M(a) \cup Ann_M(b) \tag{2.1}$$

while $Ann_M(a) = \{m \in M \mid am = 0\}$. Due to the commutative property of $\cup$ of an R-module over a commutative unital ring, the annihilator graph $AG(M)$ is an undirected graph.

Note the terminology of $Ann_M(a) \subseteq M$ and the Definition 1.8, $Ann_R(m) \subseteq R$ where $a \in R, m \in M$. Respectively, we denote the annihilator element in R relative to M and the annihilator element in M relative to R. Both are similar in definition, but not equal in position.

Besides the torsion elements, the main components in forming an annihilator graph of a reduced module are the annihilators and residuals. These three terms play an important role in determining the adjacency between vertices in the annihilator graph $AG(M_R)$.

The adjacency of vertices in an annihilator graph $AG(M_R)$ is developed from $AG(M)$ based on (2.1). The right-hand side of inequality (2.1) can be adopted directly, the annihilator element in R relative to M is replaced by the annihilator

element in M relative to R. However, the left-hand side in (2.1) cannot be adopted directly because there is no multiplicative operation in M_R. Two non-zero torsion elements x, y, can be sequentially multiplied by the ideals of R of the form $[y : M_R]$ and $[x : M_R]$. Based on Definition 1.10, the left-hand side of (2.1) can be replaced by $Ann_R(x[y : M_R])$ or $Ann_R(y[x : M_R])$. More completely, the annihilator graph of a reduced module is defined as follows.

Definition 2.1. An annihilator graph of a reduced module M_R is an undirected graph with $T(M_R) \setminus \{0\}$ as its vertex set. Two vertices x, y are connected by an edge, denoted by $x \sim y$ if

$$Ann_R(y[x : M_R]) \neq Ann_R(x) \cup Ann_R(y)$$
$$\text{or } Ann_R(x[y : M_R]) \neq Ann_R(x) \cup Ann_R(y).$$

Based on Example 1.4, we give an example of an annihilator graph of a reduced module $\mathbb{Z}_6$ over itself. An annihilator graph of a reduced module $\mathbb{Z}_6$ over itself consistently denoted by $AG(\mathbb{Z}_6)$ is discussed in the following example.

Example 2.2. Given the reduced module $\mathbb{Z}_6$ over the ring $\mathbb{Z}_6$. The set of all torsion elements of $\mathbb{Z}_6$ is $T(\mathbb{Z}_6) = \{0, 2, 3, 4\}$ so the vertex set of $AG(\mathbb{Z}_6)$ is $\{2, 3, 4\}$. Next, we investigate the adjacency of the vertices of $AG(\mathbb{Z}_6)$. Note that, $Ann_{\mathbb{Z}_6}(3[2 : \mathbb{Z}_6]) = \mathbb{Z}_6$ and $Ann_{\mathbb{Z}_6}(2) \cup Ann_{\mathbb{Z}_6}(3) = \{0, 2, 3, 4\}$. It means that, $2 \sim 3$. Similarly, we get $3 \sim 4$. However, $2 \not\sim 4$ because $Ann_{\mathbb{Z}_6}(2[4 : \mathbb{Z}_6]) = Ann_{\mathbb{Z}_6}(4[2 : \mathbb{Z}_6]) = \{0, 3\} = Ann_{\mathbb{Z}_6}(2) = Ann_{\mathbb{Z}_6}(4)$. Thus, the graph $AG(\mathbb{Z}_6)$ is a simple graph with 3 vertices and 2 edges as shown in Figure 2.

Apart from being a reduced module, $\mathbb{Z}_6$ is also a module over the commutative unital ring $\mathbb{Z}_6$. The ring $\mathbb{Z}_6$ has a non-trivial zero-divisor set $\{2, 3, 4\}$ as its vertex set. Based on (2.1), the annihilator graph is exactly the same as depicted in Figure 2. We have $2 \sim 3 \sim 4$, and $2 \not\sim 4$, because

$$Ann_{\mathbb{Z}_6}(2 \cdot 3) = Ann_{\mathbb{Z}_6}(4 \cdot 3) = \mathbb{Z}_6$$
$$\neq Ann_{\mathbb{Z}_6}(2) \cup Ann_{\mathbb{Z}_6}(3) = Ann_{\mathbb{Z}_6}(4) \cup Ann_{\mathbb{Z}_6}(3) = \{0, 2, 3, 4\},$$
$$\text{and } Ann_{\mathbb{Z}_6}(2 \cdot 4) = Ann_{\mathbb{Z}_6}(2) = Ann_{\mathbb{Z}_6}(4) = \{0, 3\} = Ann_{\mathbb{Z}_6}(2) \cup Ann_{\mathbb{Z}_6}(4).$$

