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REACTION–DIFFUSION ON INFINITE GRAPHS: STABILITY AND BLOW-UP

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Abstract. We consider bounded initial data on finite and countably infinite graphs with uniformly bounded vertex degrees. Using Dini derivatives and almost-extremizing vertices, we establish comparison principles without assuming spatial extrema are attained. We obtain positivity, invariant intervals, uniform absorption, and convergence under a strict restoring condition. For superlinear reactions, scalar comparisons give blow-up-time bounds without boundary constraints, while a first-eigenfunction argument yields a Dirichlet blow-up criterion on finite graphs. A Lyapunov identity describes bounded trajectories on finite graphs. Numerical experiments illustrate dissipative absorption and blow-up.

Keywords. reaction–diffusion equations; infinite graphs; Dini derivatives; discrete maximum principle; finite-time blow-up.
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1. Introduction

Reaction-diffusion equations describe the evolution of a quantity subject simultaneously to a local reaction and spatial transport. On a continuous domain $\Omega \subset \mathbb{R}^d$, they take the classical form

$$\partial_t u = D\Delta u + f(u),$$

a framework treated extensively, for example, by Evans [6], Henry [9], Rothe [14], Quittner and Souplet [13], and Temam [15].

In many applications, the underlying geometry is not continuous and is more naturally represented by a network, as in communication systems, biological interactions, and contact structures in epidemiology. Values are attached to vertices, while edges describe the interactions. Replacing the continuous Laplacian by the graph Laplacian Δ_G leads to the Cauchy problem

$$\frac{\mathrm{d}u_i}{\mathrm{d}t}(t) = D(\Delta_G u(t))_i + f(u_i(t)), \qquad i \in V, \qquad u(0) = u^0 \in \ell^\infty(V). \tag{P}$$

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We take $D > 0$ and a locally Lipschitz reaction $f : \mathbb{R} \to \mathbb{R}$. Unless otherwise stated, $G = (V, E)$ denotes a finite or countably infinite, locally finite, undirected graph with uniformly bounded degree. For $i \in V$, let $N(i)$ be its neighborhood and let $d_i = \#N(i)$. The discrete Laplacian is

$$(\Delta_G u)_i = \sum_{j \in N(i)} (u_j - u_i),$$

and the evolution is studied in the uniform space $\ell^\infty(V)$. The assumption $d^* = \sup_{i \in V} d_i < +\infty$ ensures that Δ_G is a bounded operator on $\ell^\infty(V)$.

On infinite graphs, spatial extrema need not be attained and bounded sets in the uniform norm are not compact. The spectral theory of the discrete Laplacian is presented systematically by Chung [3]. Mugnolo [11] develops a semigroup framework for linear diffusion equations on networks. Grigor'yan [7] develops heat-kernel methods on manifolds, while Keller, Lenz, and Wojciechowski [10] study Dirichlet forms on graphs. Coulhon et al. [5] establish integral maximum principles. In another direction, Chow, Mallet-Paret, and Shen [2] show that reaction-diffusion fronts on lattices may remain pinned.

The present paper works in $\ell^\infty(V)$ and assumes only local Lipschitz continuity of the reaction and a uniform degree bound. This choice covers finite graphs and a broad class of infinite graphs, while allowing bounded initial data that need not satisfy any summability condition.

Local well-posedness, continuation, and the finite-dimensional gradient formulation follow from standard results for differential equations in Banach spaces [4, 12]. They are stated explicitly to fix the assumptions and notation used throughout the paper. The graph-dependent part of the analysis is the adaptation of the maximum principle to $\ell^\infty(V)$ when the supremum is not attained. The main tool combines Dini derivatives with almost-maximizing sequences and yields comparison estimates, stability, absorption, and blow-up results in a common uniform-norm framework.

A distinction used repeatedly is essential. Invariance of an interval gives global existence and, under a dissipativity condition, entry into a bounded ball of $\ell^\infty(V)$. It does not by itself imply asymptotic convergence. On an infinite graph, a trajectory may remain bounded in $\ell^\infty(V)$ without possessing a convergent subsequence. This lack of compactness, already central in the theory of dissipative systems [8, 15], is explicitly taken into account.

Section 2 presents the functional setting and local well-posedness. Section 3 establishes the discrete maximum principle. Sections 4 and 5 derive stability under a strict restoring condition and dissipative absorption, respectively. Section 6 treats gradient systems on finite graphs, Section 7 studies superlinear blow-up, and Section 8 presents the numerical tests.

2. Functional Framework and Local Well-Posedness

We first introduce the notation used throughout the paper. The assumptions cover both finite graphs and infinite graphs of uniformly bounded degree while keeping the Laplacian bounded on $\ell^\infty(V)$.

Graphs. Let $G = (V, E)$ be an undirected graph whose vertex set V is finite or countably infinite. For $i \in V$, define

$$N(i) = \{j \in V : \{i, j\} \in E\}, \qquad d_i = \#N(i).$$

We impose the structural assumptions

(G0): $d_i < +\infty$ for every $i \in V$;
(G1): $d^* := \sup_{i \in V} d_i < +\infty$.

Discrete Laplacian. The unweighted discrete Laplacian is

$$(\Delta_G u)_i = \sum_{j \in N(i)} (u_j - u_i).$$

We write $L = -\Delta_G$. Under (G1), Δ_G is bounded on $\ell^\infty(V)$ and

$$\|\Delta_G u\|_\infty \le 2d^* \|u\|_\infty .$$

The same framework extends to weighted graphs when the weighted degrees are uniformly bounded.

