Gulf Journal of Mathematics
Online First, accepted manuscript

ISSN: 2309-4966
https://doi.org/10.56947/g4cdze11

FINITE-DEGREE PROBABILISTIC ZERO BOUNDS FROM AN IMPROVED CAUCHY RADIUS

SAJAD A. SHEIKH[1], HILAL AHMAD KHANDAY[2] and ABDUL HASEEB GANIE[3*]

Abstract. We locate all zeros of a monic random polynomial of fixed degree whose non-leading coefficients are independent and uniform on the unit interval. Using the Joyal–Labelle–Rahman refinement of the Cauchy radius, we find the exact distribution of the resulting random radius and hence a closed-form lower bound for the probability that every zero lies in a disc of radius between one and two. A Vieta-type argument gives a complementary upper bound for radii below one. This confidence function strictly dominates the one given by the classical Cauchy radius. As the degree grows, it tends to zero for radii up to the golden ratio φ and stays positive beyond it, a threshold shared by every continuous coefficient law on the unit interval with support endpoints zero and one. We also bound the mean modulus of the outermost zero, find the limit of this bound, and compare all results with simulations.

Keywords. Random polynomials, probabilistic zero bounds, Joyal–Labelle–Rahman bound, order statistics, golden ratio
2020 Mathematics Subject Classification. Primary 30C15, 26C10; Secondary 60E15, 60G70

1. Introduction

Random polynomials connect probability with complex analysis. The classical questions concern the number of real zeros, the spatial distribution of the complex zeros, and how both depend on the law of the coefficients. The subject goes back to Bloch and Pólya [2] and to Littlewood and Offord [11]; Kac's formula [9] made the expected number of real zeros computable, and Hammersley [7] studied the complex zeros. Edelman and Kostlan [4] and Kostlan [10] recast part of the theory geometrically, and the monograph [1] together with the survey [20] records much of the classical material. Current work on local universality, on ensembles with an intermediate number of real zeros and on variance asymptotics continues to extend this picture [14, 12, 3].

Most of these results are asymptotic: one lets the degree n tend to infinity and studies empirical zero measures or real-zero statistics. In that regime it is well

Date: Received: Oct 13, 2026; Revised: Nov 7, 2026; Accepted: Nov 22, 2026.
* Corresponding author
© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

known that the zeros of polynomials with independent and identically distributed coefficients accumulate near the unit circle [7, 5, 19]. The question asked here is different and is posed at a *fixed* degree: given n and a radius u, how confident can one be that *all* zeros lie in the closed disc $\{|z| \le u\}$? Exact finite-n formulas for this global containment event are rare. In [17], probabilistic zero bounds were obtained for Gaussian coefficients by combining tail estimates with Cauchy's classical radius, and probabilistic versions of the Eneström–Kakeya theorem and of perturbative root estimates were developed in [18, 13]. In the present paper the same question is pursued for the bounded monic ensemble

$$P_n(z) = z^n + A_{n-1}z^{n-1} + \cdots + A_0, \qquad A_0, \dots, A_{n-1} \text{ independent}, \ A_j \sim \mathrm{U}(0,1). \tag{1.1}$$

The uniform law is bounded and non-Gaussian, and its order statistics can be written down exactly. This allows the containment probability produced by the Joyal–Labelle–Rahman refinement of the Cauchy radius [8] to be computed in closed form. Unlike the Cauchy radius, which uses only the largest coefficient modulus, the Joyal–Labelle–Rahman radius treats the subleading coefficient A_{n-1} separately and controls the remaining coefficients through their maximum. This separation is what produces a sharper confidence radius.

Contributions and significance. The paper makes the following contributions.

(i) *Exact law of the Joyal–Labelle–Rahman radius.* We compute the distribution function of the random radius H of (2.4) in closed form (Theorem 3.1). This gives an explicit lower confidence function $F_n(u)$ for $1 \le u \le 2$ and, by inversion, explicit probabilistic zero bounds at any prescribed confidence level (Corollary 3.3).

(ii) *Upper bounds below the unit circle.* Using all of Vieta's relations together with the independence of the coefficients, we prove an upper bound for the containment probability on $0 \le u \le 1$ that refines the bound u^n (Theorem 3.4).

(iii) *Comparison with the Cauchy radius.* The new confidence function strictly dominates the classical Cauchy confidence function $(u-1)^n$ throughout $(1, 2)$ (Theorem 4.1).

(iv) *A golden-ratio threshold and its universality.* As $n \to \infty$, $F_n(u) \to 0$ for $1 \le u \le \varphi = (1+\sqrt{5})/2$, whereas $F_n(u) \to u - 1/(u-1) > 0$ for $\varphi < u < 2$ (Theorem 4.2). We explain this threshold by an almost sure limit of the random radius (Proposition 4.3) and show that its location does not depend on the coefficient law, provided the law is supported in $[0, 1]$ with endpoints 0 and 1 (Theorem 4.5).

(v) *The outermost zero in mean.* We give an explicit upper bound for the expected modulus of the outermost zero (Theorem 5.1) and determine the exact large-degree limit $\varphi^2/2 + \log\varphi$ of the expected Joyal–Labelle–Rahman radius (Theorem 5.2).

All formulas are checked against Monte Carlo simulations in Section 6. The finite-degree point of view complements the asymptotic theory: universality and variance results describe typical behaviour for large degree [14, 12, 3], whereas the

results below give a certified confidence region containing *every* zero of a polynomial of given degree. They also show concretely how a sharper deterministic zero bound becomes a sharper probabilistic statement.

Organization. Section 2 fixes notation and collects the deterministic and probabilistic tools. Section 3 gives the exact law of the Joyal–Labelle–Rahman radius and the finite-degree confidence bounds. Section 4 compares the new bound with the Cauchy bound and studies the golden-ratio threshold. Section 5 treats the expected modulus of the outermost zero. Section 6 reports the numerical experiments, and Section 7 contains concluding remarks and open problems.

2. Preliminaries

Throughout, $n \geq 2$ (the case $n = 1$ is immediate), $P = P_n$ is the random polynomial (1.1), $\zeta_1, \dots, \zeta_n$ are its zeros counted with multiplicity, and

$$R := \max_{1 \leq k \leq n} |\zeta_k|$$

is the modulus of its outermost zero. We write $\varphi = (1 + \sqrt{5})/2$ for the golden ratio and recall that $\varphi^2 = \varphi + 1$ and $\varphi - 1 = 1/\varphi$.