Based on Proposition 1.7, the module $\mathbb{Z}_6$ is also a reduced module over $\mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain, $\mathbb{Z}$ does not contain non-trivial zero-divisors. Hence, $AG(\mathbb{Z}_6)$ over $\mathbb{Z}$ as a commutative unital ring has no vertices under the classical definition (2.1). However, the reduced module $\mathbb{Z}_6$ over $\mathbb{Z}$ can be represented by our proposed annihilator graph $AG(\mathbb{Z}_6)$ with the vertex set $T(\mathbb{Z}_6) \setminus \{0\} = \{1, 2, 3, 4, 5\}$, which is structurally different from the one in Figure 2. According to Definition 1.10, we get the residuals,

$$[1 : \mathbb{Z}_6] = [5 : \mathbb{Z}_6] = \mathbb{Z}, [2 : \mathbb{Z}_6] = [4 : \mathbb{Z}_6] = 2\mathbb{Z}, [3 : \mathbb{Z}_6] = 3\mathbb{Z}.$$

On the other hand,

$$Ann_{\mathbb{Z}}(1) = Ann_{\mathbb{Z}}(5) = \mathbb{Z}, Ann_{\mathbb{Z}}(3) = 2\mathbb{Z}, Ann_{\mathbb{Z}}(2) = Ann_{\mathbb{Z}}(4) = 3\mathbb{Z}.$$

We obtain the adjacency between vertices in $AG(\mathbb{Z}_6)$ over $\mathbb{Z}$ as:

$$1 \sim 2, 1 \sim 3, 1 \sim 4, 2 \sim 3, 2 \sim 4, 2 \sim 5, 3 \sim 5, 4 \sim 5, \text{ while } 1 \not\sim 5, 3 \not\sim 4.$$

Thus, the graph $AG(\mathbb{Z}_6)$ over $\mathbb{Z}$ can be described as shown in Figure 2.

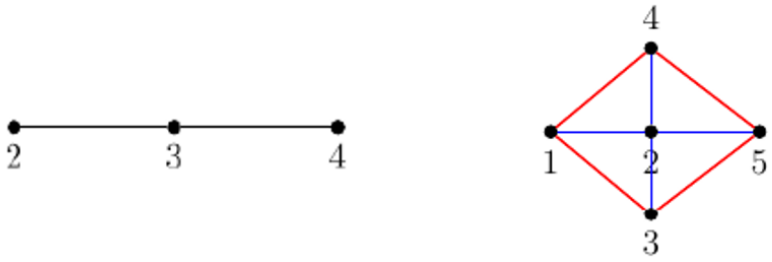

Fig. 2. On the left is annihilator graph $AG(\mathbb{Z}_6)$ over $\mathbb{Z}_6$, and on the right is annihilator graph $AG(\mathbb{Z}_6)$ over $\mathbb{Z}$

Figure 2 shows that reduced modules represented in an annihilator graph over different rings can yield different forms. It is also shown that the graph $AG(\mathbb{Z}_6)$ over the ring Z contains a cycle with a girth of 3 or 4. In addition to the reduced module $\mathbb{Z}_6$, examples of the other representation of graph $AG(M_R)$ are given below.

Example 2.3. Given a module $11\mathbb{Z}_{121}$ over the ring $\mathbb{Z}_{121}$. Based on Proposition 1.5, $11\mathbb{Z}_{121}$ is a reduced module over $\mathbb{Z}_{121}$. The set of non-zero torsion elements is $T(11\mathbb{Z}_{121}) \setminus \{0\} = \{11, 22, 33, 44, 55, 66, 77, 88, 99, 110\}$. Next, we investigate the adjacency of vertices in $AG(11\mathbb{Z}_{121})$. Note that, for any two vertices x, y,

$$Ann_{\mathbb{Z}_{121}}(x[y : 11\mathbb{Z}_{121}]) = Ann_{\mathbb{Z}_{121}}(x \cdot \mathbb{Z}_{121}) = 11\mathbb{Z}_{121}.$$

Meanwhile, $Ann_{\mathbb{Z}_{121}}(x) = 11\mathbb{Z}_{121}$ for any $x \in T(11\mathbb{Z}_{121}) \setminus \{0\}$. Thus, we obtain $Ann_{\mathbb{Z}_{121}}(x[y : 11\mathbb{Z}_{121}]) = Ann_{\mathbb{Z}_{121}}(x) \cup Ann_{\mathbb{Z}_{121}}(y)$, for every vertices x, y. This implies that no two vertices are adjacent. Hence, the graph $AG(11\mathbb{Z}_{121})$ is null graph (a graph which no vertex connected to any other vertex), as shown in Figure 3.