Proposition 2.1. *Let $\omega_{ij} = \omega_{ji} \ge 0$ be symmetric weights and let $\mu_i > 0$ be vertex masses. Define*

$$(\Delta_{\omega,\mu} u)_i = \frac{1}{\mu_i} \sum_{j \sim i} \omega_{ij}(u_j - u_i).$$

If

$$d^*_{\omega,\mu} := \sup_i \frac{1}{\mu_i} \sum_{j \sim i} \omega_{ij} < +\infty,$$

then $\Delta_{\omega,\mu}$ is bounded on $\ell^\infty(V)$. Every argument that uses only boundedness of the Laplacian and the sign of vertex differences remains valid with d^* replaced by $d^*_{\omega,\mu}$. Spectral and Lyapunov results require additional assumptions adapted to the relevant inner product.

Proof. The estimate $\|\Delta_{\omega,\mu} u\|_\infty \le 2d^*_{\omega,\mu} \|u\|_\infty$ is immediate. If (i_k) is almost maximizing and $u_{i_k} \to M := \sup_V u$, then

$$(\Delta_{\omega,\mu} u)_{i_k} \le \frac{1}{\mu_{i_k}} \sum_{j \sim i_k} \omega_{i_k j}(M - u_{i_k}) \le d^*_{\omega,\mu}(M - u_{i_k}).$$

Hence $\limsup_{k\to\infty} (\Delta_{\omega,\mu} u)_{i_k} \le 0$. The corresponding estimate near the infimum follows symmetrically. The Dini-derivative, comparison, invariance, and absorption arguments therefore retain the same bounds. Local well-posedness then follows from the Cauchy-Lipschitz theorem in $\ell^\infty(V)$. □

Homogeneous Dirichlet condition. Let $\overline{S} = S \cup \partial S$, where S is the set of interior vertices and ∂S is the boundary vertex set. The homogeneous Dirichlet condition is $v_b(t) = 0$ for $b \in \partial S$. The zero extension of v to $\overline{S}$ is denoted by $\widetilde{v}$. For $i \in S$, define

$$(\Delta_D v)_i = \sum_{\substack{j \sim i \\ j \in S}} (v_j - v_i) + \sum_{\substack{b \sim i \\ b \in \partial S}} (0 - v_i). \tag{2.1}$$

The homogeneous Dirichlet problem is

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}(t) = D(\Delta_D v(t))_i + f(v_i(t)), \quad i \in S, \quad v^0 \in \ell^\infty(S), \quad v_b = 0 \text{ on } \partial S. \quad \text{(PD)}$$

The prescribed boundary value requires barriers that are compatible with zero on ∂S.

Proposition 2.2. *Assume $d_D^* := \sup_{i\in S} \#\{j \in S \cup \partial S : j \sim i\} < +\infty$. Then Δ_D is a bounded linear operator from $\ell^\infty(S)$ into itself. Moreover:*

- *(i) let $M_0 \geq 0$. If $A := \sup_{i\in S}(v_i - M_0) > 0$ and $(i_k) \subset S$ satisfies $v_{i_k} - M_0 \to A$, then $\limsup_k (\Delta_D v)_{i_k} \leq 0$;*
- *(ii) let $m_0 \leq 0$. If $B := \inf_{i\in S}(v_i - m_0) < 0$ and $(i_k) \subset S$ satisfies $v_{i_k} - m_0 \to B$, then $\liminf_k (\Delta_D v)_{i_k} \geq 0$;*
- *(iii) if $m_0 \leq 0 \leq M_0$, the constant barriers m_0 and M_0 are compatible with the homogeneous boundary condition.*

Proof. The estimate $|(\Delta_D v)_i| \leq 2d_D^* \|v\|_\infty$ follows from (2.1). For the upper estimate, set $w_i = v_i - M_0$ on S and $w_b = -M_0$ on ∂S. Since $M_0 \geq 0$ and $A > 0$, we have $w_b \leq 0 < A$. Every interior or boundary neighbor of i_k satisfies $w_j \leq A$, and therefore

$$(\Delta_D v)_{i_k} = (\Delta_D w)_{i_k} \leq d_D^*(A - w_{i_k}).$$

Taking the upper limit gives $\limsup_k (\Delta_D v)_{i_k} \leq 0$.

For the lower estimate, set $z_i = v_i - m_0$ on S and $z_b = -m_0$ on ∂S. Since $m_0 \leq 0$ and $B < 0$, we have $z_b \geq 0 > B$. Every interior or boundary neighbor of i_k therefore satisfies $z_j \geq B$, so

$$(\Delta_D v)_{i_k} = (\Delta_D z)_{i_k} \geq d_D^*(B - z_{i_k}).$$

Taking the lower limit gives $\liminf_k (\Delta_D v)_{i_k} \geq 0$. Finally, $m_0 \leq 0 \leq M_0$ ensures that the prescribed boundary value belongs to $[m_0, M_0]$. □

These estimates transfer the relevant barrier arguments to the Dirichlet problem, subject to the natural restriction imposed by the boundary value.