Definition 2.1 (Probabilistic zero bound [17])**.** Let $\mathcal{P}_n$ be a family of random polynomials of degree n defined on a probability space. A number $c > 0$ is a *probabilistic zero bound at confidence level* p if

$$\mathbb{P}\big(\text{every zero of } P \text{ satisfies } |z| \leq c\big) \geq p.$$

Lemma 2.2 (Probabilistic transitivity [17])**.** *Let X and Y be random variables on a common probability space with $X \leq Y$ almost surely, and let F_Y be the distribution function of Y. Then $\mathbb{P}(X \leq c) \geq F_Y(c)$ for every $c \in \mathbb{R}$.*

Lemma 2.3 (Maximum of independent variables)**.** *Let $X_1, \dots, X_m$ be independent with common distribution function F, and let $Y = \max_{1 \leq j \leq m} X_j$. Then $\mathbb{P}(Y \leq y) = F(y)^m$ for all $y \in \mathbb{R}$. In particular, if $X_j \sim \mathrm{U}(0,1)$, then*

$$\mathbb{P}(Y \leq y) = y^m \quad (0 \leq y \leq 1) \qquad \text{and} \qquad \mathbb{E}Y = \frac{m}{m+1}.$$

Proof. By independence, $\mathbb{P}(Y \leq y) = \prod_{j=1}^{m} \mathbb{P}(X_j \leq y) = F(y)^m$. In the uniform case Y has density $m y^{m-1}$ on $[0,1]$, so $\mathbb{E}Y = \int_0^1 m y^m \, dy = m/(m+1)$; see also [6, 16]. □

Lemma 2.4 (Cauchy radius [15, Chapter 8])**.** *Every zero ζ of a monic polynomial $Q(z) = z^n + B_{n-1} z^{n-1} + \cdots + B_0$ satisfies $|\zeta| \leq 1 + \max_{0 \leq j \leq n-1} |B_j|$.*

Lemma 2.5 (Joyal–Labelle–Rahman radius [8])**.** *Every zero ζ of a monic polynomial $Q(z) = z^n + B_{n-1} z^{n-1} + \cdots + B_0$ satisfies*

$$|\zeta| \leq \frac{1}{2}\Big(1 + |B_{n-1}| + \sqrt{(1 - |B_{n-1}|)^2 + 4 \max_{0 \leq j \leq n-2} |B_j|}\Big).$$

For $(v,t) \in [0,1]^2$ put

$$h(v,t) := \frac{1}{2}\Big(1+v+\sqrt{(1-v)^2+4t}\Big). \tag{2.1}$$

The next lemma collects the elementary properties of h used repeatedly below.

Lemma 2.6. *The function h in (2.1) has the following properties.*

(a) *h is nondecreasing in each variable on $[0,1]^2$.*
(b) *$1 \le h(v,t) \le 1+\max\{v,t\}$ for all $(v,t) \in [0,1]^2$.*
(c) *For every $u \ge 1$ and $(v,t) \in [0,1]^2$,*

$$h(v,t) \le u \quad \Longleftrightarrow \quad t \le (u-1)(u-v). \tag{2.2}$$

Proof. (a) Monotonicity in t is obvious. Put $a = (1-v)/2 \in [0, \frac{1}{2}]$; then $h(v,t) = 1 - a + \sqrt{a^2+t}$. For $t > 0$ we have $\sqrt{a^2+t} - a = t/(\sqrt{a^2+t}+a)$, which is decreasing in a; for $t = 0$ the difference is 0. Since a decreases as v increases, h is nondecreasing in v.

(b) The lower bound follows from $\sqrt{a^2+t} \ge a$. For the upper bound let $M = \max\{v,t\}$. By (a),

$$h(v,t) \le h(M,M) = \tfrac{1}{2}\big(1+M+\sqrt{(1-M)^2+4M}\big) = \tfrac{1}{2}\big(1+M+(1+M)\big) = 1+M.$$

(c) We have $h(v,t) \le u$ if and only if $\sqrt{(1-v)^2+4t} \le 2u-1-v$. Since $v \le 1 \le u$, the right-hand side satisfies $2u-1-v \ge 2u-2 \ge 0$, so squaring is an equivalence. Thus the condition reads $(1-v)^2+4t \le (2u-1-v)^2$, and $(2u-1-v)^2 - (1-v)^2 = (2u-2)(2u-2v) = 4(u-1)(u-v)$. □

For the random polynomial (1.1) we define

$$V := A_{n-1}, \qquad T := \max_{0 \le j \le n-2} A_j, \tag{2.3}$$

and the *Joyal–Labelle–Rahman random radius*

$$H := h(V,T) = \frac{1}{2}\Big(1+V+\sqrt{(1-V)^2+4T}\Big). \tag{2.4}$$

The variables V and T are independent, $V \sim \mathrm{U}(0,1)$, and T is the maximum of $n-1$ independent $\mathrm{U}(0,1)$ variables. By Lemma 2.5, every zero of P satisfies $|\zeta| \le H$, so that $R \le H$ almost surely; by Lemma 2.6(b), $1 \le H \le 2$ almost surely.

3. Exact law of the random radius and finite-degree bounds

Theorem 3.1. *Let P be the random polynomial (1.1). For every $u \in [1,2]$,*

$$\mathbb{P}\big(\text{all zeros of } P \text{ lie in } |z| \le u\big) \;\ge\; F_n(u) := \mathbb{P}(H \le u),$$

where

$$F_n(u) = \begin{cases} \dfrac{(u-1)^{n-1}}{n}\big(u^n - (u-1)^n\big), & 1 \le u < \varphi, \\[2ex] \dfrac{u^2-u-1}{u-1} + \dfrac{1}{n}\Big(\dfrac{1}{u-1} - (u-1)^{2n-1}\Big), & \varphi \le u \le 2. \end{cases} \tag{3.1}$$

Proof. Since $R \le H$ almost surely, Lemma 2.2 gives $\mathbb{P}(R \le u) \ge \mathbb{P}(H \le u) = F_n(u)$. It remains to compute F_n. Fix $u \in [1,2]$ and put $x = u - 1 \in [0,1]$. By Lemma 2.6(c), $\{H \le u\} = \{T \le x(u - V)\}$. Conditioning on V and using independence and Lemma 2.3,

$$\begin{aligned} F_n(u) &= \int_0^1 \mathbb{P}\big(T \le x(u-v)\big)\, dv = \int_0^1 \min\big\{1, \big(x(u-v)\big)^{n-1}\big\}\, dv \\ &= \int_x^u \min\big\{1, x^{n-1}s^{n-1}\big\}\, ds, \end{aligned}$$

where we substituted $s = u - v$.