Example 2.2 and 2.3 illustrate two reduced modules with different annihilator graph representations. The graph $AG(\mathbb{Z}_6)$ in Example 2.2 is a connected graph, while in Example 2.3, the graph $AG(11\mathbb{Z}_{121})$ is a null graph. The following is an example of a reduced module with a different representation from Example 2.2 and 2.3.

Example 2.4. Let $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ be a reduced module over $\mathbb{Z}$. Then we have $T(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This means the vertex set of $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is $\{(0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$. Note that for any two vertices x, y,

$$Ann_{\mathbb{Z}}(x[y : (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)]) = Ann_{\mathbb{Z}}(x \cdot 2\mathbb{Z}) = Ann_{\mathbb{Z}}((0,0,0)) = \mathbb{Z}.$$

Meanwhile, $Ann_{\mathbb{Z}}(x) = 2\mathbb{Z}$ for every vertex x. Consequently, for every pair of vertices x, y in $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ it holds that $Ann_{\mathbb{Z}}(x[y : (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)]) \neq$

$Ann_{\mathbb{Z}}(x) \cup Ann_{\mathbb{Z}}(y)$. This means that any two vertices are adjacent. Thus, the annihilator graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is a complete graph.

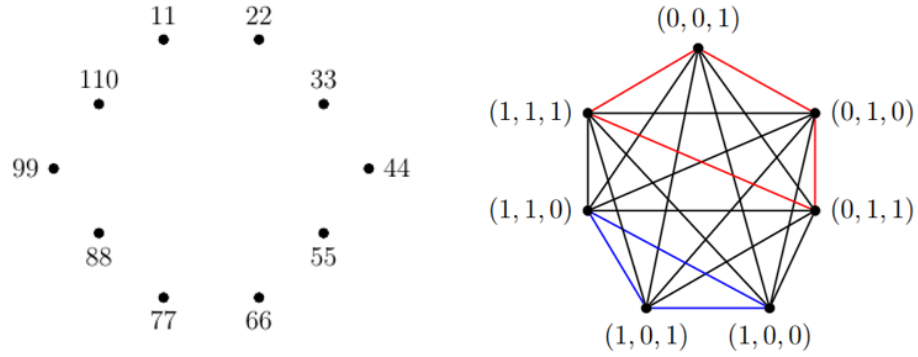

Fig. 3. [4] On the left is annihilator graph $AG(11\mathbb{Z}_{121})$ over $\mathbb{Z}_{121}$, and on the right is annihilator graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ over $\mathbb{Z}$.

Graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ over $\mathbb{Z}$ on Figure 3 is an example of the annihilator graph of a reduced module that is not a multiplication module. Note that the annihilator graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ contains cycles with a girth of 3 or 4.

Example 2.4 leads to the conjecture that any $AG(M_R)$ of a reduced module containing cycles has a girth of either 3 or 4. Proving this property requires the following supporting lemmas.

Lemma 2.5. *Let M_R be a reduced module and $x, y \in T(M_R) \setminus \{0\}$. If $y[x : M_R] = \{0\}$ then $(xR \setminus \{0\}), (yR \setminus \{0\}) \subseteq T(M_R) \setminus \{0\}$ as well as $(xR \setminus \{0\})$ and $(yR \setminus \{0\})$ fully connected in $AG(M_R)$.*

Proof. Let M_R be a reduced module, $x, y \in T(M_R) \setminus \{0\}$ and $y[x : M_R] = \{0\}$.

(a) We will show that $(xR \setminus \{0\}), (yR \setminus \{0\}) \subseteq T(M_R) \setminus \{0\}$. Let $z \in (xR \setminus \{0\})$, and $w \in (yR \setminus \{0\})$, then there exist $r, s \in R$ such that $z = xr, w = ys$. Since $x, y \in T(M_R) \setminus \{0\}$, then $xa = 0$ and $yb = 0$ with $a, b \in R \setminus \{0\}$ (Definition 1.8). Note that, $za = (xr)a = x(ra) = x(ar) = (xa)r = 0$, $wb = (ys)b = y(sb) = y(bs) = (yb)s = 0$. Since $0 \neq a, b \in R$ and $z, w \neq 0$, we obtain $z, w \in T(M_R) \setminus \{0\}$. Consequently, $(xR \setminus \{0\}), (yR \setminus \{0\}) \subseteq T(M_R) \setminus \{0\}$.