Corollary 2.3. *Under $d_D^* < +\infty$, local well-posedness, uniqueness, and the continuation criterion remain valid for (PD). Barrier arguments also yield positivity, invariance, and dissipative absorption whenever the intervals $[m, M]$ satisfy $m \leq 0 \leq M$. In contrast, a blow-up estimate based on a global comparison with the infimum does not transfer to the Dirichlet problem.*

Remark 2.4. Under homogeneous Dirichlet conditions, a nonzero constant c is generally not stationary because its zero boundary extension creates boundary contributions in Δ_D. Stationary states of (PD) satisfy $D\Delta_D \bar{v} + f(\bar{v}) = 0$ in S and $\bar{v} = 0$ on ∂S.

Solutions, stationary states, and blow-up. The following notions fix the terminology used below.

Definition 2.5. A classical solution of (P) on $[0, T)$ is a function $u \in C^1([0, T); \ell^\infty(V))$ satisfying the equation componentwise. A stationary state is a function $\bar{u} : V \to \mathbb{R}$ such that $D\Delta_G \bar{u} + f(\bar{u}) = 0$. In the

boundaryless case, every constant c satisfying $f(c) = 0$ is stationary. A maximal solution blows up in finite time if $T_{\max} < +\infty$ and

$$\|u(t)\|_\infty \longrightarrow +\infty \qquad \text{as } t \uparrow T_{\max}.$$

Local well-posedness. Define $\Phi : \ell^\infty(V) \to \ell^\infty(V)$ by

$$(\Phi(u))_i = D(\Delta_G u)_i + f(u_i).$$

Then (P) becomes $u' = \Phi(u)$ in the Banach space $\ell^\infty(V)$. We first verify the local regularity of the vector field.

Lemma 2.6. *Under (G1), if f is locally Lipschitz, then for every $R > 0$ there exists $L_R = 2Dd^* + \ell_R$ such that*

$$\|\Phi(u) - \Phi(v)\|_\infty \le L_R \|u - v\|_\infty$$

whenever $\|u\|_\infty, \|v\|_\infty \le R$.

Proof. We have $\|\Delta_G u - \Delta_G v\|_\infty \le 2d^* \|u - v\|_\infty$. If ℓ_R is a Lipschitz constant of f on $[-R, R]$, then $\|f(u) - f(v)\|_\infty \le \ell_R \|u - v\|_\infty$. □

The Cauchy-Lipschitz theorem in Banach spaces now applies directly.

Theorem 2.7. *For every $u^0 \in \ell^\infty(V)$, there exist $T_{\max} \in (0, +\infty]$ and a unique solution $u \in C^1([0, T_{\max}); \ell^\infty(V))$ of (P).*

Proof. The local Lipschitz continuity of Φ on $\ell^\infty(V)$ permits application of the Cauchy-Lipschitz theorem in a Banach space [4, 12]. This gives existence, uniqueness, and the maximal time $T_{\max}$. □

The continuation criterion below uses a uniform bound on the trajectory.

Lemma 2.8. *If $\sup_{0 \le t < T_{\max}} \|u(t)\|_\infty < +\infty$, then $T_{\max} = +\infty$. Moreover, if $T_{\max} < +\infty$, then $\|u(t)\|_\infty \to +\infty$ as $t \uparrow T_{\max}$.*

Proof. The local Lipschitz bounds for Φ, together with its boundedness on bounded balls, give a common positive existence time for solutions starting in any fixed ball of $\ell^\infty(V)$. If $T_{\max} < +\infty$ and some sequence $t_n \uparrow T_{\max}$ satisfied $\|u(t_n)\|_\infty \le R$, restarting at t_n would extend the solution beyond $T_{\max}$ for large n. This is impossible. In particular, $\|u(t)\|_\infty \to +\infty$ as $t \uparrow T_{\max}$. □

3. Discrete Maximum Principle

With local well-posedness established, we turn to the central comparison tool. On an infinite graph, the usual argument based on evaluating the equation at a maximizing vertex is unavailable. It is replaced by Dini estimates constructed from almost-maximizing sequences.

Dini derivatives of the supremum and infimum. We first control the temporal regularity of the supremum and infimum. For a locally Lipschitz function y, we use the right Dini derivatives

$$D^+ y(t) = \limsup_{h \downarrow 0} \frac{y(t+h) - y(t)}{h}, \qquad D_+ y(t) = \liminf_{h \downarrow 0} \frac{y(t+h) - y(t)}{h}.$$

Lemma 3.1. *Let $u \in C^1([0,T);\ell^\infty(V))$ and set*

$$M(t) = \sup_{i\in V} u_i(t), \qquad m(t) = \inf_{i\in V} u_i(t).$$

Then M and m are locally Lipschitz and

$$D^+M(t) \le \limsup_{\varepsilon\downarrow 0}\{\dot u_i(t) : u_i(t) > M(t)-\varepsilon\}, \tag{3.1}$$

$$D_+m(t) \ge \liminf_{\varepsilon\downarrow 0}\{\dot u_i(t) : u_i(t) < m(t)+\varepsilon\}. \tag{3.2}$$

Proof. On every compact interval $[0,T_0]$, the quantity $C_{T_0} := \sup_{[0,T_0]} \|\dot u(t)\|_\infty$ is finite. Hence $|u_i(t)-u_i(s)| \le C_{T_0}|t-s|$ uniformly in i, and therefore $|M(t)-M(s)| \le C_{T_0}|t-s|$.