Case $1 \le u < \varphi$. Here $xu = u^2 - u < 1$, so $xs < 1$ for every $s \in [x,u]$ and

$$F_n(u) = x^{n-1}\int_x^u s^{n-1}\, ds = \frac{x^{n-1}}{n}\big(u^n - x^n\big).$$

Case $\varphi \le u \le 2$. Now $xu \ge 1$ and $x^2 \le 1$, so $1/x \in [x,u]$ and the integrand equals $x^{n-1}s^{n-1}$ on $[x, 1/x]$ and 1 on $[1/x, u]$. Hence

$$F_n(u) = \frac{x^{n-1}}{n}\big(x^{-n} - x^n\big) + u - \frac{1}{x} = \frac{1}{n}\Big(\frac{1}{x} - x^{2n-1}\Big) + \frac{u^2 - u - 1}{u - 1}.$$

This is (3.1). □

Remark 3.2. (a) Both expressions in (3.1) equal $\frac{1}{n}\big(\varphi - \varphi^{-(2n-1)}\big)$ at $u = \varphi$, so F_n is continuous; moreover $F_n(1) = 0$ and $F_n(2) = 1$. Being the distribution function of H, F_n is nondecreasing; it is in fact strictly increasing on $[1,2]$, since its derivative $\int_0^1 (n-1)\, x^{n-2}(u-v)^{n-2}(2u - 1 - v)\, \mathbf{1}\{x(u-v) < 1\}\, dv$ (with $x = u - 1$) is positive for $1 < u < 2$.

(b) Since F_n is the *exact* distribution function of H, it is the best lower confidence function that can be obtained from the Joyal–Labelle–Rahman radius alone. Any further improvement must use more information about P than the moduli $|A_{n-1}|$ and $\max_{j \le n-2} |A_j|$.

Inverting F_n gives explicit probabilistic zero bounds in the sense of Definition 2.1.

Corollary 3.3. *For $p \in (0,1]$ let $u_n(p) \in (1,2]$ be the unique solution of $F_n(u) = p$. Then $u_n(p)$ is a probabilistic zero bound for P_n at confidence level p, and $u_n(p) \le 1 + p^{1/n}$, the corresponding radius obtained from the Cauchy bound; the inequality is strict for $0 < p < 1$.*

Proof. The first statement is Theorem 3.1. The Cauchy radius gives confidence $G_n(u) = (u-1)^n$, and $G_n(1 + p^{1/n}) = p$. By Theorem 4.1 below, $F_n(1 + p^{1/n}) > p$ when $0 < p < 1$, so the strict monotonicity of F_n gives $u_n(p) < 1 + p^{1/n}$. For $p = 1$ both radii equal 2. □

We now turn to radii below the unit circle, where the question is how *unlikely* it is that all zeros are small.

Theorem 3.4. *Let P be the random polynomial (1.1). For every $u \in [0,1]$,*

$$\mathbb{P}\big(\text{all zeros of } P \text{ lie in } |z| \le u\big) \;\le\; \prod_{k=1}^{n} \min\Big\{1, \binom{n}{k} u^k\Big\} \;\le\; u^n. \tag{3.2}$$

The middle bound is strictly smaller than u^n whenever $0 < u < n^{-1/(n-1)}$.

Proof. Let e_k denote the kth elementary symmetric function of $\zeta_1, \dots, \zeta_n$. By Vieta's formulas, $A_{n-k} = (-1)^k e_k(\zeta_1, \dots, \zeta_n)$ for $1 \le k \le n$. Since e_k is a sum of $\binom{n}{k}$ products of k zeros, each of modulus at most R^k,

$$0 \le A_{n-k} = |e_k| \le \binom{n}{k} R^k \qquad (1 \le k \le n).$$

Hence the event $\{R \le u\}$ is contained in $\bigcap_{k=1}^{n} \{A_{n-k} \le \binom{n}{k} u^k\}$. The coefficients are independent and $\mathbb{P}(A_j \le c) = \min\{1, c\}$ for $c \ge 0$, which gives the first inequality in (3.2). The second inequality follows because the factor with $k = n$ equals u^n (as $u \le 1$) and the other factors are at most 1. Finally, the factor with $k = n-1$ equals $\min\{1, nu^{n-1}\}$, which is smaller than 1 precisely when $u < n^{-1/(n-1)}$; for $u > 0$ this makes the product strictly smaller than u^n. □

Remark 3.5. The two estimates play opposite roles. For $u \in [1,2]$, Theorem 3.1 guarantees a minimum probability that all zeros lie in $\{|z| \le u\}$; for $u \in [0,1]$, Theorem 3.4 caps that probability. Together they bracket the localization problem over the whole range $0 \le u \le 2$. The inequality $A_0 \le R^n$ alone already shows that the zeros cannot all be packed into a small disc with high probability; using the remaining coefficients sharpens this for small radii.

4. Comparison with the Cauchy radius and the golden-ratio threshold

Theorem 4.1. *Let $G_n(u) := (u-1)^n$, $1 \le u \le 2$, be the confidence function obtained from the Cauchy radius (Lemma 2.4). Then $F_n(u) \ge G_n(u)$ for all $u \in [1,2]$, with strict inequality for every $u \in (1,2)$.*

Proof. Let $M := \max_{0 \le j \le n-1} A_j = \max\{V, T\}$. By Lemma 2.6(b), $H \le 1 + M$, so $\{1 + M \le u\} \subseteq \{H \le u\}$ and, by Lemma 2.3,

$$F_n(u) = \mathbb{P}(H \le u) \ge \mathbb{P}(M \le u - 1) = (u-1)^n = G_n(u).$$

For strictness fix $u \in (1,2)$ and consider

$$E_u := \big\{u - 1 < V < 1,\; T \le (u-1)(u-V)\big\}.$$

For $v \in (u-1, 1)$ we have $0 < (u-1)(u-v) < (u-1) \cdot 1 < 1$, so by independence and Lemma 2.3,

$$\mathbb{P}(E_u) = \int_{u-1}^{1} \big((u-1)(u-v)\big)^{n-1}\, dv > 0.$$

On E_u we have $H \le u$ by Lemma 2.6(c), while $1 + M \ge 1 + V > u$. Thus $E_u \subseteq \{H \le u\} \setminus \{1 + M \le u\}$, and therefore $F_n(u) \ge G_n(u) + \mathbb{P}(E_u) > G_n(u)$. □

Theorem 4.2. *For every fixed $u \in [1,2]$,*

$$\lim_{n\to\infty} F_n(u) = \begin{cases} 0, & 1 \le u \le \varphi, \\ u - \dfrac{1}{u-1}, & \varphi < u \le 2. \end{cases}$$

In particular, the limit is strictly positive exactly for $u \in (\varphi, 2]$, and it equals 1 at $u = 2$.