(b) We will show that $(xR \setminus \{0\})$ and $(yR \setminus \{0\})$ are fully connected in $AG(M_R)$. Let $p \in (xR \setminus \{0\})$ and $q \in (yR \setminus \{0\})$. Since $p \in xR$, then $pR \subseteq xR$. Based on Definition 1.10, note that $[p : M_R] = [pR : M_R] = \{t \in R \mid M_R t \subseteq pR\}$. If $t \in [p : M_R]$, then $M_R t \subseteq pR \subseteq xR$, such that $t \in [x : M_R]$. Hence, we have $[p : M_R] \subseteq [x : M_R]$. Next, since $q \in (yR \setminus \{0\})$, it means $q = ys$ for some $s \in R$. Note that, $q[p : M_R] = ys[p : M_R] = y[p : M_R]$ because $[p : M_R]$ is an ideal of R. Since $[p : M_R] \subseteq [x : M_R]$ and we know that $y[x : M_R] = \{0\}$, then

$y[p : M_R] \subseteq y[x : M_R] = \{0\}$, yielding $y[p : M_R] = \{0\}$. Consequently, $q[p : M_R] = \{0\}$, such that $Ann_R(q[p : M_R]) = Ann_R(0) = R$. Since $p, q \neq 0$, then $1 \notin Ann_R(p)$ and $1 \notin Ann_R(q)$, so $Ann_R(p) \cup Ann_R(q) \neq R$. Thus, $Ann_R(p[q : M_R]) \neq Ann_R(p) \cup Ann_R(q)$, which means p connected to q in $AG(M_R)$. Since p and q are arbitrary, then $(xR \setminus \{0\})$ and $(yR \setminus \{0\})$ are fully connected in $AG(M_R)$.

□

Based on the explanation of Example 2.4, that $(0,1,0)[(0,0,1) : (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)] = \{(0,0,0)\}$ then $(0,1,0)\mathbb{Z} = \{(0,0,0),(0,1,0)\} \subset T(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2), (0,0,1)\mathbb{Z} = \{(0,0,0),(0,0,1)\} \subset T(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. In addition, $(0,1,0)\mathbb{Z} \setminus \{(0,0,0)\}, (0,0,1)\mathbb{Z} \setminus \{(0,0,0)\}$ are fully connected in $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$. the properties of the annihilator graph AG(MR) containing a cycle will be presented.

Lemma 2.6. *Let M_R be a reduced module and $AG(M_R)$ be an annihilator graph of M_R. If the graph $AG(M_R)$ contains a cycle of length n, where $5 \leq n < \infty$ and $y, z \in T(M_R) \setminus \{0\}$, then there exists $z' \in T(M_R) \setminus \{0\}$ such that $z'[y : M_R] = \{0\}$.*