To prove (3.1), choose a sequence $h_n \downarrow 0$ such that

$$\frac{M(t+h_n)-M(t)}{h_n} \longrightarrow D^+M(t).$$

Select i_n so that $u_{i_n}(t+h_n) > M(t+h_n) - h_n^2$. Since u is Lipschitz in time with values in $\ell^\infty(V)$, for all sufficiently large n,

$$u_{i_n}(t) > M(t) - h_n^2 - 2C_{T_0}h_n.$$

Thus (i_n) is almost maximizing at time t. Moreover,

$$\frac{M(t+h_n)-M(t)}{h_n} \le \frac{u_{i_n}(t+h_n)-u_{i_n}(t)}{h_n} + h_n.$$

The continuity of u' in $\ell^\infty(V)$ then gives

$$D^+M(t) \le \limsup_{n\to\infty} \dot u_{i_n}(t) \le \limsup_{\varepsilon\downarrow 0}\{\dot u_i(t) : u_i(t) > M(t)-\varepsilon\}.$$

Inequality (3.2) follows by applying the same argument to $-u$. □

Discrete maximum up to ε. We next determine the sign of the graph Laplacian along almost-extremizing sequences.

Lemma 3.2. *Let $u : V \to \mathbb{R}$ be bounded. If (i_k) is almost maximizing and $u_{i_k} \to M := \sup_i u_i$, then $\limsup_k (\Delta_G u)_{i_k} \le 0$. If (j_k) is almost minimizing and $u_{j_k} \to m := \inf_i u_i$, then $\liminf_k (\Delta_G u)_{j_k} \ge 0$.*

Proof. For every $i \in V$,

$$(\Delta_G u)_i = \sum_{j\in N(i)} (u_j - u_i) \le d_i(M-u_i) \le d^*(M-u_i).$$

Taking $i = i_k$ and then letting $k \to \infty$ gives the first inequality. The second follows symmetrically from the infimum. □

Remark 3.3. The correct conclusion is an inequality for the lim sup, not convergence of $(\Delta_G u)_{i_k}$ to zero. The Laplacian may remain strictly negative along an almost-maximizing sequence.

Combining the Dini-derivative estimates with the discrete Laplacian bounds yields the differential inequalities that replace the pointwise maximum principle.

Lemma 3.4. *If $u \in C^1([0,T);\ell^\infty(V))$ solves (P), then for every $t \in [0,T)$,*

$$D^+M(t) \le f(M(t)), \qquad D_+m(t) \ge f(m(t)). \tag{3.3}$$

Proof. Fix $t \in [0,T)$ and $\varepsilon > 0$. If $u_i(t) > M(t) - \varepsilon$, then

$$(\Delta_G u(t))_i \le d_i\Big(M(t) - u_i(t)\Big) \le d^*\varepsilon.$$

Consequently,

$$\sup_{u_i(t) > M(t)-\varepsilon} \dot{u}_i(t) \le Dd^*\varepsilon + \sup_{s\in[M(t)-\varepsilon, M(t)]} f(s).$$

The continuity of f and (3.1) give $D^+M(t) \le f(M(t))$ as $\varepsilon \downarrow 0$.

Similarly, if $u_i(t) < m(t) + \varepsilon$, then $(\Delta_G u(t))_i \ge -d^*\varepsilon$ and

$$\inf_{u_i(t) < m(t)+\varepsilon} \dot{u}_i(t) \ge -Dd^*\varepsilon + \inf_{s\in[m(t), m(t)+\varepsilon]} f(s).$$

Inequality (3.2) therefore gives $D_+m(t) \ge f(m(t))$. □

We shall use the following scalar comparison principle formulated in terms of Dini derivatives.

Lemma 3.5. *Let $y : [0,T) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz functions. If $D^+y(t) \le g(y(t))$ and $y(0) \le z_0$, then $y(t) \le z(t)$ on their common interval of existence, where $z' = g(z)$ and $z(0) = z_0$. Symmetrically, if $D_+y(t) \ge g(y(t))$ and $y(0) \ge z_0$, then $y(t) \ge z(t)$.*

Proof. Fix a compact interval $[0, T_0] \subset [0,T)$ on which y and z take values in a compact set K, and let L_K be a Lipschitz constant of g on K. The functions y and z are locally Lipschitz and hence absolutely continuous on $[0,T_0]$. Set

$$q(t) = (y(t) - z(t))_+.$$

For almost every t, the ordinary derivatives exist. If $y(t) > z(t)$, the Dini inequality gives $y'(t) \le g(y(t))$ and

$$q'(t) = y'(t) - z'(t) \le g(y(t)) - g(z(t)) \le L_K q(t).$$

If $y(t) < z(t)$, then $q'(t) = 0$. At differentiability points where $y(t) = z(t)$, one has $y'(t) - z'(t) \le 0$, so again $q'(t) \le L_K q(t)$. Since $q(0) = 0$, Gronwall's lemma gives $q \equiv 0$ on $[0,T_0]$. Because T_0 is arbitrary, $y(t) \le z(t)$ throughout the common interval of existence. The symmetric statement follows by applying the same argument to $(z - y)_+$. □

This comparison principle yields invariance of intervals compatible with the signs of the reaction term.