Proof. Put $x = u - 1$. If $1 \le u < \varphi$, then $xu < 1$ and, by (3.1), $0 \le F_n(u) \le x^{n-1}u^n/n = u\,(xu)^{n-1}/n \to 0$. If $u = \varphi$, then $F_n(\varphi) = \frac{1}{n}\big(\varphi - \varphi^{-(2n-1)}\big) \to 0$. If $\varphi < u \le 2$, then $0 < x \le 1$, so $x^{2n-1}/n \to 0$ and $1/(nx) \to 0$; hence $F_n(u) \to (u^2-u-1)/(u-1) = u-1/(u-1)$, which is positive because $u^2-u-1 > 0$ for $u > \varphi$, and equals 1 at $u = 2$. □

The threshold has a transparent probabilistic origin, described next.

Proposition 4.3. *Let $U, U_1, U_2, \ldots$ be independent $\mathrm{U}(0,1)$ variables, $M_m := \max_{1\le j\le m} U_j$, and $H'_n := h(U, M_{n-1})$. Then:*

(a) *H'_n has the same law as H in (2.4), for every $n \ge 2$;*
(b) *H'_n is nondecreasing in n and $H'_n \to H_\infty := h(U,1)$ almost surely; consequently $F_{n+1}(u) \le F_n(u)$ for all u and n;*
(c) *H_∞ takes values in $[\varphi, 2]$ and $\mathbb{P}(H_\infty \le u) = u - 1/(u-1)$ for $\varphi \le u \le 2$.*

Proof. (a) (U, M_{n-1}) and (V, T) have the same joint law. (b) M_m is nondecreasing in m and $M_m \to 1$ almost surely, because $\mathbb{P}(M_m \le 1-\varepsilon) = (1-\varepsilon)^m \to 0$ for every $\varepsilon \in (0,1)$. The assertion follows from Lemma 2.6(a) and the continuity of h. Since $H'_{n+1} \ge H'_n$, we get $F_{n+1}(u) = \mathbb{P}(H'_{n+1} \le u) \le \mathbb{P}(H'_n \le u) = F_n(u)$. (c) By Lemma 2.6(a), $h(0,1) \le h(U,1) \le h(1,1)$, and $h(0,1) = (1+\sqrt{5})/2 = \varphi$, $h(1,1) = 2$. By Lemma 2.6(c), $h(U,1) \le u$ if and only if $(u-1)(u-U) \ge 1$, that is, $U \le u - 1/(u-1)$, and this number lies in $[0,1]$ for $u \in [\varphi, 2]$. □

Remark 4.4. Proposition 4.3 explains Theorem 4.2. As n grows, the largest of the lower coefficients tends to 1, so the Joyal–Labelle–Rahman radius stabilizes at $h(V,1)$, which can never be smaller than $h(0,1) = \varphi$. Deterministically, φ is the Joyal–Labelle–Rahman radius of $z^n + z^{n-2} + z^{n-3} + \cdots + 1$, the case in which the subleading coefficient vanishes and all other coefficients are extremal. Hence the golden ratio is already encoded in the finite-degree geometry of the bound; no asymptotic zero-distribution law is involved. Because $H'_n \uparrow H_\infty$, the events $\{H'_n \le u\}$ decrease to $\{H_\infty \le u\}$, so continuity of probability gives $F_n(u) \downarrow \mathbb{P}(H_\infty \le u)$; this is a second proof of Theorem 4.2.

The location of the threshold does not depend on the uniform law. Let the coefficients in (1.1) be independent with a common distribution function F on $[0,1]$, and let $F_n^F(u) := \mathbb{P}(H \le u)$, with H still defined by (2.3)–(2.4).

Theorem 4.5. *Assume that $F(0) = 0$, $F(1) = 1$, $F(t) > 0$ for $t > 0$ and $F(t) < 1$ for $t < 1$; that is, the law has no atom at 0 and the endpoints of its*

support are 0 *and* 1. *Then for every* $u \in [1,2]$,

$$\mathbb{P}\big(\textit{all zeros of } P \textit{ lie in } |z| \le u\big) \;\ge\; F_n^F(u) = \int_{[0,1]} F\big((u-1)(u-v)\big)^{n-1}\, dF(v), \tag{4.1}$$

where $F(s) := 1$ *for* $s \ge 1$, *and*

$$\lim_{n\to\infty} F_n^F(u) = \begin{cases} 0, & 1 \le u \le \varphi, \\ F\Big(u - \dfrac{1}{u-1}\Big) > 0, & \varphi < u \le 2. \end{cases}$$

Proof. The inequality and (4.1) follow as in the proof of Theorem 3.1, using Lemma 2.6(c), the independence of V and T, and the identity $\mathbb{P}(T \le s) = F(s)^{n-1}$ from Lemma 2.3. Fix u, put $x = u-1$, and let $g_n(v) := F\big(x(u-v)\big)^{n-1}$ for $v \in [0,1]$. If $x(u-v) \ge 1$, then $g_n(v) = 1$ for all n; if $x(u-v) < 1$, then $F(x(u-v)) < 1$ and $g_n(v) \to 0$. Since $0 \le g_n \le 1$, dominated convergence gives

$$\lim_{n\to\infty} F_n^F(u) = \mathbb{P}\big(x(u-V) \ge 1\big).$$

If $1 \le u < \varphi$, then $x(u-V) \le xu < 1$, and the limit is 0. If $u = \varphi$, then $x(u-V) \ge 1$ forces $V = 0$, an event of probability $F(0) = 0$. If $\varphi < u \le 2$, then $x(u-V) \ge 1$ if and only if $V \le c(u) := u - 1/(u-1)$, where $c(u) \in (0,1]$, so the limit is $F(c(u))$, which is positive because $c(u) > 0$. □

Remark 4.6. For the uniform law, Theorem 4.5 reduces to Theorem 4.2. The *location* φ of the transition depends only on the endpoints of the support, while the limiting confidence profile $F(u - 1/(u-1))$ on $(\varphi, 2]$ carries the shape of the coefficient law.