Proof. Let graph $AG(M_R)$ with M_R is a reduced module. We know $y, z \in T(M_R) \setminus \{0\}$ and the length of the cycle in $AG(M_R)$ is n with $5 \leq n < \infty$. We will prove that there exists $z' \in T(M_R) \setminus \{0\}$ such that $z'[y : M_R] = \{0\}$. Suppose $AG(M_R)$ has a cycle $x \sim y \sim z \sim w \sim ... \sim k \sim x$, where $x, y, z, w, k \in T(M_R) \setminus \{0\}$. Since y and z are adjacent, then it holds $Ann_R(z[y : M_R]) \neq Ann_R(y) \cup Ann_R(z)$ or $Ann_R(y[z : M_R]) \neq Ann_R(z) \cup Ann_R(y)$. We consider two cases:
Case 1: $Ann_R(z[y : M_R]) \neq Ann_R(y) \cup Ann_R(z)$.
Let $s \in Ann_R(z[y : M_R]) \setminus (Ann_R(y) \cup Ann_R(z))$. Suppose $z' = zs$, we have $z'[y : M_R] = zs[y : M_R] = (z[y : M_R])\, s = \{0\}$. On the other hand, $zs \neq 0$ because $s \notin (Ann_R(y) \cup Ann_R(z))$. We obtain $z' \in T(M_R)$ and $z'[y : M_R] = \{0\}$.
Case 2: $Ann_R(y[z : M_R]) \neq Ann_R(z) \cup Ann_R(y)$.
Let $s \in Ann_R(y[z : M_R]) \setminus (Ann_R(z) \cup Ann_R(y))$ with $ys \neq 0$ and $zs \neq 0$, so we obtain $(y[z : M_R])\, s = \{0\}$. We will show that $(M_R[z : M_R])\, s \neq \{0\}$. Suppose $(M_R[z : M_R])\, s = \{0\}$, then for every $m \in M_R$, it holds that $m\,(s[z : M_R]) = \{0\}$. This means, $s[z : M_R] \subseteq Ann_R(M_R)$. As a result, a new connection will be formed between the vertices zs, ys, or other elements. For example, $x \sim y \sim z \sim w \sim zs \sim k \sim x$ would arise, even though $w \sim k$ was previously known. Therefore, a cycle of length less than 5 is formed in $AG(M_R)$. This contradicts the assumption that the shortest cycle length is $n \geq 5$. Consequently, the supposition is false and it must be $M_R\,(s[z : M_R]) \neq \{0\}$. Hence, there is $w \in M_R[z : M_R]$ such that $ws \neq 0$. Next, we define $u' = ws$ and $u' \neq 0$. Let $r \in [y : M_R]$, then $yr = 0$. Note that $u'r = (ws)r = w(rs)$. Since $w \in M_R[z : M_R]$, then $wr \in y[z : M_R] \cdot R$, such that $(wr)s \in (u'[z : M_R])\, s = \{0\}$. From both cases, it is established that if $u'r = 0$ for every $r \in [y : M_R]$. Thus, $u'[y : M_R] = \{0\}$. □

Note for those vertices $(0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,1,1)$ in Figure 3, which form a cycle of length 5. It appears that $(0,1,1)$ is connected to the other four vertices, thus forming a cycle with a girth 4, namely $(0,1,1) \sim (0,1,0) \sim (0,0,1) \sim (1,1,1) \sim (0,1,1)$. There is also a vertex $x = (1,0,1)$ connected to all

five vertices, thus producing several cycles with the shortest length (girth) of 3, such as:

$$(1,0,1) \sim (1,0,0) \sim (0,0,1) \sim (1,0,1)$$

$$(1,0,1) \sim (1,1,1) \sim (0,0,1) \sim (1,0,1).$$

Moreover, $x[x : (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)] = x \cdot 2\mathbb{Z} = \{(0,0,0)\}$ then $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)[x : (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)] = \{(0,0,0)\}$. We also obtained that $x = (1,0,1)$ is connected to all vertices in $T(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \setminus \{(0,0,0)\}$. This inspires the following property.

Proposition 2.7. *Let M_R be a reduced module over a commutative unital ring and $x \in T(M_R) \setminus \{0\}$. If $x[x : M_R] = \{0\}$ then $M_R[x : M_R] = \{0\}$ such that x is connected for every vertex in $T(M_R) \setminus \{0\}$.*

Proof. Given a reduced module M_R and $x \in T(M_R) \setminus \{0\}$ with $x[x : M_R] = \{0\}$. We will prove that $M_R[x : M_R] = \{0\}$ such that x is connected for every element of $T(M_R) \setminus \{0\}$. Let $r \in [x : M_R]$ and $m \in M_R$. Since $r \in [x : M_R]$ then $M_R r \subseteq xR$, so there exists $a \in R$ such that $mr = xa$. Furthermore, $r \in [x : M_R]$ implies that $xr \in x[x : M_R]$. Recall the assumption that $x[x : M_R] = \{0\}$, hence $xr = 0$. Note that

$$mr = xa$$

$$(mr)r = (xa)r$$

$$mrr = xar$$

$$mr^2 = xra$$

$$mr^2 = (xr)a$$

$$mr^2 = a \cdot 0$$

$$mr^2 = 0$$

Since $mr^2 = 0$ holds for every $m \in M_R$, we have $r^2 \in Ann_R(M_R)$. By Lemma 1.6(2), since M_R is reduced module, $mr^2 = 0$ implies $mr = 0$. Thus, $M_R r = \{0\}$, which leads to $M_R[x : M_R] = \{0\}$. Now, let $y \in T(M_R) \setminus \{0\}$ with $y \neq x$. Since $y[x : M_R] \subseteq M_R[x : M_R] = \{0\}$, we obtain $y[x : M_R] = \{0\}$ implying that $Ann_R(y[x : M_R]) = Ann_R(0) = R$. Meanwhile $Ann_R(x) \cup Ann_R(y) \not\subseteq R$. Therefore, $Ann_R(y[x : M_R]) \neq Ann_R(x) \cup Ann_R(y)$. Thus, x is connected to y for every $y \in T(M_R) \setminus \{0\}$. □