Proposition 3.6. *Assume $m < M$, $f(m) \ge 0$, $f(M) \le 0$, and $m \le u_i^0 \le M$ for every $i \in V$. Then $m \le u_i(t) \le M$ for every $i \in V$ and every $t \in [0, T_{\max})$. In particular, $T_{\max} = +\infty$.*

Proof. By (3.3), $D^+M(t) \le f(M(t))$. Let z_M solve $z' = f(z)$ with $z_M(0) = M$. Since $f(M) \le 0$, a first-crossing argument shows that $z_M(t) \le M$ throughout its interval of existence. Scalar comparison therefore gives $M(t) \le z_M(t) \le M$. Similarly, if $z_m(0) = m$, the condition $f(m) \ge 0$ implies $z_m(t) \ge m$, and the lower

comparison gives $m(t) \geq z_m(t) \geq m$. Hence every component remains in $[m, M]$. The continuation criterion then yields global existence. □

4. Global Stability under a Strict Restoring Condition

The maximum principle has an immediate asymptotic consequence: if the reaction strictly drives every value toward the same level c, then convergence is uniform in $\ell^\infty(V)$.

Theorem 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and assume*

$$(s - c)f(s) < 0 \qquad \text{for every } s \neq c. \tag{4.1}$$

Then, for every $u^0 \in \ell^\infty(V)$, the solution of (P) is defined on $[0, +\infty)$ and

$$\lim_{t \to +\infty} \|u(t) - c\|_{\ell^\infty(V)} = 0.$$

Proof. Set $A = \max\{c, \|u^0\|_\infty\}$ and $B = \min\{c, -\|u^0\|_\infty\}$. Condition (4.1) implies $f(A) \leq 0$ and $f(B) \geq 0$. Invariance of $[B, A]$ gives global existence.

Fix $\varepsilon > 0$. If $A > c + \varepsilon$, compactness of $[c + \varepsilon, A]$ gives

$$\alpha_\varepsilon := - \max_{s \in [c+\varepsilon, A]} f(s) > 0.$$

As long as $M(t) \geq c + \varepsilon$, one has $D^+M(t) \leq -\alpha_\varepsilon$. Comparison with a decreasing affine function shows that M reaches $c + \varepsilon$ no later than

$$T_\varepsilon^+ \leq \frac{(M(0) - c - \varepsilon)_+}{\alpha_\varepsilon},$$

and cannot subsequently cross this level upward. If $A \leq c + \varepsilon$, this conclusion is immediate.

Similarly, if $B < c - \varepsilon$, the strict restoring condition gives

$$\beta_\varepsilon := \min_{s \in [B, c-\varepsilon]} f(s) > 0.$$

As long as $m(t) \leq c - \varepsilon$, one has $D_+m(t) \geq \beta_\varepsilon$, and therefore

$$T_\varepsilon^- \leq \frac{(c - \varepsilon - m(0))_+}{\beta_\varepsilon}.$$

If $B \geq c - \varepsilon$, the lower estimate already holds. For $t \geq \max\{T_\varepsilon^+, T_\varepsilon^-\}$,

$$c - \varepsilon \leq m(t) \leq M(t) \leq c + \varepsilon.$$

Since ε is arbitrary, the convergence to c is uniform. □

Remark 4.2. Condition (4.1) excludes multistable nonlinearities such as $f(s) = s - s^3$, whose zeros -1, 0, and 1 produce several constant equilibria. In that case, the long-time behavior depends more delicately on the initial data.

5. Dissipativity and Uniform Absorption

We now weaken the global restoring assumption. A dissipative condition at large amplitudes prevents blow-up of the uniform norm and produces an absorbing ball.

Theorem 5.1. *Assume*

$$\exists R > 0,\ \exists a > 0 : \qquad f(s) \le -a \text{ for } s \ge R, \qquad f(s) \ge a \text{ for } s \le -R. \tag{5.1}$$

Then $T_{\max} = +\infty$ *and, for every* $\varepsilon > 0$, *there exists* $T_\varepsilon \ge 0$ *such that*

$$\|u(t)\|_{\ell^\infty(V)} \le R + \varepsilon \qquad \text{for every } t \ge T_\varepsilon.$$

Proof. Let $M(t) = \sup_i u_i(t)$ and $m(t) = \inf_i u_i(t)$, and set $C_0 = \max\{R, \|u^0\|_\infty\}$. By (5.1), $f(C_0) \le 0$ and $f(-C_0) \ge 0$. Hence $[-C_0, C_0]$ is invariant and

$$\|u(t)\|_\infty \le C_0 \qquad (0 \le t < T_{\max}).$$

The continuation criterion gives $T_{\max} = +\infty$.

Fix $\varepsilon > 0$. As long as $M(t) \ge R + \varepsilon$, condition (5.1) and the Dini inequality give

$$D^+ M(t) \le f(M(t)) \le -a.$$

Thus M reaches $R + \varepsilon$ no later than

$$T_M \le \frac{(M(0) - R - \varepsilon)_+}{a},$$

and cannot subsequently cross this level upward. The same argument applied to the minimum gives $D_+ m(t) \ge a$ whenever $m(t) \le -R - \varepsilon$, and hence

$$T_m \le \frac{(-R - \varepsilon - m(0))_+}{a}.$$

With $T_\varepsilon = \max\{T_M, T_m\}$, one obtains

$$-R - \varepsilon \le m(t) \le M(t) \le R + \varepsilon$$

for every $t \ge T_\varepsilon$. □

Remark 5.2. On an infinite graph, the ball $\|u\|_\infty \le R + \varepsilon$ is not relatively compact in $\ell^\infty(V)$. Absorption alone therefore yields neither a compact attractor nor asymptotic convergence without an additional assumption.

Remark 5.3. If f has several zeros, coupling through Δ_G may produce nonconstant stationary states satisfying $D\Delta_G \bar{u} + f(\bar{u}) = 0$. Their existence depends on the graph geometry and the diffusion intensity.