5. The expected modulus of the outermost zero

Theorem 5.1. *For every* $n \ge 2$,

$$\mathbb{E}R \;\le\; \mathbb{E}H \;\le\; J_n := \frac{3}{4} + \frac{1}{2}\sqrt{\frac{1}{3} + \frac{4(n-1)}{n}} = \frac{3}{4} + \frac{1}{2}\sqrt{\frac{13}{3} - \frac{4}{n}}.$$

Proof. Since $R \le H$ almost surely, $\mathbb{E}R \le \mathbb{E}H$. By (2.4),

$$\mathbb{E}H = \frac{1}{2}\Big(1 + \mathbb{E}V + \mathbb{E}\sqrt{(1-V)^2 + 4T}\Big).$$

The square-root function is concave on $[0,\infty)$, so Jensen's inequality [6] gives

$$\mathbb{E}\sqrt{(1-V)^2 + 4T} \le \sqrt{\mathbb{E}(1-V)^2 + 4\,\mathbb{E}T}.$$

Since $V \sim \mathrm{U}(0,1)$, we have $\mathbb{E}V = \frac{1}{2}$ and $\mathbb{E}(1-V)^2 = \int_0^1 (1-v)^2\, dv = \frac{1}{3}$. Lemma 2.3, applied to T, the largest of the $n-1$ independent uniform coefficients $A_0, \dots, A_{n-2}$, yields $\mathbb{E}T = (n-1)/n$. Therefore

$$\mathbb{E}H \le \frac{1}{2}\Big(1 + \frac{1}{2} + \sqrt{\frac{1}{3} + \frac{4(n-1)}{n}}\Big) = J_n,$$

which is the assertion. □

Theorem 5.2. *The sequence $\mathbb{E}H = \mathbb{E}H^{(n)}$ is nondecreasing in n, and*

$$\lim_{n\to\infty} \mathbb{E}H^{(n)} = \frac{\varphi^2}{2} + \log\varphi \approx 1.790229.$$

Consequently $\mathbb{E}R \le \min\{J_n,\ \varphi^2/2 + \log\varphi\}$ for every $n \ge 2$.

Proof. By Proposition 4.3, $\mathbb{E}H^{(n)} = \mathbb{E}H'_n$, where H'_n is nondecreasing in n, bounded by 2, and converges almost surely to $H_\infty = h(U,1)$. By monotone convergence, $\mathbb{E}H^{(n)} \uparrow \mathbb{E}H_\infty$. With $w = 1-U \sim \mathrm{U}(0,1)$,

$$\mathbb{E}H_\infty = \frac{1}{2}\Big(1 + \frac{1}{2} + \int_0^1 \sqrt{w^2+4}\,dw\Big).$$

Since $\frac{d}{dw}\big[\frac{w}{2}\sqrt{w^2+4} + 2\log\big(w+\sqrt{w^2+4}\big)\big] = \sqrt{w^2+4}$, we obtain

$$\int_0^1 \sqrt{w^2+4}\,dw = \frac{\sqrt{5}}{2} + 2\log\frac{1+\sqrt{5}}{2} = \frac{\sqrt{5}}{2} + 2\log\varphi.$$

Hence $\mathbb{E}H_\infty = \frac{3}{4} + \frac{\sqrt{5}}{4} + \log\varphi = \frac{3+\sqrt{5}}{4} + \log\varphi = \frac{\varphi^2}{2} + \log\varphi$, because $\varphi^2 = (3+\sqrt{5})/2$. The last statement follows from Theorem 5.1 and $\mathbb{E}R \le \mathbb{E}H^{(n)} \le \mathbb{E}H_\infty$. □

Remark 5.3. (a) The Jensen bound is almost sharp for large n: $\lim_n J_n = \frac{3}{4} + \frac{1}{2}\sqrt{13/3} \approx 1.790833$ exceeds the exact limit $\varphi^2/2 + \log\varphi$ by less than 6.1×10^{-4}. A direct computation shows that $J_n \le \varphi^2/2 + \log\varphi$ exactly when $n \le 795$, so the degree-free bound of Theorem 5.2 is the better one for $n \ge 796$.

(b) For each n the exact value of $\mathbb{E}H$ is available from the distribution function (3.1): since $1 \le H \le 2$,

$$\mathbb{E}H = \int_0^2 \mathbb{P}(H > u)\,du = 1 + \int_1^2 \big(1 - F_n(u)\big)\,du,$$

which can be evaluated numerically to any accuracy; see Table 3.

6. Numerical experiments and discussion

The simulations are intended to check scale and shape; they do not replace the proofs. For each degree we generated independent samples of (1.1) (with 4×10^5 polynomials for $n = 5$ and 2×10^5 for $n = 10, 20$), computed all zeros as eigenvalues of the companion matrix, and recorded both the outermost modulus R and the radius H of (2.4). As a sanity check, the inequality $R \le H$ of Lemma 2.5 held in every one of the 4×10^5 samples for $n = 5$. The figures and tables below were all produced by a single script with a fixed random seed.

Lower confidence bounds. Figure 1 shows F_5 from Theorem 3.1 together with the Cauchy function G_5; the shaded gap is the strict improvement of Theorem 4.1. Figure 2 shows F_n for several degrees and the limit of Theorem 4.2. The curves decrease in n (Proposition 4.3(b)), collapse to zero to the left of φ, and approach the positive profile $u - 1/(u-1)$ to the right of it. Table 1 compares G_n, F_n, the empirical distribution function of H, and the empirical containment probability $\mathbb{P}(R \le u)$.

The column for $\mathbb{P}(H \le u)$ agrees with $F_n(u)$ to within Monte Carlo error. This confirms that (3.1) is the exact law of the Joyal–Labelle–Rahman radius. The true containment probability is much larger (Figure 3). The reason is that H depends only on the *moduli* of the coefficients, whereas for the present ensemble the coefficients are positive and the zeros concentrate near the unit circle [5, 19]. Closing this gap requires deterministic bounds that are sensitive to the signs or the monotonicity of the coefficients, such as Eneström–Kakeya-type bounds [18]; this is a natural continuation of the present work. Table 2 lists the confidence radii $u_n(p)$ of Corollary 3.3 next to the Cauchy radii $1 + p^{1/n}$.

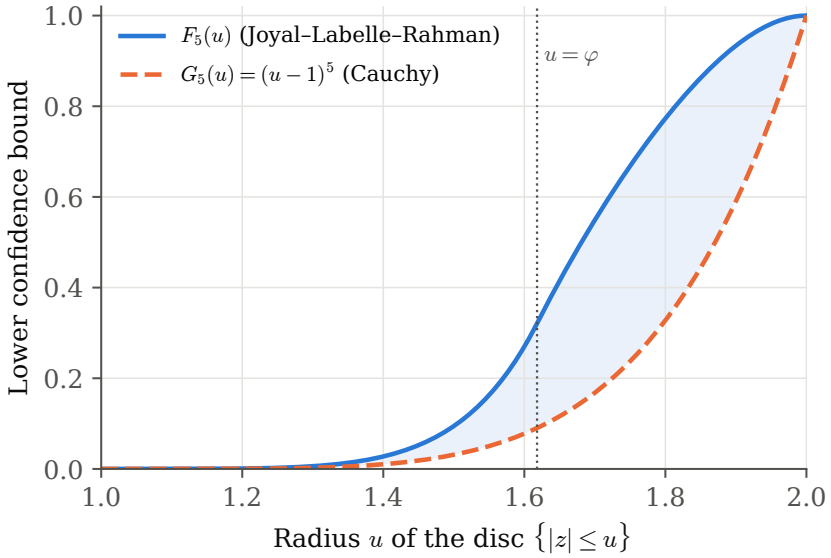

Fig. 1. Lower confidence functions for the event that all zeros of a degree-5 polynomial (1.1) lie in $\{|z| \le u\}$, $1 \le u \le 2$. Solid: $F_5(u)$ from Theorem 3.1, built on the Joyal–Labelle–Rahman radius. Dashed: $G_5(u) = (u-1)^5$, built on the classical Cauchy radius. The shaded region is the gain $F_5 - G_5 > 0$ of Theorem 4.1. The dotted vertical line marks $u = \varphi$, where the formula (3.1) changes branch.