The cycle marked in blue in Figure 3 is an example of a cycle of girth 3, while the cycle marked in red is an example of a cycle of girth 4. There is no girth 2 because the graph $AG(M_R)$ is an undirected graph. The annihilator graph of reduced modules cannot have girth 1 or 2 because the existing definition of adjacency in Definition 2.1 applies to two different vertices. This inductively yields the following theorem regarding the structure of the annihilator graph.

Theorem 2.8. *Let M_R be a reduced module over a ring R. If the graph $AG(M_R)$ has a cycle, then the girth of the graph is either 3 or 4.*

Proof. Given a reduced module M_R. It is known that $AG(M_R)$ has a cycle. We will prove that $AG(M_R)$ has only girth 3 or 4. Suppose $AG(M_R)$ has a girth $n \geq 5$, consisting of those vertices $x_1, x_2, x_3, ..., x_n$ where $x_1 \sim x_2$, $x_2 \sim x_3$, and so on until $x_n \sim x_1$. This configuration implies that $x_{i+2}[x_i : M_R] = 0$ for each index i modulo n. Since the cycle is shortest cycle by definition, there is no edge connecting two unconnected vertices in the cycle. Hence, x_1 is not connected to x_3. Next, we choose $y = x_1$ and $z = x_3$. Since $x_1, x_3 \in T(M_R) \setminus \{0\}$ are not connected, then by Lemma 2.5, there exists $z' \in T(M_R) \setminus \{0\}$ such that $z'[x_1 : M_R] = \{0\}$. Based on Lemma 2.6, setting $x = x_1$ and $y = z'$, we obtained the set $S = z'R \setminus \{0\}$. Furthermore, we investigate the adjacency between z' and each vertex on the cycle. If $z' \sim x_2$ then we obtain a cycle of the form $x_1 \sim x_2 \sim z' \sim x_1$, while if $z' \sim x_3$ then we obtain a cycle of the form $x_1 \sim x_2 \sim x_3 \sim z' \sim x_1$. According to Proposition 2.7 and the complete graph structure constructed based on Lemma 2.6, a cycle of length less than 5 can always be formed. This contradicts the supposition that the girth $n \geq 5$. As a result, the girth must be either 3 or 4. □

The application of Theorem 2.8 can be seen in Figure 3 as a complete graph. The graph $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ can form a cycle because the 7 vertices on the graph are fully connected to each other. Similarly, the annihilator graph $AG(\mathbb{Z}_6)$ over $\mathbb{Z}$ in Figure 2 contains cycles, even though it is not a complete graph. Furthermore, we obtain a girth of 3 (for the cycle marked in blue) and a girth of 4 (for the cycle marked in red).

3. Conclusion

This research inductively, through several examples of reduced modules, provides inspiration and illustrations of the properties of reduced modules, including:

(1) module $p^{n-1}Z_{p^n}$ is a reduced module over the ring Z_{p^n} with p is a prime number;
(2) for a reduced module M_R and for every $m \in M_R, a \in R$, then the following properties holds:
   (a) if $ma = 0$ then $mRa = \{0\}$;
   (b) if $ma^2 = 0$ then $ma = 0$;
(3) the set of all torsion elements of a reduced module M_R, denoted by $T(M_R)$ is a reduced submodule over an integral domain.

The vertices of the annihilator graph of a module over a commutative unital ring and those of a reduced module are, respectively, the non-trivial zero-divisors of the ring, and non-trivial zero torsion elements in a module. The definition of adjacency vertices in an annihilator graph of a reduced module is the extension of the annihilator graph of an R-module over a commutative unital ring. The extension process can be seen from the annihilator terminology of inequality (2.1) which is integrated with the residuals yR and xR by M_R, leading up to Definition 2.1.