6. Lyapunov Structure on Finite Graphs

The preceding results do not provide orbital compactness on an infinite graph. When the graph is finite, this difficulty disappears and the analysis can be completed by a Lyapunov structure. Throughout this section, V is finite and $f = F'$ with $F \in C^1(\mathbb{R})$.

Boundaryless case. Define

$$E(u) = \frac{D}{2} \sum_{\{i,j\}\in E} (u_i - u_j)^2 - \sum_{i\in V} F(u_i). \tag{6.1}$$

Its derivative along solutions yields an exact dissipation identity.

Proposition 6.1. *For every classical solution on a finite graph,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) = -\sum_{i\in V} |\dot{u}_i(t)|^2 \leq 0.$$

Proof. Differentiating (6.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u) = D\langle Lu, \dot{u}\rangle - \sum_i F'(u_i)\dot{u}_i.$$

Since $F' = f$ and $L = -\Delta_G$, one has $D(Lu)_i - f(u_i) = -\dot{u}_i$, which gives the identity. □

Homogeneous Dirichlet condition. The energy adapted to the boundary is

$$E_D(v) = \frac{D}{2} \sum_{\{i,j\}\in E(\overline{S})} (\widetilde{v}_i - \widetilde{v}_j)^2 - \sum_{i\in S} F(v_i), \tag{6.2}$$

where the diffusion sum includes edges joining S to ∂S. Omitting these edges would invalidate the dissipation identity at vertices adjacent to the boundary.

Proposition 6.2. *For every classical solution of (PD),*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_D(v(t)) = -\sum_{i\in S} |\dot{v}_i(t)|^2 \leq 0.$$

Proof. The discrete Green identity with zero boundary extension gives

$$\sum_{\{i,j\}\in E(\overline{S})} (\widetilde{v}_i - \widetilde{v}_j)(\widetilde{w}_i - \widetilde{w}_j) = \langle L^D v, w\rangle_{\mathbb{R}^S}.$$

Differentiating (6.2) and using $L^D = -\Delta_D$ and $F' = f$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_D(v) = \sum_{i\in S} (D(L^D v)_i - f(v_i))\dot{v}_i = -\sum_{i\in S} |\dot{v}_i|^2.$$

□

Energy dissipation and finite-dimensional compactness describe the asymptotic behavior of bounded trajectories. We state the result in the boundaryless case; the Dirichlet version follows by replacing E with E_D.

Theorem 6.3. *Assume that V is finite and $f = F'$. Let $u(t)$ be a bounded global solution. Its ω-limit set is nonempty, compact, connected, invariant, and consists only of stationary states. If the set of stationary states is finite, then there exists a stationary state v^∞ such that $\|u(t) - v^\infty\|_\infty \to 0$ as $t \to +\infty$.*

Proof. Because V is finite, $\ell^\infty(V)$ is isomorphic to $\mathbb{R}^{|V|}$. The bounded trajectory is precompact and the energy is nonincreasing. Standard gradient-flow theory [8] then gives the stated properties of the ω-limit set. Since the energy is constant on $\omega(u^0)$, the dissipation identity forces $\dot{u} \equiv 0$ on this set. If the stationary states are finite in number, connectedness reduces $\omega(u^0)$ to a singleton. □

Laplacian spectrum. If G is finite, connected, and boundaryless, then $L = -\Delta_G$ is symmetric positive semidefinite, with eigenvalues

$$0 = \mu_1 < \mu_2 \le \cdots \le \mu_{|V|}.$$

For the path P_N without boundary conditions,

$$\mu_k = 2\left(1 - \cos\frac{(k-1)\pi}{N}\right), \qquad k = 1, \dots, N.$$

For a path with N interior vertices and homogeneous Dirichlet conditions at both endpoints [3],

$$\mu_k^D = 2\left(1 - \cos\frac{k\pi}{N+1}\right), \qquad k = 1, \dots, N.$$

Then $\mu_1^D > 0$, and constant functions are not admissible modes. This distinction is essential when interpreting the simulations.

If, in addition, f is C^1 near a stationary state v, the linearized equation $\dot{z} = D\Delta_G z + f'(v)z$ has operator

$$L_v = D\Delta_G + \operatorname{diag}(f'(v_i)).$$

If every eigenvalue of L_v is strictly negative, then v is locally exponentially stable.

7. Finite-Time Blow-Up for a Superlinear Reaction

We finally consider the regime opposite to the dissipative case, in which a superlinear reaction may cause finite-time blow-up. The analysis is carried out in the positive cone. When $p > 1$ is not an integer, the vector field can be defined on all of $\mathbb{R}$ by the locally Lipschitz extension $f(s) = (s_+)^p$, which agrees with s^p for $s \ge 0$. Positivity makes this extension immaterial for the solutions considered below.

Lemma 7.1. *The solution of $z' = z^p$, $z(0) = z_0 > 0$, $p > 1$, is*

$$z(t) = z_0\left[1 - (p-1)z_0^{p-1}t\right]^{-1/(p-1)},$$

and blows up at $T_{\mathrm{scal}}(z_0) = [(p-1)z_0^{p-1}]^{-1}$.

In the boundaryless case, estimates on the supremum and infimum directly bracket the maximal existence time.