Upper bounds below the unit circle. Figure 4 compares the two upper bounds of Theorem 3.4 with the simulated containment probability for $n = 5$. The bound that uses all coefficients improves on u^5 for $u < 5^{-1/4} \approx 0.669$, and both bounds dominate the simulated curve, as they must. For larger u the two bounds coincide, because only the constant term carries information there.

The outermost zero in mean. Figure 5 and Table 3 compare the Jensen bound J_n of Theorem 5.1, the exact value of $\mathbb{E}H$ from Remark 5.3(b), the limit $\varphi^2/2 + \log\varphi$ of Theorem 5.2, and Monte Carlo estimates of $\mathbb{E}R$. The Jensen bound is within 0.04 of $\mathbb{E}H$ for $n = 2$ and within 10^{-3} for $n \ge 20$, so it captures the mean of the Joyal–Labelle–Rahman radius essentially exactly. The remaining

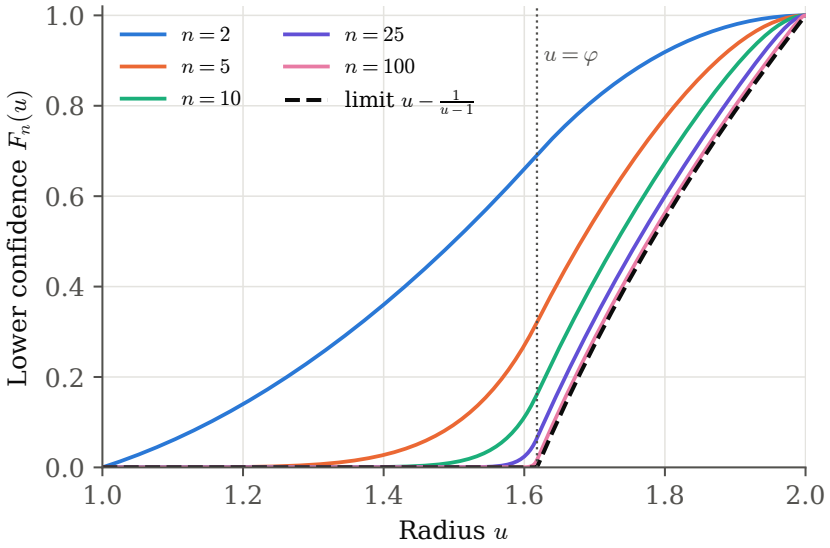

Fig. 2. The lower confidence function $F_n(u)$ of Theorem 3.1 for $n = 2, 5, 10, 25, 100$, together with its limit (dashed), which is 0 on $[1, \varphi]$ and $u - 1/(u-1)$ on $(\varphi, 2]$ (Theorem 4.2). For fixed u the curves decrease in n (Proposition 4.3). The dotted vertical line marks the golden-ratio threshold $u = \varphi$.

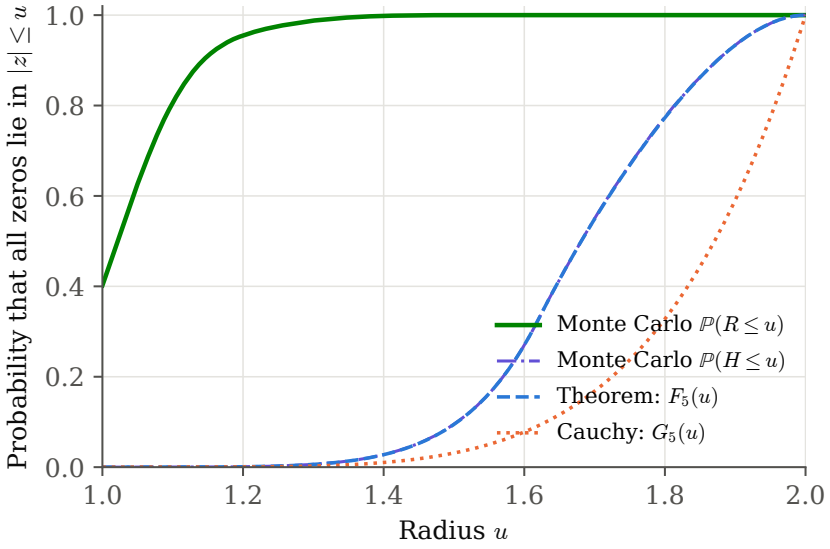

Fig. 3. Monte Carlo check of the lower bound for $n = 5$ (4×10^5 samples). Solid: empirical probability $\mathbb{P}(R \le u)$ that all zeros lie in $\{|z| \le u\}$. Dash-dotted: empirical distribution function of the Joyal–Labelle–Rahman radius H; it coincides with the exact formula $F_5(u)$ of Theorem 3.1 (dashed). Dotted: Cauchy confidence function $G_5(u)$. All theoretical curves lie below the empirical containment probability, as they must.

Table 1. Lower confidence functions versus simulation. G_n: Cauchy confidence function; F_n: Theorem 3.1; $\widehat{\mathbb{P}}(H \le u)$ and $\widehat{\mathbb{P}}(R \le u)$: Monte Carlo estimates (4×10^5 samples for $n = 5$, 2×10^5 for $n = 10, 20$).

n	u	$G_n(u)$	$F_n(u)$	$\widehat{\mathbb{P}}(H \le u)$	$\widehat{\mathbb{P}}(R \le u)$
5	1.2	0.0003	0.0008	0.0008	0.9551
	1.4	0.0102	0.0275	0.0276	0.9984
	φ	0.0902	0.3210	0.3204	1.0000
	1.7	0.1681	0.5491	0.5503	1.0000
	1.8	0.3277	0.7732	0.7744	1.0000
	1.9	0.5905	0.9336	0.9335	1.0000
10	1.2	0.0000	0.0000	0.0000	0.9642
	1.4	0.0001	0.0008	0.0008	0.9990
	φ	0.0081	0.1618	0.1607	1.0000
	1.7	0.0282	0.4142	0.4136	1.0000
	1.8	0.1074	0.6736	0.6747	1.0000
	1.9	0.3487	0.8865	0.8870	1.0000
20	1.2	0.0000	0.0000	0.0000	0.9549
	1.4	0.0000	0.0000	0.0000	0.9988
	φ	0.0001	0.0809	0.0817	1.0000
	1.7	0.0008	0.3429	0.3437	1.0000
	1.8	0.0115	0.6125	0.6110	1.0000
	1.9	0.1216	0.8436	0.8421	1.0000

Table 2. Probabilistic zero bounds at confidence level p: the radius $u_n(p)$ of Corollary 3.3 (Joyal–Labelle–Rahman) and the radius $1 + p^{1/n}$ obtained from the Cauchy bound. The last row gives the limit $\lim_n u_n(p)$, the solution of $u - 1/(u-1) = p$.