The annihilator graph representations of some reduced modules have different forms. Some of them cannot be represented as an annihilator graph of a module over a commutative unital ring with the definition of [22]. An annihilator graph

$AG(\mathbb{Z}_6)$ over $\mathbb{Z}_6$ or over $\mathbb{Z}$, $AG(11\mathbb{Z}_{121})$ over $\mathbb{Z}_{121}$, and $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ over Z, respectively, are simple graphs, null graphs, and complete graphs, and there are other examples of reduced (multiplication) modules that contain isolation vertices.

Both of simple graph in Figure 2 and the complete graph in Figure 3 contain cycles, inspire the main result of this research of annihilator graphs of reduced modules M_R. Namely, if an annihilator graph $AG(M_R)$ contains a cycle, then the girth of the graph is only 3 or 4. A formal proof of this main results requires the support of the following annihilator graph properties of reduced modules:

(1) If $x, y \in T(M_R) \setminus \{0\}$ and $y[x : M_R] = \{0\}$ then $(xR \setminus \{0\}), (yR \setminus \{0\}) \subseteq T(M_R) \setminus \{0\}$ also $(xR \setminus \{0\})$ and $(yR \setminus \{0\})$ are fully connected in $AG(M_R)$.
(2) If the annihilator graph $AG(M_R)$ has a cycle with length n where $5 \leq n < \infty$ and $y, z \in T(M_R) \setminus \{0\}$, then there exists $z' \in T(M_R) \setminus \{0\}$ such that $z'[y : M_R] = \{0\}$.
(3) If $x \in T(M_R) \setminus \{0\}$ and $x[x : M_R] = \{0_M\}$ then there exists $M_R[x : M_R] = \{0_M\}$, such that x connected to every vertices in $T(M_R) \setminus \{0\}$.

An opportunity for further research lies in providing a more detailed characterization of the annihilator graph parameters of reduced modules. It is anticipated that this will reveal graph-related properties and potential special cases that could arise in the representation of reduced modules. Furthermore, the torsion graph representations of reduced modules are also interesting to study.

Acknowledgement. The authors would like to thank the Mathematics Study Program, Sunan Kalijaga State Islamic University, for providing the authors with the opportunity to present at the International Conference on Science and Engineering (ICSE) 2025. The authors are also grateful to the anonymous reviewers for their helpful comments and careful evaluation of the manuscript.

References

1. Abdollah, Z., Rad, P.M., Ghalandarzadeh, S., & Shahriari, S. (2022). On Torsion Elements and Their Annihilators. Journal of Algebra, 610, 199–222. https://doi.org/10.1016/j.jalgebra.2022.07.009
2. Adhikari, A. & Banarjee, S. (2022). Prime coprime graph of a finite group. Journal of Mathematics, 52(2), 41-59. https://doi.org/10.30755/NSJOM.11151
3. Agayev, N., Halicioglu, S., & Harmanci, A. (2009). On Reduced Modules. Community Faculty Science University Ankara, 58(1), 9–16. https://doi.org/10.1501/Commua1_0000000642
4. 'Aisy, N.R. (2026). Beberapa Modul Tereduksi dan Representasi Graf Annihilator (Some Properties of Reduced Module and Annihilator Graph Representation). Undergraduate Thesis of UIN Sunan Kalijaga Yogyakarta. DOI not found
5. Ali, F., Fatima, S., & Wang, W. (2020). On the power graphs of certain finite groups. Linear and Multilinear Algebra, 70(19), 3803-3817. https://doi.org/10.1080/03081087.2020.1856028
6. Alireza, D., Ahmad, E., & Abbas, J. (2014). Some results on the power graphs of finite groups. ScienceAsia, 41(2015), 73-78. https://doi.org/10.2306/scienceasia1513-1874.2015.41.073