Theorem 7.2. *Let $f(u) = u^p$, $p > 1$, let $D > 0$, and assume that G has bounded degree. If $m_0 := \inf_{i\in V} u_i^0 > 0$ and $M_0 := \|u^0\|_\infty$, then*

$$\frac{1}{(p-1)M_0^{p-1}} \le T_{\max} \le \frac{1}{(p-1)m_0^{p-1}}. \tag{7.1}$$

Moreover, $\|u(t)\|_\infty \to +\infty$ as $t \uparrow T_{\max}$.

Proof. Positivity follows by comparison with the zero solution. The inequality $D_+m(t) \geq m(t)^p$ gives $m(t) \geq z(t)$, where $z' = z^p$ and $z(0) = m_0$. As z becomes unbounded at $[(p-1)m_0^{p-1}]^{-1}$, the vector solution cannot exist beyond this time; otherwise its norm would be unbounded inside its existence interval. This proves the upper bound.

Similarly, $D^+M(t) \leq M(t)^p$ gives $M(t) \leq Z(t)$ for $Z' = Z^p$, $Z(0) = M_0$. If $T_{\max}$ were smaller than the blow-up time of Z, this bound would contradict the continuation criterion. The final assertion follows from the blow-up alternative proved in Section 2. □

Remark 7.3. The bounds in (7.1) follow directly from the Dini inequalities for the supremum and infimum. No correction term of the form $2Dd^*$ is needed in the boundaryless setting.

Under homogeneous Dirichlet conditions, comparison with the infimum is no longer available. We therefore project the solution onto the first positive Dirichlet eigenfunction.

Theorem 7.4. *Assume that S is finite and connected, $D > 0$, $p > 1$, and that v is a positive solution of (PD) with $f(u) = u^p$. Let $\lambda_1^D > 0$ be the first eigenvalue of $L^D = -\Delta_D$. Irreducibility of the Dirichlet operator on the connected component yields a positive eigenfunction ϕ_1, normalized by $\sum_{i\in S} \phi_{1,i} = 1$. If*

$$Y(0) := \sum_{i\in S} \phi_{1,i} v_i^0 > (D\lambda_1^D)^{1/(p-1)}, \tag{7.2}$$

then the solution blows up in finite time.

Proof. Set $Y(t) = \sum_{i\in S} \phi_{1,i} v_i(t)$. Then

$$Y'(t) = \sum_{i\in S} \phi_{1,i} \Big[D(\Delta_D v)_i + v_i^p \Big] = -D\lambda_1^D Y(t) + \sum_{i\in S} \phi_{1,i} v_i^p.$$

Since $\phi_{1,i} > 0$, $\sum_i \phi_{1,i} = 1$, and $v_i(t) \geq 0$, Jensen's inequality gives

$$\sum_{i\in S} \phi_{1,i} v_i^p \geq Y(t)^p.$$

Hence $Y'(t) \geq -D\lambda_1^D Y(t) + Y(t)^p$. Condition (7.2) places $Y(0)$ above the positive equilibrium of $z' = -D\lambda_1^D z + z^p$. For $z \geq Y(0)$, the right-hand side is positive, and $\int_{Y(0)}^{\infty} (s^p - D\lambda_1^D s)^{-1}\, \mathrm{d}s < +\infty$ because $p > 1$. Thus the scalar solution blows up in finite time. Comparison gives $Y(t) \geq z(t)$, so $T_{\max}$ is finite. The blow-up alternative then yields $\|v(t)\|_\infty \to +\infty$. □

8. Numerical Validation

We test two conclusions on the finite path with $N = 50$ interior vertices and zero Dirichlet boundary values. These computations do not reproduce the full geometry of an infinite graph. They are consistency checks for the absorption and blow-up results. The Dirichlet Laplacian is $\Delta_D = -L$, where $L = \mathrm{tridiag}(-1, 2, -1)$, and the semi-implicit scheme is

$$(I + \Delta t\, DL)u^{n+1} = u^n + \Delta t\, f(u^n). \tag{8.1}$$

The diffusion step has a positive, sub-Markov resolvent. As a result, invariance of an interval also holds for the numerical scheme when the explicit reaction map preserves that interval.

Proposition 8.1. *Assume $m < M$, $m \le 0 \le M$, $f(m) \ge 0$, $f(M) \le 0$, and that the map $s \mapsto s + \Delta t f(s)$ preserves $[m, M]$. Then scheme (8.1) preserves $[m, M]$.*

Proof. Set $A_{\Delta t} = (I + \Delta t DL)^{-1}$. The matrix $I + \Delta t DL$ is an M-matrix [1]; hence $A_{\Delta t}$ is entrywise nonnegative. Under homogeneous Dirichlet conditions, $L\mathbf{1} \ge 0$, and therefore

$$(I + \Delta t DL)\mathbf{1} \ge \mathbf{1}.$$

Positivity of the inverse gives $0 \le A_{\Delta t}\mathbf{1} \le \mathbf{1}$. If $v \in [m, M]^N$, then, because $m \le 0 \le M$,

$$A_{\Delta t} v \le M A_{\Delta t}\mathbf{1} \le M\mathbf{1}, \qquad A_{\Delta t} v \ge m A_{\Delta t}\mathbf{1} \ge m\mathbf{1}.$$

By assumption, the explicit reaction step maps $[m, M]^N$ into itself. The conclusion follows by induction. □

Dissipative absorption. We take $f(u) = u - u^3$, $D = 0.1$, $T = 20$, and $u_i^0 = 4\sin(\pi i/51)$. The absorbing radius is $R = 2$ and the discrete entry time is the first t_n after which $\|u^k\|_\infty \le R$ for all later steps. Table 1 gives the observed values for decreasing time steps.