	$p = 0.90$		$p = 0.95$		$p = 0.99$	
n	$u_n(p)$	$1 + p^{1/n}$	$u_n(p)$	$1 + p^{1/n}$	$u_n(p)$	$1 + p^{1/n}$
5	1.8746	1.9791	1.9143	1.9898	1.9634	1.9980
10	1.9075	1.9895	1.9388	1.9949	1.9749	1.9990
20	1.9272	1.9947	1.9539	1.9974	1.9820	1.9995
50	1.9412	1.9979	1.9656	1.9990	1.9880	1.9998
100	1.9462	1.9989	1.9703	1.9995	1.9908	1.9999
∞	1.9512	2	1.9753	2	1.9950	2

gap to $\mathbb{E}R$ has the same source as the gap in Figure 3: the deterministic radius ignores the signs of the coefficients.

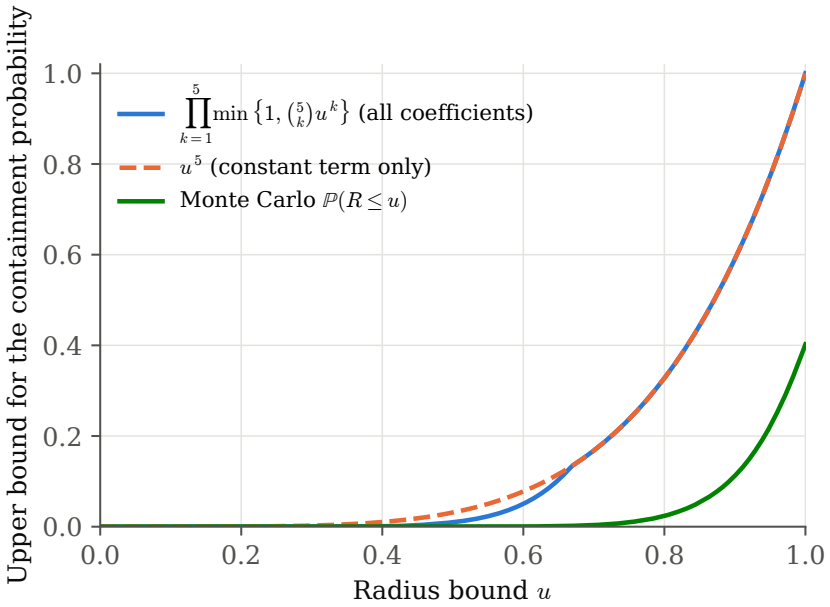

Fig. 4. Upper bounds for the probability that all zeros of a degree-5 polynomial (1.1) lie in $\{|z| \le u\}$, $0 \le u \le 1$ (Theorem 3.4). Dashed: u^5, from the constant term alone. Solid blue: $\prod_{k=1}^{5} \min\{1, \binom{5}{k} u^k\}$, from all coefficients; it is strictly smaller than u^5 for $0 < u < 5^{-1/4}$ and equal to it afterwards. Green: Monte Carlo estimate of $\mathbb{P}(R \le u)$ (4×10^5 samples).

Table 3. Expected modulus of the outermost zero: Jensen bound J_n (Theorem 5.1), exact $\mathbb{E}H$ (Remark 5.3), and Monte Carlo estimate of $\mathbb{E}R$ with two standard errors (4×10^4 samples per degree). The limit of $\mathbb{E}H$ is $\varphi^2/2 + \log\varphi \approx 1.7902$.

n	J_n	$\mathbb{E}H$	$\widehat{\mathbb{E}}R$
2	1.5138	1.4822	0.6972 ± 0.0020
5	1.6899	1.6843	1.0214 ± 0.0010
10	1.7416	1.7398	1.0788 ± 0.0006
20	1.7665	1.7656	1.1027 ± 0.0006
50	1.7812	1.7805	1.1081 ± 0.0004
100	1.7860	1.7854	1.1084 ± 0.0004

7. Conclusion and future work

We have studied a fixed-degree ensemble of monic polynomials whose non-leading coefficients are independent and uniform on $(0, 1)$. Combining the Joyal–Labelle–Rahman refinement of Cauchy's radius with the exact order statistics of the coefficients gives the exact distribution of the refined random radius. This yields a closed-form lower confidence function on $[1, 2]$ and explicit probabilistic zero bounds at every confidence level. All of Vieta's relations together give

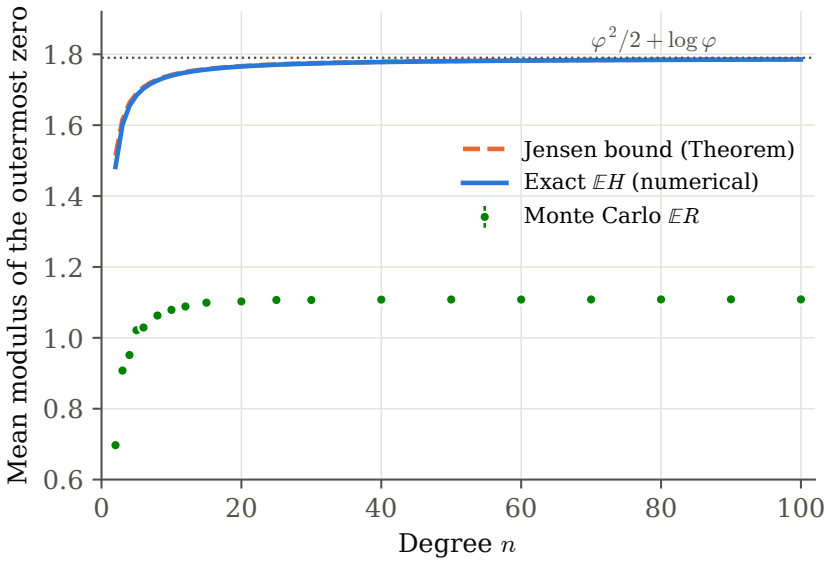

Fig. 5. Mean modulus of the outermost zero as a function of the degree n. Dashed: the Jensen bound J_n of Theorem 5.1. Solid: the exact value $\mathbb{E}H = 1 + \int_1^2 (1 - F_n(u))\,du$ (Remark 5.3). Dotted horizontal line: the limit $\varphi^2/2 + \log\varphi$ (Theorem 5.2). Markers: Monte Carlo estimates of $\mathbb{E}R$ (4×10^4 samples per degree; error bars of two standard errors are smaller than the markers).