7. Alsuraiheed, T. & Bavula, V.V. (2021). Criteria for a direct sum of modules to be a multiplication module over noncommutative rings. Journal of Algebra, 584, 69–88. https://doi.org/10.1016/j.jalgebra.2021.05.006
8. Andari, A., Mahanani, D.M., & Purnamaadi, D.Z. (2025). Correspondence Between Torsion and Annihilator Graph of Modules Zn Over Commutative Rings. Cauchy: Jurnal Matematika Murni Dan Aplikasi, 10(1), 403–414. https://doi.org/10.18860/cauchy.v10i1.30002
9. Anderson, D.F. & Lagrange, J.D. (2016). Some remarks on the compressed zero-divisor graph. Journal of Algebra, 447, 297–321. http://dx.doi.org/10.1016/j.jalgebra.2015.08.021
10. Anderson, D.F. & Lewis, E.F. (2016). A General Theory of Zero-Divisor Graphs over a Commutative Ring. International Electronic Journal of Algebra, 20, 111–135. https://doi.org/10.24330/ieja.266187
11. Anderson, D.F. & Mcclurkin, G. (2020). Generalizations of The Zero-Divisor Graph. International Electronic Journal of Algebra, 27(27), 237–262. https://doi.org/10.24330/ieja.663079
12. Atmaja, I.A., Erfanian, A., Susanti, Y., Huda, M.N., & Suparwanto, A. (2024). Prime Square Order Cayley Graph of Cyclic Groups of Particular Valency. Asian-European Journal of Mathematics, 17(2), 1–17. https://doi.org/10.1142/S1793557124500037
13. Başer, M. & Agayev, N. (2006). On reduced and semicommutative modules. Turkish Journal of Mathematics, 30(3), 285–291. DOI not found
14. Bordbar, H. (2023). Torsion Elements and Torsionable Hypermodules. Mathematics, 11(21), 4525. https://doi.org/10.3390/math11214525
15. Cahyati, E.S., Maya, R., Maharani, I., & Nurhayati, S. (2022). Generalized Non-Braid Graphs of Rings. Journal of Fundamental Mathematics and Applications (JFMA), 5(2), 192–201. https://doi.org/10.14710/jfma.v5i2.14152
16. Djuang, F.S., Wijayanti, I.E., & Susanti, Y. (2026). Clean Graphs and Idempotent Graphs over Finite Rings: An Approach Based on Zn. Communications in Combinatorics and Optimization, 1-19. https://doi.org/10.22049/cco.2026.31307.2811
17. Kaleel, S.M., Mohammed, H.N., Alghamdi, A.M., Ali, A.M., & Almotairi, E.S. (2026). Digraph and Graph of Regular Rings with Some Properties. Baghdad Science Journal, 23(6), 2216-2232. https://doi.org/10.21123/2411-7986.5335
18. Kyomuhangi, A. & Ssevviiri, D. (2022). Generalised reduced modules. Rendiconti Del Circolo Matematico Palermo Series 2, 72, 421–431. https://doi.org/10.1007/s12215-021-00686-8
19. Lee, T.-K. & Zhou, Y. (2004). Reduced Modules. https://doi.org/10.1201/9780824750817.ch29
20. Muhammad, H., Qonita, N., Fibriyanti, R.A.W., & Susanti, Y. (2023). Bipartite Graph Associated with Elements and Cosets of Subrings of Finite Rings. Barekeng: Journal of Mathematics and Its Applications, 17(2), 667–672. https://doi.org/10.30598/barekengvol17iss2pp0667-0672
21. Munandar, A. (2023). Some Properties on Coprime Graph of Generalized Quaternion Groups. Barekeng: Journal of Mathematics and Its Application, 17(3), 1373–1380. https://doi.org/10.30598/barekengvol17iss3pp1373-1380
22. Nozari, K. & Payrovi, S. (2021). The Annihilator Graph for Modules over Commutative Rings. Journal of Algebraic Systems, 9(1), 1–12. https://doi.org/10.22044/jas.2020.9194.1448
23. Suparwati, T., Susanti, Y., Wahyuni, S., & Erfanian, A. (2026). Prime cubic order Cayley graphs over cyclic groups. Asian-European Journal of Mathematics, 19 (11). https//doi.org/10.1142/S1793557126500038
24. Tavallaee, H.A. & Mahtabi, R. (2017). Some Properties of Multiplication Modules. Journal of Indonesian Mathematical Society (JIMS), 23(2), 47–53. https://doi.org/10.22342/jims.23.2.276.47-53

25. Tolooei, Y. (2020). Multiplication Modules That are Finitely Generated. Journal of Algebraic Systems, 8(1), 1–5. https://doi.org/10.22044/jas.2019.8699.1421
26. Yalçın, N. F., & Kırğıl, Y. (2020). Identity graphs of finite cyclic groups. Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 22(1), 149-158. https://doi.org/10.25092/baunfbed.679578
27. Yaraneri, E. (2013). Intersection Graph of a Module. Journal of Algebra and Its Applications, 12(5). https://doi.org/10.1142/S0219488812502180

[1] Department of Mathematics, Sunan Kalijaga State Islamic University, Yogyakarta.
Email address: nisrinaaisy12@gmail.com

[2] Department of Mathematics, Sunan Kalijaga State Islamic University, Yogyakarta.
Email address: khurul.wardati@uin-suka.ac.id