Table 1. Numerical entry time into $\|u\|_\infty \le 2$ for $f(u) = u - u^3$.

Δt	10^{-3}	5×10^{-4}	2.5×10^{-4}	1.25×10^{-4}
$T_{\mathrm{abs}}(2)$	0.1110	0.1110	0.1113	0.1115

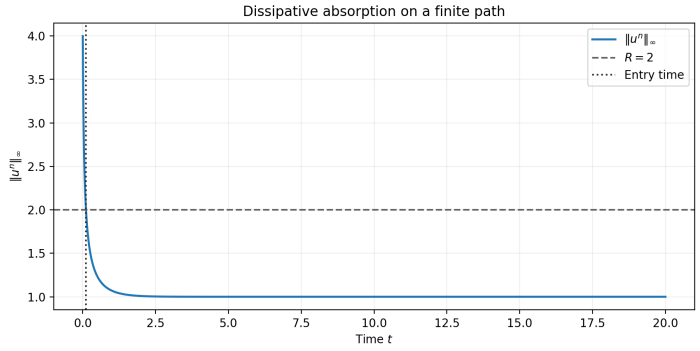

Figure 1. Decay of the maximum norm for $\Delta t = 10^{-3}$. The horizontal line marks $R = 2$, and the vertical line marks the entry time $t = 0.111$.

All four step sizes give nearly the same entry time. The numerical trajectory crosses the radius and remains below it, in agreement with the absorption theorem. This observation is not a numerical proof of an infinite-graph absorbing set.

Superlinear blow-up. For $f(u) = u^2$, $D = 0.1$, and $u_i^0 = A(1 + \sin^2(\pi i/51))$, the lowest Dirichlet eigenvalue is

$$\lambda_1^D = 2\left(1 - \cos\frac{\pi}{51}\right) \simeq 0.003793, \qquad D\lambda_1^D \simeq 0.000379.$$

The first-eigenfunction averages exceed $D\lambda_1^D$ for all tested amplitudes, so the Kaplan condition holds. With $\Delta t = 10^{-5}$, computation stops at $\|u^n\|_\infty \geq 10^6$.

Table 2. Kaplan criterion and numerical stopping times for u^2 reaction.

A	$Y(0)$	Kaplan condition	T_{num}
0.5	0.8334	satisfied	1.00098
1.0	1.6669	satisfied	0.50045
1.5	2.5003	satisfied	0.33365
2.0	3.3338	satisfied	0.25026

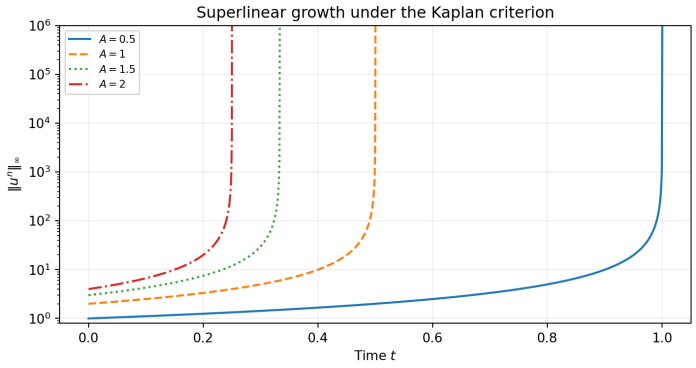

Figure 2. Superlinear growth of the discrete maximum norm for four amplitudes. Each computation stops at the first time step for which the discrete maximum norm reaches or exceeds 10^6.

Increasing A shortens the numerical stopping time, as expected from the scalar comparison. Since the stop is based on a finite threshold, T_{num} must not be identified with the exact blow-up time.

Conclusion and perspectives. We have developed a comparison and uniform-boundedness framework for reaction-diffusion equations on graphs of uniformly bounded degree in $\ell^\infty(V)$. The main technical difficulty is that a bounded vertex function on an infinite graph need not attain its supremum. Dini derivatives and almost-maximizing sequences replace the usual maximum-vertex argument and yield comparison estimates in the uniform norm.

The resulting maximum principle gives interval invariance, positivity, global stability under a strict restoring condition, and uniform absorption. In the superlinear regime $f(u) = u^p$, $p > 1$, the boundaryless and homogeneous Dirichlet problems must be distinguished. Without a boundary, direct comparison of the supremum and infimum brackets the blow-up time. Under homogeneous Dirichlet

conditions, this global infimum comparison is unavailable, and a spectral Kaplan criterion based on the first Dirichlet eigenvalue provides a sufficient condition for blow-up.

For finite graphs with gradient structure, the Lyapunov energy describes the ω-limit set and connects dissipation with convergence toward stationary states. On infinite graphs, however, absorption in $\ell^\infty(V)$ does not imply orbital compactness. Additional assumptions are therefore needed to obtain compact attractors or convergence to a stationary state.

The bounded-degree hypothesis excludes graphs with unbounded degree. Scalar lower-envelope blow-up bounds apply to boundaryless graphs, whereas the Dirichlet Kaplan condition is only sufficient. Front pinning is not studied, and the simulations concern finite graphs.

Further work may examine conditions ensuring orbital precompactness on infinite graphs, the dependence of a discrete Fujita exponent on graph geometry and growth, and schemes that preserve comparison and energy dissipation.

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