an upper bound below the unit circle, and the mean modulus of the outermost zero admits both an explicit degree-dependent bound and a degree-free bound $\varphi^2/2 + \log\varphi$ that is attained in the limit by the refined radius. Two structural facts give these formulas their meaning. Over the whole interval $(1, 2)$ the refined radius gives strictly more confidence than the classical Cauchy radius, so sharpening a deterministic bound produces a genuine, not cosmetic, probabilistic improvement. The large-degree behaviour of the confidence function changes at the golden ratio, which is the Joyal–Labelle–Rahman radius of an extremal configuration; the location of this transition depends only on the endpoints of the support of the coefficient law, not on its shape.

The computations do not use special features of the uniform law beyond the tractability of its order statistics, so other bounded laws, and more generally laws for which the auxiliary random radius remains analysable, are natural next cases; Theorem 4.5 is a first step in that direction. The simulations show that the containment probability is considerably larger than any bound that depends only on coefficient moduli, which points towards probabilistic versions of sign-sensitive bounds of Eneström–Kakeya type. Further open problems are probabilistic lower bounds for the smallest zero modulus via reciprocal polynomials, annular localization, and bounds for the probability that a prescribed number of zeros lies in a given disc, which would move the theory from the localization of all zeros to partial zero counting.

Acknowledgement. The authors thank the editor and the anonymous reviewer for their careful reading and constructive comments, which improved the paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Data availability. No external data were used. The Python script that generates all figures and tables is available from the corresponding author upon reasonable request.

References

[1] Bharucha-Reid, A. T., & Sambandham, M. (1986). *Random polynomials*. Academic Press.

[2] Bloch, A., & Pólya, G. (1932). On the roots of certain algebraic equations. *Proceedings of the London Mathematical Society*, *s2-33*(1), 102–114. https://doi.org/10.1112/plms/s2-33.1.102

[3] Do, Y. Q., & Nguyen, N. D. V. (2025). Real roots of random polynomials: Asymptotics of the variance. *Electronic Journal of Probability*, *30*, 1–37. https://doi.org/10.1214/25-EJP1375

[4] Edelman, A., & Kostlan, E. (1995). How many zeros of a random polynomial are real? *Bulletin of the American Mathematical Society*, *32*(1), 1–37. https://doi.org/10.1090/S0273-0979-1995-00571-9

[5] Erdős, P., & Turán, P. (1950). On the distribution of roots of polynomials. *Annals of Mathematics*, *51*(1), 105–119. https://doi.org/10.2307/1969500

[6] Feller, W. (1968). *An introduction to probability theory and its applications* (3rd ed., Vol. 1). Wiley.

[7] Hammersley, J. M. (1956). The zeros of a random polynomial. In J. Neyman (Ed.), *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability* (Vol. 2, pp. 89–111). University of California Press.

[8] Joyal, A., Labelle, G., & Rahman, Q. I. (1967). On the location of zeros of polynomials. *Canadian Mathematical Bulletin*, *10*(1), 53–63. https://doi.org/10.4153/CMB-1967-006-3

[9] Kac, M. (1943). On the average number of real roots of a random algebraic equation. *Bulletin of the American Mathematical Society*, *49*(4), 314–320. https://doi.org/10.1090/S0002-9904-1943-07912-8

[10] Kostlan, E. (1993). On the distribution of roots of random polynomials. In M. W. Hirsch, J. E. Marsden, & M. Shub (Eds.), *From topology to computation: Proceedings of the Smalefest* (pp. 419–431). Springer. https://doi.org/10.1007/978-1-4612-2740-3_38

[11] Littlewood, J. E., & Offord, A. C. (1938). On the number of real roots of a random algebraic equation. *Journal of the London Mathematical Society*, *s1-13*(4), 288–295. https://doi.org/10.1112/jlms/s1-13.4.288

[12] Michelen, M., & O'Rourke, S. (2024). On random polynomials with an intermediate number of real roots. *Proceedings of the American Mathematical Society*, *152*(11), 4933–4942. https://doi.org/10.1090/proc/16999

[13] Nafisah, I. A., Sheikh, S. A., Alshahrani, M. A., Almazah, M. M. A., Alnssyan, B., & Dar, J. G. (2024). Perturbation approach to polynomial root estimation and expected maximum modulus of zeros with uniform perturbations. *Mathematics*, *12*(19), 2993. https://doi.org/10.3390/math12192993

[14] Nguyen, O., & Vu, V. (2022). Roots of random functions: A framework for local universality. *American Journal of Mathematics*, *144*(1), 1–74. https://doi.org/10.1353/ajm.2022.0000

[15] Rahman, Q. I., & Schmeisser, G. (2002). *Analytic theory of polynomials*. Oxford University Press. https://doi.org/10.1093/oso/9780198534938.001.0001

[16] Ross, S. M. (2014). *Introduction to probability models* (11th ed.). Academic Press.
[17] Sheikh, S. A., & Mir, M. I. (2024). Probabilistic zero bounds of certain random polynomials. *Journal of Applied Mathematics, Statistics and Informatics*, *20*(1), 53–66. https://doi.org/10.2478/jamsi-2024-0004
[18] Sheikh, S. A., Mir, M. I., Dar, J. G., Almanjahie, I. M., & Alshahrani, F. (2023). A probabilistic version of Eneström–Kakeya theorem for certain random polynomials. *Mathematics*, *11*(19), 4061. https://doi.org/10.3390/math11194061
[19] Shepp, L. A., & Vanderbei, R. J. (1995). The complex zeros of random polynomials. *Transactions of the American Mathematical Society*, *347*(11), 4365–4384. https://doi.org/10.1090/S0002-9947-1995-1308023-8
[20] Thangaraj, V., & Sambandham, M. (2021). On random polynomials–II: A survey. *Neural, Parallel, and Scientific Computations*, *29*(4), 230–251. https://doi.org/10.46719/npsc20212944

[1] Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India.
Email address: sajadsheikh@uok.edu.in

[2] Department of Computer Science, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India.
Email address: hilalhyder@gmail.com

[3] Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur 303007, Rajasthan, India.
Email address: abdul.ganie@jaipur.manipal.edu; ahg110605@gmail.com